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This paper looks at a critical episode in the history of logic through the eyes of a philosophically motivated participant, Rudolf Carnap. As is generally known, Carnap's interest shifted, after the *Aufbau* (1928), to questions of logic and the foundations of mathematics. In 1929 he published the textbook *Abriß der Logistik*, and in 1930 a summary proof of what he called the *Gabelbarkeitssatz*, which says, roughly, that a consistent set of axioms is categorical just if it is complete. This not only sounds false, but raises further questions: What flaw in Carnap's conceptual framework enabled him to arrive at such a result? Why did the logicians to whom he showed it -- among them Fraenkel, Zermelo, Gödel -- not see what had gone wrong? And why did Carnap get interested in completeness and categoricity in the first place?

A closer look at the *Gabelbarkeitssatz* not only yields tentative answers to these questions, but casts a new and perhaps surprising light on the development of logic in the period leading up to Gödel's celebrated discoveries. Moreover, it affords a better understanding of the nature and motivations of Carnap's logicism which, we hope, points to a more unified and coherent view of his intellectual development.

We begin by describing Carnap's logical work in 1927-29, to which the *Gabelbarkeitssatz* was central, and show that it grew directly out of the *Aufbau* project. In section II, we focus on Carnap's version of logicism, which is the basis of the underlying continuity between his epistemological work of the earlier 1920's and his logical writings of 1927-9. In section III, we examine Carnap's logical investigations of this period, and focus on his proof of the *Gabelbarkeitssatz*. In section IV, we take a more modern view of these topics, pinpointing what went wrong in Carnap's investigations. But we also show that Carnap was not as mistaken as he initially appears to be. His understanding of the concepts involved, especially the concept(s) of completeness, was different from the modern one, and in his terms the proof was correct. In fact, it turns out that an interesting and neglected problem arises from Carnap's investigation. Finally, in section V, we conclude our story by following the development of his thought to the point where he abandoned the *Gabelbarkeitssatz* in 1930 -- though not before directing Gödel's attention to the problem that would lead him to his incompleteness theorems.

I. The *Untersuchungen zur allgemeinen Axiomatik* of 1927-9

Even before the *Aufbau* had been seen through the press, Carnap began devoting the bulk of his time to questions of logic and the foundations of mathematics. In 1927 and 1928 he worked steadily on what was evidently planned as a comprehensive treatise building on (and in scale comparable to) *Principia Mathematica* or Frege's *Grundgesetze*. It bore the suitably ambitious title *Untersuchungen zur allgemeinen Axiomatik*. What survives of it in his papers are two main parts, in different stages of completion. Part One, of something over 100 typed pages, was reworked at least twice, and circulated to a number of colleagues in late 1928.[1] Part Two consists of about 100 pages of shorthand notes and drafts.[2] Neither part was ever published, though a very brief summary of Part One was printed in *Erkenntnis* in 1930, almost exactly at the time when the whole project was abandoned (Carnap 1930d).[3]

What overall goals motivated the *Axiomatics* (as we will henceforth refer to this project)? This question is not as easy to answer as might appear[4], but at least it seems clear what role Part One was to play in the finished work. Its main theme is to make precise the interrelations among three different notions of completeness of an axiom system, as discussed in Fraenkel's *Einleitung in die Mengenlehre*[5] -- indeed, to show that the three are equivalent.

The three notions of completeness in question were, roughly, the following: (1) categoricity (which Carnap called *Monomorphie*); (2) decidability[6] (*Entscheidungsdefinitheit*); and (3) a notion that Carnap called "non-forkability" (*Nichtgabelbarkeit*) — roughly, any two models satisfy all the same sentences. The idea behind this odd-sounding term was that some axiom systems (e.g. Euclidean geometry without the parallel axiom) can be added to in ways that are incompatible with each other (e.g. the parallel axiom or Lobachevski's axiom). Such a system can be said to reach a "fork" (in the road) at that point. *Gabel* is German for "fork", *gabelbar* might be rendered as "forkable", and *Gabelbarkeit* as "forkability". We stick to Carnap's original, and call the statement that the three forms of completeness are equivalent the *Gabelbarkeitssatz*[7].

We will discuss Carnap's proof of this proposition in sections III and IV below. Here we will pursue the question how Carnap -- the epistemologist and philosopher of physics -- got interested in the problem of completeness to begin with. This will help to pinpoint the place of the *Gabelbarkeitssatz* in Carnap's overall conception of his task in the late 1920's.

What was the connection between the *Aufbau* program and the *Axiomatics*? He answers this question in his paper "Eigentliche und Uneigentliche Begriffe" ("Proper and Improper Concepts"; Carnap 1927), written at almost exactly the time of this transition. The constitution system of the *Aufbau*, he writes, requires explicit definitions; only by means of explicit definitions can all the concepts of the special sciences (the *Realbegriffe* or empirical concepts) -- "the most important sort of concepts, for the sake of which in the end all science is pursued" -- be brought into a genuine deductive interrelation [*Ableitungszusammenhang*], i.e. "they can be derived from one another using definitions [*definitorisch auseinander abgeleitet*]" and thus find their place within a single constitution system of the whole of science". (ibid., p. 357) Explicitly defined concepts are *eigentliche Begriffe* or "proper" concepts (ibid., p. 355), in contrast to those defined by an axiom system (defined implicitly), which are *uneigentliche Begriffe* -- "improper" concepts:

> Logically, the implicitly defined concepts differ so radically from proper concepts that one may well have doubts about even calling them "concepts" at all. We will retain this name, however, in view of common usage, especially within mathematics . . . There [in mathematics] one talks as if one were dealing with concepts -- "point", "line", "between", etc. -- that meet all the requirements of a legitimate concept. Since this is not the case, we will limit our terminological concession to usage by calling implicitly defined concepts "*improper concepts*". (ibid., pp. 366-7)

Some commentators have concluded from such statements that Carnap opposed the use of implicit definitions in science and mathematics altogether, thus taking a direct position against Schlick's *Allgemeine Erkenntnislehre*[8]. But we take these statements of Carnap's rather as the recognition of a *tension*, or the statement of a *problem*, at the basis of the kind of scientific philosophy the Vienna Circle was striving toward. Let us try to see this problem as it presented itself to Carnap in 1927.

On the one hand, no one could have been more enthusiastic about axiomatic formalization than Carnap. His first thesis proposal was entitled "Axiomatic Foundations of Kinematics". Though it was rejected by the physics department in Jena as too philosophical and the philosophy department as belonging to physics, Carnap eventually did carry it out (only an informal summary was ever published; Carnap 1925).[9] The logic textbook *Abriß der Logistik* (1929) contains many examples of axiomatizations of non-mathematical fields, including again the "axiom system of spacetime topology" (Carnap 1929a, pp. 80-85). There is no question that, throughout this period, Carnap fully understood the power of "axiomatic thinking" for unifying science and showing how various parts of human knowledge fit together[10] -- not just formal science, but empirical science as well.

On the other hand, as we have seen, Carnap did not regard implicitly defined concepts as proper concepts, because of their inherent ambiguity. They could not be employed in a constitution system of the concepts of science (whether on a phenomenalistic, a physicalistic, or any other basis). And we will see in the next section that Carnap doubted that they could, on their own, permit a deductive connection of "atomic sentences" -- in Wittgenstein's or any other sense -- to statements of physics containing numbers or other mathematical expressions.

The *Axiomatics,* culminating in the *Gabelbarkeitssatz,* was Carnap's answer (in 1927-30) to the question how, in view of this tension, to integrate axiom systems and implicit definitions into the *Aufbau* framework. In the 1927 paper we have been discussing, Carnap follows Frege's denial (in the dispute with Hilbert over the foundations of geometry) that implicit definitions enable the concepts they define to be applied unambiguously to objects. But a set of axioms can, he says, define a *second-order concept* explicitly, though this is of course a different concept and cannot be used in place of one defined implicitly by those same axioms.[11] (Carnap 1927, p. 368) Still following Frege, he stresses that the implicitly defined concepts are not constants, as explicitly defined concepts are, but *variables* that range over many different possible models, both formal and empirical (ibid., pp. 370-72).

Moreover, this is precisely what makes them useful in the construction of science; their deductive development "is a means of producing empty theories to be kept in reserve for later use" (ibid., p. 373). This useful property appears also to raise an epistemological problem, though; in contrast to proper concepts, whether empirical or mathematical, improper ones are not constituted from basics, but float loosely in space, one might say, unmoored to a source of definite meaning:

> Empirical concepts are constituted step by step in the systematic construction (*Aufbau*) of our world-knowledge. Each empirical concept, as a component of this structure, has a direct connection to reality. In contrast, the improper concepts hang in the air, so to speak, awaiting instructions. They are introduced by an axiom system, but that system doesn't relate directly to anything real. The axioms of this system and the theorems deduced from it don't properly form a theory (as they aren't actually *about* anything in particular), but rather just a theory-schema, an empty framework for possible theories. (ibid., p. 372)

There are actually two aspects to this, depending whether one views an axiom system from within, figuratively speaking, or from outside. From within the system, the problem is to ensure that it will not at any point in its deductive development permit ambiguities. For some implicitly defined concepts, as Carnap says, "*the principle of excluded middle does not hold*" (ibid., p. 364), i.e. we cannot establish either *p* or not-*p* for every sentence *p* containing the concept. But the *Gabelbarkeitssatz* can pick out the cases where this problem occurs, as Carnap immediately points out (ibid., pp. 364-65, where he also presents a brief argument for it), since if we can show that a given axiom system is categorical — like (higher-order) Peano arithmetic — then by the *Gabelbarkeitssatz* we also know that it is decidable; so the law of excluded middle holds for concepts implicitly defined by it.

Viewing an axiom system from outside, we encounter a different problem; we need to know how to distinguish the objects (in the physical or conceptual world) that fall under the concepts implicitly defined by the axiom system from those which do not. This is an instance of Frege's problem about the criterion for a proper definition of number; he thought a concept should be so defined that it could tell him whether Julius Caesar, for instance, or his pocket watch, fell under it (this might also be called the problem of *Eindeutigkeit*[12]). Carnap, Frege's attentive student, considers this question particularly with respect to implicit definitions:

> It belongs to the essence of a proper concept that for every object we can in principle decide whether it falls under that concept or not; and for sufficiently well-known objects the decision can be carried out in practice as well. For the empirical concept horse, for instance, and any visible object we can unambiguously [*eindeutig*] decide -- insofar as the concept has sufficiently sharp boundaries and the object is sufficiently well-known -- whether the object satisfies the concept, i.e. whether or not it is a horse. But for an improper concept the question whether a particular object falls under it is not decidable and thus has no sense. (Carnap 1927, p. 367)

Carnap addresses this problem in two steps. First, he says that although it is meaningless to say of one particular, isolated pocket watch (or particle, or sphere, or whatever) that it falls or does not fall under an improper concept, this *does* make sense if the object in question can be recognized as belonging to a structure that falls under the second-order concept explicitly defined by the axiom system (e.g. as an element of a progression). (ibid., p. 367) Without the establishment of such a context, the problem of *Eindeutigkeit* does not even make sense. What *does* have sense is the question whether a given collection of first-order objects, equipped with suitable operations and relations, falls under the second-order concept defined by the axiom system, i.e. (in modern terms) whether a structure of the right kind is a model of the axiom system.

This brings to mind Frege's context principle ("the meaning of words is to be sought only in the context of sentences [*im Satzzusammenhang*], not in isolation"). And in the second step of Carnap's argument, a more precise explication of the context principle gives it a more positive turn, endowing the notion of a context with more force. When the axiom system in question is (consistent and) decidable, thus specifying all the properties of the improper concepts it defines, the question remains whether it has defined such concepts uniquely (whether the objects have been successfully characterized). Once again, the *Gabelbarkeitssatz* plays a role in Carnap's solution, but now in the other direction of the equivalence it states. For by that entailment, if the axiom system is decidable, it is categorical and so has a unique model (up to isomorphism)[13], and the implicit definition has indeed given an unambiguous characterization of the objects falling under the concept it defines. This use of the *Gabelbarkeitssatz* that can be regarded as a formal analogue of the context principle: if we know the truth value of every sentence in which the names of the purported first-order objects occur, then we know all their possible uses (all the logical properties of the objects), and have thereby uniquely determined their meanings.[14]

Thus the *Gabelbarkeitssatz* has at least two important roles in Carnap's *Aufbau* project of "constituting" the objective world of science from subjective sense data (or *Elementar-erlebnisse*, momentary slices of total Gestalt-experiences). The method employed in the *Aufbau* was that of "purely structural designation" [*rein strukturelle Kennzeichnung*], memorably exemplified by Carnap in his image of a railroad map as a "structural" representation of actual railway connections among towns.

Such a structure is determinate and unambiguous [*eindeutig*] if it matches up uniquely with the actual phenomenon it is to represent, and not with any other. There is no certainty, however, that this will be the case even for a finite structure (like the railway map) that employs only explicit definitions. As Carnap points out, even when we have explored the whole map, there could still be points that are structurally indistinguishable.[15] We still have no guarantee that the empirical concepts in question (which are, after all, "the proper object of science", while mathematics and logic "serve only as aids for representing the knowledge of empirical concepts"; Carnap 1927, p. 373) are uniquely represented. For that to be the case, we must find an empirically constituted concept "for which it can be shown that it has the formal character [*formale Beschaffenheit*] of the improper concept given by the axiom system in question" (ibid., p. 372), and assigning this empirical concept to the improper concept as a value to a variable.

> Through the contact between the empirical concept and the axioms (in which the former satisfies the latter), a connection is created, by a single stroke, with the whole of the theory-schema resting on the axiom system. The blood of empirical reality streams in through the point of connection and flows into the most ramified capillaries of the hitherto empty schema, which is thereby transformed into a genuine theory. (ibid., p. 373)

Of course "it can be shown" only by induction and empirical testing ("constitution on the basis of the given", in the language of the *Aufbau* program) what "formal character" an empirical concept has; the assignment in question can only hold tentatively. But this does not, in Carnap's view, put improper concepts in a worse position than any concept whatever (even the simplest sensory qualities), regarding empirical assignments to them.[16]

Thanks to the *Gabelbarkeitssatz*, the application of axiom systems to empirical reality is, for Carnap, essentially no different from the application of *any* sort of abstract representation to empirical reality (even a finite one like the railway map). We will see now that it also has a role in this more general question of how mathematics can be applied to empirical reality, which turns out to be a central concern of Carnap's overall logicist program.

II. Carnap's Logicism

Logicism is generally seen as a program for the reduction of mathematics to logic (Stein 1998). Carnap's writings of 1929-30, especially "The Logicist Foundations of Mathematics"[17] confirm that he shared this program. We saw in the previous section, however, that Carnap's plunge into the philosophy of mathematics was motivated by epistemological problems arising from the *Aufbau* program. How did Frege-Russell logicism relate to these problems, and what role did the *Gabelbarkeitssatz* play within his logicism?

It was the Vienna Circle that gave logicism the philosophical significance it is still largely taken to possess. Frege-Russell reduction of mathematics to logic was combined, there, with Wittgenstein's diagnosis of both as holding vacuously or tautologically; they were pure artefacts of reason and thus empty of empirical content or meaning. Not only did this eliminate Kant's problem "How is pure mathematics possible?", it also gave a precise (and precisely limited) place to reason in knowledge. On the one hand reason was essential for proceeding from one empirical statement to others, i.e. for giving coherence and unity to empirical science. But by itself reason was vacuous; the only source of content was observation, and without observational content, all reasoning was empty. For the Vienna Circle, this conception pulled the rug out from the entire tradition of purely speculative philosophy in the style of Descartes's *Meditations* or Hegel's *Logic*.

Carnap's 1929-30 writings reflect this idea[18], but if we examine them more carefully, we can see that there was another dimension to Carnap's logicism. This is particularly evident in his Königsberg discussion remarks following the three papers by himself, Heyting, and von Neumann (representing logicism, intuitionism, and formalism, respectively), that were published along with those papers in *Erkenntnis* (Hahn *et al.* 1931). The way Carnap positions his own logicism here is telling; he portrays it as the standpoint of the *physicist* -- as opposed to those both of the *logician* and of the *mathematician*. The former he identifies with Frege's and Russell's logicism but also, interestingly, with intuitionism; both these approaches, he says, make the demand that "every sign of the language . . . must have a definite, specifiable meaning". The standpoint of the *mathematician* Carnap attributes to Hilbert, who says in effect, "We feel no obligation to be held accountable for the meaning of mathematical signs; we demand the right to operate axiomatically in freedom, i.e. to set up axioms and operational specifications for a mathematical field and then to find the consequences formalistically" (ibid., p. 141). These opposed standpoints of the logician and the mathematician appear irreconcilable.

But Carnap sees his own standpoint -- his own *logicist* standpoint (i.e. that of the physicist!) -- as a way of reconciling them;

> [Logicism] demands of the logico-mathematical system not only that it be consistent in itself, but that it be applicable in the field of empirical science. For it is the true purpose [*der eigentliche Sinn*] of this system to say how consequences can be deduced, i.e. which transformations of sentences are allowed. (ibid.)

Specifically, we want to be able to infer quantitative theoretical statements from qualitative atomic statements. So just as we demand of a deductive system that it get us from "All persons are mortal" and "All Greeks are persons" to "All Greeks are mortals" -- a condition that all usable deductive systems meet -- we are justified in demanding that we be able to transform qualitative into quantitative statements, i.e into statements involving numbers: We want to be able to get from the sentence "In this room there are only the persons Hans and Peter" to the sentence "In this room there are two persons". Otherwise, Carnap says, we can't apply arithmetic to empirical matters.[19] Frege-Russell logicism meets this requirement better than formalism: "With the Frege-Russell definition of number the required deduction is possible", while by Hilbert's method that is not so clearly the case, though "the exact form of the axiom system hasn't been given yet". Carnap ventures that a logical analysis of the formalist construction (once completed) would find that "formalism can't get around giving operational rules for mathematical signs, i.e. rules that determine the use of these signs not only within mathematics but also in empirical science. And by this determination the meaning of all signs is then implicitly fixed." (ibid., p. 142) More specifically:

> My supposition . . . is that this logical analysis of the formalistic system will have the following result (if this supposition turns out to hold, then despite the formalist method of construction, logicism would be justified and the opposition between the two approaches would be overcome):
>
> 1. For every mathematical sign one or more *interpretations* are found, and in fact purely logical interpretations.
>
> 2. If the axiom system is consistent, then upon replacing each mathematical sign by its logical interpretation (or any one of its various interpretations), every *mathematical formula* becomes a *tautology* (a generally valid sentence).
>
> 3. If the axiom system is complete (in Hilbert's sense: no non-derivable formula can be added without contradiction), then the interpretation is unique [*eindeutig*]; every sign has exactly one interpretation, and with that the formalist construction is transformed into a logicist one. (ibid., pp. 143-4)

This depends, once again, on the *Gabelbarkeitssatz*; in the last point quite explicitly. So not only does this theorem underlie Carnap's incorporation of implicitly defined concepts into his epistemological program, as we saw in the previous section; it also has a role in reconciling logicism with formalism, and more generally in the philosophical significance accorded by the Vienna Circle to logic (and logicism). The *Gabelbarkeitssatz*, then, occupies an absolutely central position in Carnap's thought during his Vienna period, especially the years 1927-1931, the classical and constitutive years of the Vienna Circle.

It may still be wondered why he should have given the name "logicist" to his position -- which in this discussion he called the standpoint "of the physicist", after all, in explicit *contrast* to Frege-Russell logicism, which is lumped together with Brouwer as the standpoint "of the logician". Logicism was a name for something to which Carnap claimed allegiance throughout his career, from 1920 to 1970 (Bohnert 1975). What he meant by this term changed over time, in its details, but the underlying motive remained constant: it was the need for a consistent and integral framework for the whole of science, empirical as well as formal -- the whole of thought, in Carnap's understanding.[20] It was his life-long preoccupation, from *Der Raum* through *The Continuum of Inductive Methods* and beyond, to create frameworks in which various competing philosophical or scientific positions could be viewed from a larger perspective, their differences specified precisely, and their respective costs and benefits weighed as rationally as possible. This required an over-arching framework of logic.

In the 1920's, this underlying motivation drew Carnap naturally to follow his teacher Frege and his guiding star Russell in their "universal" conception of logic, described by van Heijenoort as follows:

> The universality of logic expresses itself in an important feature of Frege's system. In that system the quantifiers binding individual variables range over all objects. [For Boole and de Morgan, the] universe of discourse comprehends only what we agree to consider at a certain time, in a certain context. For Frege it cannot be a question of changing universes. One could not even say that he restricts himself to *one* universe. His universe is *the* universe. Not necessarily the physical universe, of course, because for Frege some objects are not physical. Frege's universe consists of all that there is, and it is fixed. . . . This conception has several important consequences for logic. One, for instance, is that functions (hence, as a special case, concepts) must be defined for all objects. . . (van Heijenoort 1967, p. 325)

As Goldfarb (1979) has described, this was one of two conceptions of logic that flowed together to produce the modern model-theoretic view, the other being the "algebra of logic" due to Peirce, Schröder, and others.[21] Frege's view was dominant, however, among philosophically informed logicians in the late 1920's. Fraenkel, for instance, in the third edition of his *Einführung in die Mengenlehre* (1928) -- the very text from which Carnap took his three notions of completeness -- describes the properly "formalist" point of view regarding axiom systems as one that

> . . . takes the viewpoint of provisionally knowing nothing whatever about the objects and their interrelations, but rather of specifying those objects entirely by means of the axioms; in this way the expression "implicit definition" is given its precise meaning. However it must be emphasized, with respect to this conception, that somewhere in the whirl of axiomatic appearances there has to be a fixed pole: an absolute foundation for the formal theories that is taken as having content [*ein absoluter, inhaltlich verstandener Unterbau zu den formalen Theorien*], from which they ultimately derive their force. (Fraenkel 1928, p. 338)

In support of this, Fraenkel refers to a criticism of Hilbert by Hölder (1924, pp. 319-26), which Fraenkel claims is "generally acknowledged" to be right. He (among others) would thus appear to have shared with Carnap the view not only that mathematics and logic are something more than empty formulas, but that they must be given a *single* interpretation, an "absolute foundation" or "fixed pole".[22] Carnap's "absolute foundation" in the *Axiomatics* was a "foundation discipline" [*Grunddisziplin*] -- which we call "basic system" in section III below -- for general axiomatics:

> "General axiomatics" is the theory of axiom systems of arbitrary form . . . Every axiomatic investigation, even something as simple as deriving consequences from a single axiom system, presupposes a "basic system" [*Grunddisziplin*]: logic, arithmetic, and set theory (in the sense of "absolute" theories, in contrast to an axiomatic arithmetic and axiomatic set theory) . . . (Carnap 1930d, p. 303)

The role of the *Gabelbarkeitssatz* was to show that, given such a basic system, certain axiom systems of great importance to empirical science, like Peano arithmetic, could unambiguously be given their standard mathematical interpretations and could, as he stressed in the Königsberg discussion, be integrated seamlessly into the application of numbers to physical objects.

Carnap's somewhat cautious reference to the *Gabelbarkeitssatz* at the 1930 Königsberg conference, though, indicate that he was already beginning to have doubts. This is not surprising, as he had heard of Gödel's first incompleteness result a month before the conference, and although he had found it difficult to understand (Köhler 1991, pp. 150-51), he evidently realized that this central plank of his position might not be able to carry the enormous weight he had placed on it.

Later in the conference discussion quoted above, von Neumann challenged Carnap's claim that a consistency proof by itself guarantees the existence of a model. After an exchange on consistency in general, Gödel gives a carefully worded statement of his first incompleteness theorem; this is the first public statement of that famous result[23]. Encouraged by von Neumann, he continues:

> One can even (assuming the consistency of classical mathematics) give examples of sentences (of the same kind as Goldbach's and Fermat's) that are actually true [*inhaltlich richtig*], but

> not derivable within the formal system of classical mathematics. Adding the negation of such a sentence to the axioms of classical mathematics, one obtains a consistent system in which a sentence is derivable that is actually false [*inhaltlich falsch*]. (Hahn *et al*, p. 148)

The axiom system Gödel here uses to give an example of *Gabelbarkeit* is exactly the one (Peano arithmetic) whose decidability Carnap had argued for in the *Axiomatics* on the grounds that it was categorical. Something seems to have gone seriously wrong in Carnap's argument, to which we now direct our attention.

III. Carnap's *Axiomatics* and the Proof of the *Gabelbarkeitssatz*

The first three chapters of the *Axiomatics* are concerned with what Carnap calls the *Grunddisziplin* (which we will call "basic system"). This system, he says, is the logical basis that must be stipulated in order to conduct such an investigation. It is to consist of at least the basic principles of logic, arithmetic, and set theory. He emphasizes the need to make the laws of logic and the principles of inference explicit (ASP/RC 080-34-03, p. 5). Moreover, he says, this basic system is to be a system of sentences with "content" [*Inhalt*], i.e. with fixed meaning; they are not mere combinations of symbols. (ibid., p. 4) This he contrasts with the primitive symbols of an axiom system outside such a basic system, which have no fixed meaning:

> For that is just the essential character of an axiom system -- that it is not tied down to a particular area of application, that it deals not with objects determinate in themselves but with something indeterminate that gets its only determination through the axiom system. (ibid., p. 4)

The rest of Chapter 1 then develops the basic system, which is essentially the usual simple type theory with an axiom of infinity, as in the *Abriß der Logistik*.

In Chapter 2 Carnap defines the notion of a model and the relation of isomorphism. We summarize the main relevant steps, somewhat schematically. First, he observes (as in the 1927 article) that the primitive symbols of an axiom system can be regarded as variables, and that by conjoining the axioms one obtains a propositional function. Thus letting $R = (r_1 \ldots r_n)$ be an n-tuple of suitable (typed) variables, regarded as the primitive symbols of the axiom system in question, he can write the whole system as a single propositional function in the form $f(R) = f_1(R)$ & $f_2(R)$ & . . . & $f_m(R)$, where $f_1 \ldots f_m$ are the individual axioms. (For example we can let r_1 = point, r_2 = line, r_3 = between, . . . and let $f(R)$ be the conjunction of Hilbert's axioms for geometry.) If $g(R)$ is another propositional function in at most those free variables, g is defined to be a *consequence* of f if the proposition

(i) $$(\forall R)[f(R) \longrightarrow g(R)]$$

(also written $f \longrightarrow g$ for short) holds [*gilt*] in the basic system. Observe that since the primitive constants of the axiom system have been quantified out, (i) *is* indeed a sentence of the basic system. A *model* of f is defined to be an n-tuple $A = (a_1, \ldots, a_n)$ of logical constants (of suitable types) from the basic system which *satisfy* f, in the sense that $f(A)$ holds.

Unlike modern elementary axiomatic theories, Carnap's axiom systems have the peculiar feature that they include the language of the basic system, in which the models of the system are taken. As a result, some care is required in formulating the definitions of the different notions of completeness. Briefly, the issue is that if constants from the basic system occur in a propositional function, then satisfaction of it need not respect isomorphism of models (clearly a relevant property in this context).

Thus Carnap defines a propositional function f to be *formal* if it respects isomorphism, in the sense that if A and B are isomorphic structures of suitable type and A satisfies f then so does B (ASP/RC 080-34-03, pp. 75 f.). He then restricts attention to formal axiom systems and defines notions of interest like decidability with respect to formal propositional functions. In what follows we shall also assume without comment that the propositional functions mentioned are formal in this sense.

Chapter 3 is concerned with the *Gabelbarkeitssatz*. First, the following three notions of completeness are distinguished and named.

(1) *Decidable*. An axiom system f is *consistent* if for no propositional function g, both g and $\neg g$ are consequences of f. It is *decidable* [*entscheidungsdefinit*] if one of these two is always the case, i.e. if for every propositional function g, exactly one of g of $\neg g$ is a consequence of f.

(2) *Not gabelbar*. An axiom system f is said to be *satisfied* if the proposition $(\exists R) f(R)$ holds (in the basic system). Given a function g, the system f is called *gabelbar* ("forkable") at g if both f & g and f & $\neg g$ are satisfied. (Think for example of g as the axiom of parallels in Bolyai's "absolute" geometry, which is *gabelbar* at g because there are both Euclidean and non-Euclidean geometries.) An axiom system is said to be *gabelbar* if it is *gabelbar* at some g. If an axiom system is satisfied and not *gabelbar*, then any two models satisfy all the same propositional functions g.

(3) *Monomorphic*. An axiom system is said to be *monomorphic* if it is satisfied and any two of its models are isomorphic. It is *polymorphic* if it has non-isomorphic models.

The first two notions are easily seen to be equivalent, in Carnap's terms; the essential step of the proof is to show that any consistent axiom system is satisfied. We will return to this point in the next section. To show that all three notions of completeness are equivalent, then, it suffices to show that the last two are. And this is what the *Gabelbarkeitssatz* says; Carnap states it in the form:

(Theorem) An axiom system is *gabelbar* just if it is polymorphic.

Here is a sketch of Carnap's proof: Suppose f is *gabelbar*, say at the function g. Take models A and B such that $g(A)$ and $\neg g(B)$. Then A and B cannot be isomorphic, for otherwise we would then also have $g(B)$ (since g can be assumed to be formal). (ASP/RC 080-34-03, p. 87) Conversely, suppose f is polymorphic. Then there are non-isomorphic models A and B. Since A and B are not isomorphic, there should be some (formal) property that A has and that B does not. Suppose we have such a property g, with $g(A)$ and $\neg g(B)$. Then f is *gabelbar* at g. But such a property is easy to find: just let g be the property of being isomorphic to A, that is $g(R) =_{df} R \cong A$.

As a corollary, an axiom system is monomorphic just if it is decidable. And as an application of this, Carnap states that the system consisting of the Peano axioms for the natural numbers is decidable, since (as was well known) it is monomorphic. (ibid., pp. 90-92) This was just what Gödel had challenged at the close of section II above.

IV. What Went Wrong (Or Did It)?

It is clear in retrospect that Carnap's tools were inadequate to the uses he intended for them. Their fundamental shortcoming lay in his failure to distinguish between the axiom system under investigation and logical language used to conduct those investigations. The alien perspective that results can easily confuse the modern reader. As an example of this, let us now look at his proof that every consistent axiom system is satisfied (which sounds of course like Gödel's completeness theorem). It goes like this. Suppose the axiom system f is consistent, then by definition, for no g can we have both $f \longrightarrow g$ and $f \longrightarrow \neg g$. Thus Carnap concludes:

(iii) $\neg(\exists g)(f \longrightarrow g \;\&\; f \longrightarrow \neg g)$

so by the above definitions,

$$\neg(\exists g)\{(\forall R)[f(R) \longrightarrow g(R)] \;\&\; (\forall R)[f(R) \longrightarrow \neg g(R)]\}$$

From this, he straightforwardly infers

(iv) $(\exists R)f(R)$,

which is the definition of "f is satisfied". So the proof is already complete. (ASP 080-34-03, p. 49)

There are at least two problems with this. First, Carnap does not distinguish between the axiom system under investigation and the basic system *within* which the investigation is conducted. Thus the provability of (iii) in the basic system is not equivalent with the consistency of the axiom system

-- i.e. the absence of a formula g with the stated property. Similarly the provability of (iv) is not what we now mean by the existence of a model.

What made Carnap's definitions seem natural to him is his "universal" conception of logic, as discussed above (pp. 13-15). There is only one logic, in this conception. Not only were all possible axiom systems fragments of this universal system; all logical analysis *of* and statements *about* this system had *also* to be stated within this same universal system. So there was no distinction for Carnap between provability in an axiom system and the provability of a statement *about* axiom systems. Or, as we would now say, there was no distinction between the object language and the metalanguage.

There is a second problem, in addition to this (also part of the "universal" conception of logic discussed above (pp. 13-15), though the conflation of these two problems is not inherently necessary), and the combination of the two is what proved fatal to Carnap's goal: The sentences of the basic system, the *Grunddisziplin*, have "content" (they are interpreted, *inhaltlich*). This means that within the basic system (which is the whole of logic), we have not provability but an "absolute" concept of truth, which enables Carnap to pass from

for no $g, f \longrightarrow g$ and $f \longrightarrow \neg g$

to

$\neg(\exists g)(f \longrightarrow g \,\&\, f \longrightarrow \neg g)$ holds,

as above in (iii). Thus, in effect, "not $\vdash p$" and "$\vdash \neg p$" are synonymous and interchangeable.

The combination of these two assumptions about the universality of the single system and absolute truth within it prevented Carnap from being able to bring the *Axiomatics* to bear on the issues of provability and completeness. In general, his results are not false, indeed for the most part they are trivially true; the problem, rather, is that they fail to address the issues he intended to consider.

How do we see these issues now? What specifically is the relationship between categoricity and completeness, in the meanings we now give these terms?

A theory in first-order logic can only be categorical if it has no infinite models, by the Löwenheim-Skolem theorem. A categorical theory in first-order logic is obviously complete, but the converse is false since there are complete theories with infinite models. So the *Gabelbarkeitssatz* fails for this case.

The real case of interest is that of theories in higher-order logic. In talking about completeness for such theories, we must be careful to distinguish between provability and validity, or in other words between what we will call *syntactic* and *semantic* completeness, respectively. A theory is *syntactically complete* just if every sentence in its language is either provable or refutable; it is *semantically complete* just if any two models of it (standard models, not Henkin models) satisfy all the same sentences. (The former approximates Carnap's "decidable [*entscheidungsdefinit*]"; the latter approximates "not *gabelbar*".)

For the syntactic notion, we know that the *Gabelbarkeitssatz* is false, by Gödel's first incompleteness theorem; the theory of the natural numbers is categorical, but syntactically incomplete. But what about semantic completeness? It is true that any categorical theory is semantically complete, essentially by Carnap's argument, for there is then only one isomorphism class of models, and these all satisfy the same sentences. Indeed this was shown in detail for essentially this case by Lindenbaum and Tarski in 1935, who also consider the converse implication, that every semantically complete theory is categorical. They show that it too holds, under a certain rather strong condition[24]. They mention Carnap in a footnote, saying there was hardly a resemblance between their work and his, which seems an odd way of acknowledging his priority. They also remark that they have not explored the *general* case, i.e. the question whether (semantic) completeness implies categoricity in general.

Let us consider this problem. Take for example the "empty" theory, with no non-logical constants or axioms, formulated in simple type theory over a single ground type of individuals – what is sometimes called "pure higher-order logic". A model is just a set (interpreting the type of individuals), and two models are isomorphic just if those two sets have the same cardinality. Now let us add an axiom, which is just a sentence of pure higher-order logic. Such a sentence f is (semantically) complete, regarded as a theory, if whenever two sets A and B both satisfy f they then satisfy all the same sentences:

Definition. A sentence f is *complete* if for all sets A and B with $A \models f$ and $B \models f$, for all g, $A \models f$ iff $B \models f$.

Such complete sentences are easy to find, for example the sentence expressing that a set is countably infinite has this property. So does any other sentence expressing that a set has some particular cardinality. And of course these examples are also categorical. The question is: are these *all* the complete sentences? In other words, if two sets satisfy the same complete sentence, and thus satisfy all the same sentences, are they then necessarily isomorphic?

Stated in this form, Carnap's *Gabelbarkeitssatz* of 1928 becomes an interesting problem about the logical definability of cardinal numbers. The remarkable and surprising fact is that the answer turns out to be yes -- the semantic form of the *Gabelbarkeitssatz* (for this case) is indeed true: semantic completeness implies categoricity.[25]

V. Historical Coda

We have seen that the *Gabelbarkeitssatz* had a central place in the development of Carnap's thought between the *Aufbau* and the *Syntax*; in it, many strands of his philosophical development came together. As might be expected, then, the public presentation of the *Gabelbarkeitssatz* could hardly have been more emphatic and high-profile. Carnap made it the subject of his contribution to the First Conference on Epistemology of the Exact Sciences, held in Prague in 1929. This conference occupied a special place in the Vienna Circle's campaign to promote scientific philosophy. As Philipp Frank puts it,

> In 1929, we had the feeling that from the cooperation that was centered in Vienna a definitely new type of philosophy had emerged. As every father likes to show photographs of his baby, we were looking for means of communication. We wanted to present our brain child to the world at large, to find out its reaction, and to receive new stimulation.

> We decided first to publish a monograph about our movement, next, to arrange a meeting, and eventually to get control of a philosophical journal . . . (Frank 1949, p. 38)

The journal was, of course, *Erkenntnis*. The monograph was the programmatic *Wissenschaftliche Weltauffassung: Der Wiener Kreis* (Hahn *et al* 1929). And the "meeting" was the 1929 Prague conference, which was organized by Frank himself, who happened to be the chairman of the local organizing committee for a conference of the German Physical Society that year, and used his position (against some resistance) to append a separate meeting with the topic "Epistemology of the Exact Sciences" to the main proceedings, sponsored by the Ernst Mach Verein in Vienna and the Gesellschaft für empirische Philosophie in Berlin. (ibid., p. 40)

Apart from general, wide-ranging talks by Frank, Otto Neurath, and Hans Reichenbach, there was a session devoted to the foundations of mathematics, at which the two main papers were by Fraenkel and Carnap. Fraenkel gave a broad overview of "The present controversies in the foundations of mathematics". Carnap, by contrast, gave an austerely technical outline of Part I of his *Axiomatics*, culminating in the *Gabelbarkeitssatz*.

Present at the proceedings that day were, among others, Zermelo, Fraenkel, von Neumann, and Hahn. Carnap was distinctly encouraged by the interest his work aroused. As he confided to his diary on 17 September 1929,

> My lecture: *Investigations in General Axiomatics*; just a brief summary. But the proof is requested, and acknowledged. Though it was late, a lively discussion on the basic issues afterwards; von Neumann, Zermelo, Hahn. Fraenkel said that a final judgement will only be possible when the complete proof is available. Amazing interest in my *Investigations*. (ASP/RC 025-73-03)

It was not the first time he had imparted its main contents to colleagues. In June and July of 1928, he had presented it to the Vienna Circle.[26] He had given a lecture course at the University of Vienna in 1928 entitled "Metalogik" (another name he sometimes gave his *Axiomatics* project) And as we noted earlier, he had circulated the typescript to Fraenkel, Gödel, Schlick, and others.

Despite the central importance of the *Axiomatics* through the period 1927-29, Carnap abandoned the project in early 1930, almost exactly at the time that a brief summary of his Prague presentation appeared in *Erkenntnis* (Carnap 1930d). The decisive influence was Tarski, who was in Vienna in February 1930, lecturing on metamathematics. He also met privately with Carnap, who recalls in his autobiography:

> Of special interest to me was his emphasis that certain concepts used in logical investigations, e.g. the consistency of axioms, the provability of a theorem in the deductive system, and the like, are to be expressed not in the language of the axioms (later called the object language), but in the metamathematical language (later called the metalanguage). (Carnap 1963, p. 30)

In Carnap's papers, there is a page of shorthand notes on a "conversation with Tarski about categoricity [*Monomorphie*]", dated 22 February 1930. Here Carnap saw clearly that "nicht gabelbar" could not be defined in the language of the basic system itself without incurring severe costs. He then continues, "Tarski proceeds differently. He defines [*nicht gabelbar*] *metamathematically*" (ASP/RC 090-15-03, p. 1). The definition that follows distinguishes clearly between a statement expressed by a formula and a statement *about* a formula. Also, in the new definitions for *nicht gabelbar* and *entscheidungsdefinit*, the distinction between "v" within the system and "or" in the metalanguage is explicitly stressed. (ibid.)[27]

To accept this idea was to abandon the idea of a *Grunddisziplin* and of an investigation of the kind he had put so much work into. But that is what Carnap did. The last mention of the *Axiomatics* in Carnap's diary is from 24 February 1930, two days after the above note:

> Tarski visits me . . . talked about my *Axiomatics*. It seems correct, but certain concepts don't capture what is intended; they must be defined metamathematically rather than mathematically. (ASP/RC 025-73-04)

But that is not quite the end of the story. One of the students who attended Carnap's 1928 course on "Metalogik" was the 22-year-old Kurt Gödel; it was the first (and last) logic course he ever took (Wang 1987, p. 80). It seems clear that Gödel's interest in the completeness problem dates from this course. Carnap gave him the typescript of the *Axiomatics* (part I), and not only did Gödel acknowledge it in his dissertation; he even mentions Carnap's rather trivial proof (section IV above) that the consistency of an axiom system entails the existence of a model, under what Gödel calls the "Russellian" interpretation of implication rather than syntactic consequence or provability. And even at the end of his life, when he was asked what influences had led him to the problem of completeness, Gödel singled out (along with Hilbert and Ackermann's *Grundzüge der mathematischen Logik*) "Carnap's lectures on metalogic". (Wang 1987, p. 17)

What has been less clear, however, is how Carnap's *Axiomatics* provided, in Gödel's mind, the connection between the completeness of first-order logic and the incompleteness theorems that were shortly to follow. To see this we recall, first, Gödel's close relations with the Vienna Circle (particularly Carnap) at just this time, and second, his desire to refute their doctrines. Each of these has been discussed in the literature[28]. It is only by adding Carnap's *Axiomatics* (particularly the *Gabelbarkeitssatz*) to the picture, however, that they help the connection, in Gödel's mind, between his two most famous papers fall into place.

Beginning in late 1928, Carnap met frequently with Gödel. On 13 November 1928, for instance, Carnap notes "5-9 with Gödel in the café. On foundations of mathematics. He believes the derivation from logic has failed. He thinks very penetratingly." And later that month: "Arkadencafé in the morning. Discussion: Gödel on statements about language. With Waismann, Feigl, Natkin." And two weeks later, again, a discussion on decidability. (ASP/RC 025-73-02) It was during these years, also, that Gödel regularly attended the meetings of the Vienna Circle itself. (Wang 1987, pp. 49, 76 ff.) The central role of Carnap's *Gabelbarkeitssatz* for the *Aufbau* program, and the central role of the *Aufbau* program for logical empiricism more broadly, would hardly have escaped Gödel.[29]

That Gödel was at odds with this program is now generally agreed.[30] He later insisted that although his *interest* in the foundations of mathematics had been aroused by the Vienna Circle, "the philosophical consequences of my results, as well as the heuristic principles leading to them, are anything but positivistic or empiricistic", and it seems that his motivation was the philosophical one "to refute empiricism". (Wang 1996, p. 70) So Carnap's *Gabelbarkeitssatz*, with its central place in the Vienna Circle project, would have been an obvious target for Gödel's subversive attentions.

And in fact it *was* the main target of refutation when Gödel first mentioned his incompleteness theorem in public[31] — at the same time making crystal clear the connection between that result and the completeness theorem. An application of the latter, Gödel says, is in the "general theory of axiom systems", specifically the relation between the concepts "decidable" and "categorical" (he uses Carnap's terms *"entscheidungsdefinit"* and *"monomorph"*). In first-order logic the latter entails the former, as the completeness theorem shows. If this result could be extended to higher-order logic, he goes on, then Peano Arithmetic, since it is categorical, would also have been shown to be decidable. This is clearly aimed at Carnap (who was probably in the audience). However, Gödel continues (in a tantalizingly brief conclusion), this entailment fails, so the completeness theorem can*not* be so extended; Peano Arithmetic is *not* decidable. "It would take us too far afield, however, to go into these matters more closely." (Gödel 1930, pp. 28-30)

We do not suggest that Carnap's ideas were a necessary condition for Gödel's. As Goldfarb has remarked (1979, p. 365), the implications of Löwenheim's, Skolem's, and Herbrand's work led in the direction of Gödel's results; if Gödel hadn't discovered them when he did, someone else presumably would have before very long. But for the particular course history actually took (which should be of *some* interest), it seems indisputable that, beyond whatever significance it may still have on its own merits, the *Gabelbarkeitssatz* has the hitherto unsuspected historical importance of having pointed Gödel toward the greatest logical discoveries of the century.

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Endnotes

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Chicago (February 1997), Jena (June 1997), and Pittsburgh (February 1998), particularly Dana Scott for proving the *Gabelbarkeitssatz*. Carnap's papers at the Archive of Scientific Philosophy (Hillman Library, University of Pittsburgh) are quoted in the form "ASP/RC xxx-yy-zz", meaning "Archive of Scientific Philosophy, Rudolf Carnap papers, box xxx, folder yy, item zz"; all rights to these materials are held by the Hillman Library, University of Pittsburgh, and are quoted by permission. For assistance with them we are grateful to Gerald Heverly, Curator of the Archive of Scientific Philosophy. We are also grateful to Dr. Brigitte Uhlemann of the University of Konstanz for her prompt and accurate transliteration of a difficult shorthand item from these papers.

1. The latest version (of which pp. 1-51, 71b-105 are in ASP/RC 080-34-03 and pp. 52-71 in 080-34-04), probably dating from early to mid-1929, has numerous handwritten corrections and retyped insertions, whose general tendency is to shift the book from an earlier "constructivist" standpoint (though it had retained excluded middle) to a more neutral one. (In a discussion note dated 14 December 1928, Carnap writes, "Gödel says: if I want to follow the constructivist standpoint consistently, I will either have to reject the principle of excluded middle (since 'p or not-p' is not provable) or assume a 'complete' (decidable) logic! That seems right!" ASP/RC 081-01-32). These changes follow substantially the shorthand corrections in the version ASP/RC 080-34-02, which also contains marginal comments by Schlick. In addition, Carnap had circulated Part One to Hasso Härlen (from whom he received a long letter, ASP/RC 081-01-34), Heinrich Behmann (letter of 20 July 1929, ASP/RC 028-07-02), Reinhold Baer, Gödel (see below, part V), and Fraenkel, who responded after several months with a postcard (ASP/RC 081-01-27), saying that his concerns were similar to those of Baer's (not found), and that altough he was happy to recommend the book for publication, he didn't have time for detailed comments.

2. The "preliminary work plan" for Part Two (dated 1928; ASP/RC 081-01-02) -- which the extant notes follow more or less closely -- indicates that Carnap planned to address three issues: (i) "extremal axioms" like the completeness axiom in Hilbert's geometry and the induction axiom in Peano's arithmetic; (ii) existence axioms as compared to primitive symbols; (iii) axiomatization of the basic system, specifically for (absolute) logic, arithmetic, and set theory. Carnap's preliminary work on (i) was recycled a few years later in a joint article "Über Extremalaxiome"

with Friedrich Bachmann, a graduate student at Hannover who later became a distinguished geometer (Carnap and Bachmann 1936).

3. The fragment published in *Erkenntnis* summarized a conference presentation from the previous year (see below, section V).

4. Alberto Coffa, in what is to date the only extended commentary on the *Axiomatics*, diagnoses its purpose as follows: "From our own perspective, . . . the purpose of this book could be described as that of producing a formal explication of the basic proof-theoretic and model-theoretic notions of Hilbert's metamathematics within a type-theoretic language *but* eliminating the standard assumption that an object language is under consideration. Carnap's aim was, in effect, to stand up for the Russellian monolinguistic approach, in opposition to the growing trend in metamathematics toward a Hilbertian two-language approach to foundational matters. . . . Carnap's book was thus inspired by the . . . aim of showing that everything of value in metamathematics can (or should) be expressed within the monolinguistic framework of *Principia Mathematica*." (Coffa 1991, pp. 273-4) It will be seen in what follows why we think this characterization at best incomplete.

5. The second edition (Fraenkel 1923, pp. 226-28) mentions only two, decidability and categoricity. The third edition adds *Nichtgabelbarkeit* (without any reference to the literature), and contains a much fuller discussion of all three (Fraenkel 1928, pp. 347-54). Carnap reviewed the third edition (Carnap 1929b), giving particular attention to Fraenkel's treatment of these issues; in the *Axiomatics*, he credits Fraenkel as being the first to distinguish these three notions, "though still without precise definitions". (ASP/RC 080-34-03, p. 78a) Fraenkel also discusses the connections between the three notions, without any definite conclusions; here he includes an acknowledgement to Carnap 1927 and "deeper, still unpublished work by the same author and by A. Tarski" (ibid., p. 352). In his 1928 Bologna lecture setting out four unsolved problems of the foundations of mathematics, Hilbert had also touched on the connection between categoricity and decidability in the context of his third problem, the decidability [*Vollständigkeit*, here in the sense of Carnap's *Entscheidungs-definitheit*] of the axiom systems for number theory and analysis. (Hilbert 1928, p. 320)

6. This term has a different meaning in this context, of course, from its presently standard one. Here (as is explained more precisely in section III below) an axiom system is called "decidable" if

no axiom can consistently be added to it. It should be remarked that in all three notions of completeness Carnap presupposes consistency.

7. As we will see below (section III), the actual statement of the theorem in the *Axiomatics* (and in Carnap 1930d) is somewhat narrower than this.

8. Goldfarb (1996), for instance, writes "In the end, for Carnap there is *no* place for implicitly defined concepts in knowledge . . . Ultimately, there is no place for implicit definitions in the foundations of arithmetic, either." (pp. 217-18). In the *Allgemeine Erkenntnislehre* (second edition, Schlick 1925), Schlick says ". . . an implicit definition has nothing in common with reality, no connection with it; implicit definition rejects such connection purposely, and in principle; it sticks to the realm of concepts. A system of truths created with the aid of implicit definition rests nowhere on the foundation of reality, but rather floats freely . . . In general, we deal with the abstract only to apply it to the concrete. But – and this is the point to which our investigation returns again and again – the moment a conceptual relation is applied to a concrete [*anschauliche*] example, exact rigor is no longer guaranteed. Given real objects, how can we ever know with absolute certainty whether they stand in precisely those relations to each other which are fixed in the postulates by means of which we can define our concepts?" (Schlick 1925, p. 55) This paradox is never resolved in the *Erkenntnislehre*, and Goldfarb is right to see Carnap's efforts of this period as picking up at precisely this point.

9. Carnap 1963, p. 11-15; the original proposal, dated June 1920, is preserved in ASP/RC 081-06-01. Don Howard (1996) presents a broad panorama of the efforts within various philosophical and scientific movements of the early twentieth century to come to terms with the growing divergence between the mathematical abstraction of scientific theory and the specificity of the sense-data on which it was supposed to be "based". Of particular concern in these reflections, Howard points out, was that *Eindeutigkeit* -- uniqueness or "univocalness" -- be preserved. This concern was particularly urgent in view of multiplicity of interpretations an axiomatic system could be given, and is well expressed by Schlick in the passage quoted in the preceding note. We have doubts, however, about Howard's interpretation of Carnap's 1927 article; see below, note 11.

10. Hilbert, especially his wider program for "Tieferlegung der Fundamente" expounded in the article "Axiomatisches Denken" (Hilbert 1918, cited e.g., in the *Abriß*, p. 72), was clearly an

inspiration, though Carnap's version of logicism clove, for epistemological reasons, more to what Hilbert called the "genetic method" than the "axiomatic" one Hilbert preferred. As we will see below (this section and section II), however, Carnap's efforts during this period are better seen as an attempt to *reconcile* the genetic and axiomatic methods, or *combine* their respective strengths.

11. Thus the axioms of Euclidean geometry do not explicitly define the concepts that occur in them, i.e. "point", "line", "between", etc., but they do explicitly define the second-order concept of a "Euclidean geometry".

12. Don Howard (1996) has pointed out Carnap's stress on *Eindeutigkeit* in his work of this period (see footnote 7 above). However, he seeks to show by an elaborate argument that this preoccupation "arose in the context of the so-called *Lochbetrachtung*)(hole argument) in general relativity around 1915" (p. 115), which "suggests a very different way of telling the history of the categoricity concept . . ." (p. 116). While this might indeed have been a distant influence, and he is right to draw attention to the widespread concern for *Eindeutigkeit* among philosophers and scientists in the early twentieth century (especially Carnap), Howard gives scant attention to the much more obvious source of this preoccupation in Carnap's well-informed and first-hand knowledge of Frege's logical doctrines.

13. Since isomorphic models satisfy all the same logical sentences, the best one can hope for logically is to characterize a single model *up to isomorphism*, i.e. to pick out one isomorphism class of models. (For most worries about *Eindeutigkeit* during this period, it evidently sufficed; Fraenkel (1928, p. 350), for instance, called such a characterization a "formally unambiguous determination [*formal eindeutige Festlegung*] of the basic concepts by the axioms".) And this is in fact what the (reverse) *Gabelbarkeitssatz* (decidable implies categorical) yields. In the *Axiomatics*, Carnap develops a notion of "structure" to address this issue. In a quite modern spirit, models are said to have the *same structure* just if they are isomorphic, and a *structural* property of models is one that respects isomorphism. The *Gabelbarkeitssatz* can thus be understood to say that an axiom system determines a unique structure just if it decides every definable structural property of its models. (On Carnap's definition of isomorphism, see further discussion in section III below, pp. 17-18.)

14. This view of Frege's context principle is expounded in Tait 1986 and Reck 1997.

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15. There is also the *further* problem, which Carnap never addresses or solves within the *Aufbau* framework, how we are to know what the structure is intended to represent in the first place, without having it labelled for us (as a "railway map", for instance). Carnap was certainly aware of this problem, but his ultimate solution in the *Aufbau* itself — the notorious "founded" relation (Carnap 1928, §§ 153-5) — is clearly unsatisfactory. However, the passages quoted below (in the remainder of this section) indicate a clearer awareness of the complexity of the problem, and point forward to the more adequate solutions in Carnap's later writings.

16. In principle, new deductive consequences from an axiom containing such a concept have to be tested at any point in the deductive chain where they are exposed to factual scrutiny. But assigning an empirically constituted concept to an axiomatically defined one is no different in principle, Carnap holds, from assigning an empirically consituted concept (e.g. a length) to a number, or even assigning a particular quality to a particular object (that paradigm of the observation statement!); the inductive uncertainty is of precisely the same kind in all these cases. Carnap had recently spelled this out in *Physikalische Begriffsbildung* (Carnap 1926): "All physical statements are conditional statements," he writes there (p. 8); they state conditions under which certain things will be observed. So by their very nature, "*all physical statements*, just because they are conditional statements, *claim more than has been observed*, than can ever be observed, *i.e. they claim more than one may justifiably claim*." This is the problem of induction, and the principle on which induction rests, i.e. "the same thing occurs when the conditions are the same" [*unter gleichen Bedingungen geschieht Gleiches*] can never be tested because the conditions are never the same. So "induction has no logically rigorous justification"; its legitimacy derives rather from "experiential success" [*erfahrungsmäßige Bewährung*]. (ibid.)

Thus assigning an empirical concept "of the same formal composition" to an implicitly defined one involves no loss of *Eindeutigkeit* beyond that inherent in *any* use of concepts (even the simplest sensory qualities) in empirical contexts. Carnap thus responds to worries like Schlick's (1925; see note 7 above) on the one hand, and Poincaré's or Dingler's on the other, about the inherent arbitrariness of such assignments; it is no more arbitrary to assign physical "points" to Hilbert's implicitly defined points, or a physical length to a rational number, Carnap is saying, than it is to call a rose red. There remained the entirely *formal* problem of defining *numbers* unambiguously, to ensure that *modulo* inductive uncertainty, deduction of statements containing numbers from atomic (observation) statements was possible; this is the subject of section II below.

17. "Die logizistische Grundlegung der Mathematik" (Carnap 1931) was Carnap's contribution to the Königsberg conference; see below (this section) for the discussion following these contributions, and Gödel's remarks.

18. One of the classic expressions of this view is in Carnap's own article "Die Alte und die Neue Logik" (1930a, p. 25).

19. "The mathematician does not, within his field, need to concern himself with such application. But in the framework of the whole of knowledge we must of course demand the possibility of applying arithmetic to sentences about reality; otherwise there could be no physics." (Hahn *et al.* 1931, pp. 141-42)

20. Programmatic statements of this stance are rare in Carnap's writings, but see Carnap 1934, p. 255, and Carnap 1963, pp. 927 ff., as well as the remarks quoted by Bohnert 1975, pp. 210-12.

21. He describes the complementarity between these two views in an epigrammatic summary: "The deficiencies in the two early traditions I have been discussing may be summarized thus. To arrive at metamathematics from Russell's approach we must add the "meta", that is, the possibility of examining logical systems from an external standpoint. To arrive at metamathematics from the algebra of logic we must add the "mathematics", that is, an accurate appreciation of how the system may be used to encode mathematics, and hence of how our metasystematic analyses can be taken to be about mathematics." (ibid., p. 356)

22. Carnap's understanding of this task, and the role of "metamathematics" in it (as well as the continuing influence on him of the *Tractatus*), are indicated in the conclusion of a 1930 review of Felix Kaufmann's *Das Unendliche in der Mathematik und seine Ausschaltung*: "The task (still unsolved) of the logical foundations of mathematics consists in setting up a symbolism that on the one hand makes it possible to express the mathematical knowledge we have so far attained, and on the other hand excludes everything logically impermissible by means of its syntax (i.e. by means of the metamathematical rules of the calculus). 'Automatically' here means: there should be no further need in particular cases to give attention to the content or meaning of mathematical

concepts, since that is done once for all at the outset, when the symbolism is set up." (Carnap 1930b, col. 1678)

23. Except for a brief allusion the day before in Gödel 1930 (discussed below in section V) — which, however, unlike the above-quoted discussion remark, remained unpublished until 1995.

24. Lindenbaum and Tarski 1935, pp. 390-92. Note that this restriction, that the axiom system be "effectively interpretable" (pp. 390-391), is in effect imposed by Carnap in the *Axiomatics* as well, where all sentences are interpretable in the basic system.

25. This was proved by Dana Scott in the discussion of this paper at the University of Pittsburgh Center for the Philosophy of Science. His argument is as follows: For any sentence f define the new sentence

$$f_{min} = f \,\&\, (\forall U)(f^{U} \rightarrow U \cong X)$$

$$= \text{"X is the smallest set satisfying } f\text{"}$$

where the parameter X represents the basic domain of individuals, U is a variable of type P(X), $U \cong X$ is given by the usual definition of isomorphism, and f^{U} is a new sentence derived from f by relativizing all types and quantifiers occurring in f from X to U.

If f is satisfied, then so is f_{min} (by the axiom of choice for sets). So if f is complete, then

$$f_{min} \equiv f$$

for if $S \models f$, then take $S' \models f_{min}$; since $S' \models f$, then also $S \models f_{min}$. But f_{min} is evidently categorical, so f must be so, too.

26. Outlines of these talks, on 21 June and 5 July 1928, are to be found in ASP/RC 081-01-23; their main focus is completeness and the *Gabelbarkeitssatz*.

27. Carnap still says, presumably on the basis of Tarski's comments, that *nicht gabelbar* and *monomorph* are equivalent, "but proof difficult [*Beweis aber schwierig*]", he notes laconically. This presumably explains why, despite hearing of Gödel's results, he said what he did at the discussion in Königsberg later that year.

28. The connection with the Vienna Circle by Köhler (1991); the desire to refute their doctrines by Parsons (1995), Wang (1996), and Feferman (1998).

[29]. It is stressed, for instance, as the central philosophical program in *Wissenschaftliche Weltauffassung* (Hahn *et al* 1930, pp. 90-91), and Philipp Frank says that, in the *Aufbau*, ". . . Carnap gave the new philosophy its 'classical' shape. He coined many of its terms and phrases and endowed it with subtlety and simplicity. In the form created by Carnap it became a center of interest and a target of attack on a large scale . . . Our Viennese group saw in Carnap's work the synthesis we had advocated for many years." (Frank 1949, p. 33)

[30]. Not only did Gödel insist that he had been a Platonist throughout his Vienna period, but, as Feferman points out, "he credited his enormous successes in mathematical logic during the 1930's almost entirely to his holding this point of view". (Feferman 1998, p. 150) Though much stimulated by the Vienna Circle, there is no evidence that he ever agreed with its main tenets. Once he had established his own reputation, with his famous papers of 1930 and 1931, he steadily dissociated himself from it (though he maintained personal relations with Carnap, at least). By the 1940's he was stating (e.g. in a letter to his mother) that he had always been "in many respects . . . in direct opposition to [the Schlick Circle's] principal conceptions" (Wang 1996, p. 70). Parsons (1995, pp. 48-52) is less convinced.

[31]. In the talk (Gödel 1930, pp. 28-30) summarizing his completeness theorem at the 1930 Königsberg conference, the day before the discussion quoted above (section II). We are grateful to Warren Goldfarb for pointing out this passage to us.

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