SHEAF REPRESENTATION
FOR
TOPOI

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Sheaf Representation for Topoi

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Abstract

It is shown that every (small) topos is equivalent to the category of global sections of a sheaf of so-called hyperlocal topoi, improving on a result of Lambek & Moerdijk. It follows that every boolean topos is equivalent to the global sections of a sheaf of well-pointed topoi. Completeness theorems for higher-order logic result as corollaries.

The main result of this paper is the following.

Theorem (Sheaf representation for topoi). *For any small topos $\mathcal{E}$, there is a sheaf of categories $\widetilde{\mathcal{E}}$ on a topological space, such that:*

(i) $\mathcal{E}$ is equivalent to the category of global sections of $\widetilde{\mathcal{E}}$,

(ii) every stalk of $\widetilde{\mathcal{E}}$ is a hyperlocal topos.

Moreover, $\mathcal{E}$ is boolean just if every stalk of $\widetilde{\mathcal{E}}$ is well-pointed.

Before defining the term "hyperlocal," we indicate some of the background of the theorem. The original and most familiar sheaf representations are for commutative rings (see [12, ch. 5] for a survey); e.g. a well-known theorem due to Grothendieck [9] asserts that every commutative ring is isomorphic to the ring of global sections of a sheaf of local rings. In Lambek & Moerdijk [16] it is shown that topoi admit a similar sheaf representation: every topos is equivalent to the topos of global sections of a sheaf of local topoi (cf. also [17, II.18]. A topos $\mathcal{E}$ is called *local* if the Heyting algebra $\mathrm{Sub}_{\mathcal{E}}(1)$ of subobjects of the terminal object 1 of $\mathcal{E}$ has a unique maximal ideal, in analogy with commutative rings. It is easily seen that a topos $\mathcal{E}$ is local iff 1 is indecomposable: for any $p, q \in \mathrm{Sub}_{\mathcal{E}}(1)$, if $p \vee q = 1$ then $p = 1$ or $q = 1$. In logical terms, a classifying topos $\mathcal{S}[\mathbb{T}]$ for a (possibly

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higher-order) theory $\mathbb{T}$ is thus local iff the theory $\mathbb{T}$ has the "disjunction property": for any $\mathbb{T}$-sentences p, q, if $\mathbb{T} \vdash p \vee q$ then $\mathbb{T} \vdash p$ or $\mathbb{T} \vdash q$ (cf. §3 below for classifying topoi).

A sheaf representation such as those just mentioned yields an embedding theorem, which in the case of topoi yields a logical completeness theorem (just how is shown in §3 below). From a logical point of view, however, the local topoi of the Lambek-Moerdijk representation fall short of being those of interest for completeness. For, by other methods, one can already prove logical completeness with respect to a class of topoi that are even more "**Set**-like" than local ones, in that the terminal object 1 is also projective. Such topoi, in which 1 is both indecomposable and projective, shall here be called *hyperlocal.* In logical terms, a classifying topos $\mathcal{S}[\mathbb{T}]$ is hyperlocal iff the theory $\mathbb{T}$ has both the disjunction property just mentioned and the so-called existence property: for any type X and any formula $\varphi(x)$ in at most one free variable x of type X, if $\mathbb{T} \vdash \exists_x.\varphi(x)$ then $\mathbb{T} \vdash \varphi(c)$ for some closed term c of type X. Hyperlocal topoi are called "models" in [17] (see §§17–19 for the related completeness theorem). In Lambek [15] the above-mentioned logical shortcoming of the Lambek-Moerdijk sheaf representation is noted, and the following improvement is given: for every topos $\mathcal{E}$ there is a faithful logical morphism $\mathcal{E} \to \mathcal{F}$ into a topos $\mathcal{F}$ that is equivalent to the topos of global sections of a sheaf of hyperlocal topoi. The sheaf representation theorem of this paper thus fits into this pattern of theorems; it states that every topos is equivalent to the topos of global sections of a sheaf of hyperlocal topoi. Moreover, it follows that every boolean topos is equivalent to the topos of global sections of a sheaf of well-pointed topoi. With respect to logical completeness, these are the desired results.

The paper is arranged as follows. In §1 it is shown that every topos can be represented as a sheaf of categories on a Grothendieck site (rather than a space). The sheaf in question arises most naturally, not as a sheaf, but as something more general called a "stack." Most of §1 is devoted to the technical problem of turning this (or any) stack into a sheaf. In §2 a recent covering theorem for topoi is used to transport the sheaf constructed in §1 from the site to a space. A comparison of the transported sheaf with the original one then completes the proof of the sheaf representation theorem. In §3 several logical completeness theorems are derived as corollaries.

We shall have to do with both small elementary topoi and (necessarily large) Grothendieck topoi. We maintain the convention that "topos" unqualified means the former, but we may still add the qualification "small" for emphasis when called for. We assume familiarity with the basic theory of Grothendieck topoi, e.g. as exposed in [19].

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1 Slices, stacks, and sheaves

Throughout this section, let $\mathcal{E}$ be a fixed small topos. We begin by defining the $\mathcal{E}$-indexed category $\mathcal{E}/$ (for indexed categories, see [20], [23]). Recall that an $\mathcal{E}$-indexed category $\mathbf{A}$ is essentially the same thing as a pseudofunctor $\mathbf{A} : \mathcal{E}^{\mathrm{op}} \to \mathbf{CAT}$, i.e. a contravariant "functor up to isomorphism" on $\mathcal{E}$ with values in the category $\mathbf{CAT}$ of (possibly large) categories. Since the only indexed categories to be considered here are $\mathcal{E}$-indexed, henceforth *indexed category* shall mean $\mathcal{E}$-indexed category.

The indexed category

$$\mathcal{E}/ : \mathcal{E}^{\mathrm{op}} \to \mathbf{CAT}$$

is defined as follows. For each object I of $\mathcal{E}$,

$$(\mathcal{E}/)^I =_{df} \mathcal{E}/I \quad \text{(the slice topos)}.$$

For each morphism $\alpha : J \to I$ in $\mathcal{E}$, choose a pullback functor

$$\alpha^* : \mathcal{E}/I \to \mathcal{E}/J.$$

Note that each such functor α^* is determined up to a unique natural isomorphism as the right adjoint of the composition functor $\Sigma_\alpha : \mathcal{E}/J \to \mathcal{E}/I$ along α. Thus for any composable pair of morphisms $K \xrightarrow{\beta} J \xrightarrow{\alpha} I$, there is a canonical natural isomorphism

$$\phi_{\alpha,\beta} : \beta^*\alpha^* \xrightarrow{\sim} (\alpha\beta)^*. \tag{1}$$

Furthermore, these $\phi_{\alpha,\beta}$ for each composable pair α, β then satisfy the required coherence conditions (cf. [20]), making $\mathcal{E}/$ an indexed category. Observe also that since $\mathcal{E}$ is small, each $\mathcal{E}/I$ is a small category, and so $\mathcal{E}/$ is a small indexed category.

An indexed category is called *strict* if all of its canonical natural isomorphisms (1) are identities. Thus a small, strict indexed category is the same thing as a presheaf of categories on $\mathcal{E}$, i.e. a (proper) functor $\mathcal{E}^{\mathrm{op}} \to \mathbf{Cat}$.

Now, since $\mathcal{E}/$ need not be strict, it makes no sense to ask whether it is a sheaf (of categories) for a given Grothendieck topology on $\mathcal{E}$. We shall show, however, that $\mathcal{E}/$ is equivalent as an indexed category to a strict indexed category which, furthermore, is a sheaf. The Grothendieck topology considered is the so-called finite epimorphism topology, generated by covers consisting of finite epimorphic families; when we refer to $\mathcal{E}$ as a *site* we shall always mean $\mathcal{E}$ equipped with this topology. Recall that given indexed categories $\mathbf{A}$ and $\mathbf{B}$, an indexed functor $F : \mathbf{A} \to \mathbf{B}$ such that $F^I : \mathbf{A}^I \to \mathbf{B}^I$ is an ordinary equivalence of categories for each object I of $\mathcal{E}$ is called an (indexed) *equivalence*; and that $\mathbf{A}$ and $\mathbf{B}$ are said to be *equivalent* if there exists such an equivalence (cf. [4, 1.8]). In these terms, our aim in this section is the following.

Proposition 1. *$\mathcal{E}/$ is equivalent to a sheaf.*

The proof employs the notion of a stack, which was introduced by Giraud in [8]. Roughly speaking, stacks are to indexed categories what sheaves are to presheaves. Rather than developing the theory here, we shall assume familiarity with the treatment of Bunge & Paré [4]. An adjustment of the definition of stack given there is required, however, to account for the difference in the Grothendieck topologies under consideration (cf. [14]).

Definition 2. An indexed category $\mathbf{A}$ is a *stack* if it meets the following conditions:

(S1) For any pair of objects I and J of $\mathcal{E}$, the canonical functor

$$\mathbf{A}^{I+J} \longrightarrow \mathbf{A}^I \times \mathbf{A}^J$$

is an equivalence of categories.

(S2) For any epimorphism $\alpha : J \twoheadrightarrow I$ in $\mathcal{E}$, the canonical functor

$$\mathbf{A}^I \to \mathrm{des}(\alpha)$$

is an equivalence of categories, where $\mathrm{des}(\alpha)$ is the category of objects of $\mathbf{A}^J$ equipped with descent data relative to $\alpha : J \twoheadrightarrow I$ (as in [4, 2.1]).

Remark 3. Observe that if $\mathbf{A}$ is a stack and $\mathbf{B}$ an indexed category equivalent to $\mathbf{A}$, then $\mathbf{B}$ is plainly also a stack.

Proposition 1 results directly from the following three lemmas.

Lemma 4. *$\mathcal{E}/$ is equivalent to a small, strict indexed category.*

Lemma 5. *$\mathcal{E}/$ is a stack.*

Lemma 6. *Any small, strict stack is equivalent to a sheaf.*

Proof of proposition 1: $\mathcal{E}/$ is equivalent to a small, strict stack $\mathcal{E}_1$ by (remark 3 and) lemmas 4 and 5. By lemma 6, $\mathcal{E}_1$ is equivalent to a sheaf $\mathcal{E}_2$, whence $\mathcal{E}/$ is also equivalent to $\mathcal{E}_2$. □

As shall be evident, lemma 4 holds equally for any small indexed category. Thus we have shown more generally:

Proposition 7. *Any small stack (on a topos) is equivalent to a sheaf.*

We now proceed to the proofs of the lemmas.

Proof of lemma 4: Indeed, this is true for any small indexed category $\mathbf{A}$. For let $\mathbf{A}'$ be the indexed category given by setting

$$(\mathbf{A}')^I = \underline{\mathrm{ind}}([I], \mathbf{A}),$$

where $\underline{\mathrm{ind}}(-, ?)$ is the category of indexed functors from $-$ to $?$ and indexed natural transformations between them, and the indexed category $[I]$ is the so-called "externalization" of the object I in $\mathcal{E}$, regarded as a discrete category (cf. [23]). Specifically, for each object J in $\mathcal{E}$, the category $[I]^J$ is the discrete one on the set of objects $\mathcal{E}(J, I)$,

$$([I]^J)_0 = \mathcal{E}(J, I),$$

and $[I]$ has the obvious effect on morphisms. $\mathbf{A}'$ is clearly strict, it is small since $\mathbf{A}$ is, and it is equivalent to $\mathbf{A}$ by the indexed Yoneda lemma ([23, 1.5.1]). □

Lemma 5. *$\mathcal{E}/$ is a stack.*

Proof: Condition (S2) is a special case of [4] corollary 2.6. A proof can also be given from the descent theorem of Joyal & Tierney [13]. For if a morphism $e : J \twoheadrightarrow I$ in $\mathcal{E}$ is epi, then the geometric morphism $\mathcal{E}/J \to \mathcal{E}/I$ with inverse image $e^* : \mathcal{E}/I \to \mathcal{E}/J$ is an open surjection, hence an effective descent morphism by the Joyal-Tierney theorem.
For (S1), we must consider the canonical functor

$$\mathcal{E}/(I + J) \longrightarrow \mathcal{E}/I \times \mathcal{E}/J.$$

This is seen to be an equivalence of categories by considering the quasi-inverse:

$$(X \to I,\ Y \to J) \quad \longmapsto \quad (X + Y \to I + J).$$

$\square$

Lemma 6. *Any small, strict stack is equivalent to a sheaf.*

Proof: Let $\mathbf{C}$ be a small, strict stack on $\mathcal{E}$, regarded as a presheaf of categories. We shall prove that the canonical functor $\mathbf{C} \to a\mathbf{C}$ to the associated sheaf $a\mathbf{C}$ is an equivalence of indexed categories.
First, recall that $a\mathbf{C}$ can be constructed by two successive applications of the so-called plus construction (cf. [19, III.5]). As a functor, the plus construction

$$^{+} : \mathbf{Sets}^{\mathcal{E}^{\mathrm{op}}} \to \mathbf{Sets}^{\mathcal{E}^{\mathrm{op}}}$$

preserves finite limits, and hence also category objects in $\mathbf{Sets}^{\mathcal{E}^{\mathrm{op}}}$. The canonical natural transformation with components $\eta_P : P \to P^+$ for each presheaf P therefore determines two (internal) functors in $\mathbf{Sets}^{\mathcal{E}^{\mathrm{op}}}$:

$$\begin{aligned} \eta_{\mathbf{C}} &: \mathbf{C} \to \mathbf{C}^+, \\ \eta_{\mathbf{C}^+} &: \mathbf{C}^+ \to \mathbf{C}^{++} = a\mathbf{C}, \end{aligned}$$

the composite of which is the canonical functor $\mathbf{C} \to a\mathbf{C}$. Since the property of being a stack is inherited along equivalences, it will plainly suffice to show that $\eta_{\mathbf{C}}$ is an equivalence when $\mathbf{C}$ is a stack.
Next, given any presheaf P on $\mathcal{E}$, recall that P^+ is defined by

$$P^+(I) = \varinjlim_{S \in J(I)} \mathrm{Hom}(S, P) \tag{2}$$

for each object $I \in \mathcal{E}$, where the Hom is that of the category of presheaves $\mathbf{Sets}^{\mathcal{E}^{\mathrm{op}}}$. The colimit in (2) is taken over the set $J(I)$ of all covering sieves S of I, regarded as subobjects of the representable functor $yI = \mathcal{E}(-, I)$, and ordered by reverse inclusion ("refinement"). For each such sieve S there is a category $\underline{\mathrm{Hom}}(S, \mathbf{C})$ with objects and morphisms

$$\begin{aligned} \underline{\mathrm{Hom}}(S, \mathbf{C})_0 &= \mathrm{Hom}(S, \mathbf{C}_0), \\ \underline{\mathrm{Hom}}(S, \mathbf{C})_1 &= \mathrm{Hom}(S, \mathbf{C}_1), \end{aligned}$$

and with the evident structure maps coming from those of $\mathbf{C}$. Since $J(I)$ is a filter, the colimit in (2) is filtered. Thus $\mathbf{C}^+(I)$ is the filtered colimit of the categories $\underline{\mathrm{Hom}}(S, \mathbf{C})$,

$$\mathbf{C}^+(I) \cong \varinjlim_{S \in J(I)} \underline{\mathrm{Hom}}(S, \mathbf{C}). \tag{3}$$

Now let $K(I) \subseteq J(I)$ be the set of covering sieves R of I for which there is a finite epimorphic family $(\alpha_n : A_n \to I)_n$ that generates R. We order $K(I)$ by refinement too. Since any $S \in J(I)$ has a refinement $R \subseteq S$ with $R \in K(I)$, from (3) we have:

$$\mathbf{C}^+(I) \cong \varinjlim_{R \in K(I)} \underline{\mathrm{Hom}}(R, \mathbf{C}). \tag{4}$$

We now *claim* that for each $R \in K(I)$, the canonical inclusion $R \rightarrowtail yI$ induces an equivalence of categories

$$\underline{\mathrm{Hom}}(yI, \mathbf{C}) \simeq \underline{\mathrm{Hom}}(R, \mathbf{C}). \tag{5}$$

Given this, from (4) and (5) we shall have (isomorphisms and) equivalences:

$$\begin{aligned} \mathbf{C}(I) &\cong \underline{\mathrm{Hom}}(yI, \mathbf{C}) && \text{by Yoneda,} \\ &\cong \varinjlim_{R \in K(I)} \underline{\mathrm{Hom}}(yI, \mathbf{C}), \\ &\simeq \varinjlim_{R \in K(I)} \underline{\mathrm{Hom}}(R, \mathbf{C}) && \text{by (5),} \\ &\cong \mathbf{C}^+(I) && \text{by (4).} \end{aligned}$$

Whence $\eta : \mathbf{C} \simeq \mathbf{C}^+$ as desired.
The proof of the claim is a lengthy but straightforward descent-theoretic argument, which the interested reader can find in [2]. □

2 Sheaf representation

As before, let $\mathcal{E}$ be a fixed but arbitrary small topos, equipped with the finite epi topology when regarded as a site. By proposition 1 above the indexed category $\mathcal{E}/$ is equivalent to a sheaf of categories on $\mathcal{E}$. Let us write

$$\mathcal{E}// : \mathcal{E}^{\mathrm{op}} \to \mathbf{Cat}$$

for a fixed such sheaf, with

$$\mathcal{E}//I \simeq \mathcal{E}/I, \tag{6}$$

naturally in $I \in \mathcal{E}$.

Now let

$$a : \mathbf{Sets} \to \mathrm{Sh}(\mathcal{E})$$

be a geometric morphism into the Grothendieck topos $\mathrm{Sh}(\mathcal{E})$ of sheaves on $\mathcal{E}$, and consider the effect of its inverse image $a^* : \mathrm{Sh}(\mathcal{E}) \to \mathbf{Sets}$ on the sheaf $\mathcal{E}//$, or, as we shall say more briefly, the "stalk" of $\mathcal{E}//$ at the "point" $a : \mathbf{Sets} \to \mathrm{Sh}(\mathcal{E})$.

Lemma 8. *The category $a^*(\mathcal{E}//)$ is a hyperlocal topos.*

Proof: First note that $a^*(\mathcal{E}//)$ *is* a category since $\mathcal{E}//$ is a category in $\mathrm{Sh}(\mathcal{E})$ and a^* preserves finite limits.
Next, let $A : \mathcal{E} \to \mathbf{Sets}$ be the (left exact and continuous) composite functor

$$A \;:\; \mathcal{E} \xrightarrow{\;y\;} \mathrm{Sh}(\mathcal{E}) \xrightarrow{\;a^*\;} \mathbf{Sets},$$

where y is the sheafified Yoneda embedding. Given any sheaf F on $\mathcal{E}$, the stalk $a^*(F)$ can be calculated as the colimit

$$a^*(F) = \varinjlim_{\int A} F(I) \tag{7}$$

over the category $\int A$ of elements of A (cf. [19, VII.2(13)]). Recall that an object of $\int A$ is a pair (I, x) with I an object of $\mathcal{E}$ and $x \in A(I)$; and a morphism $\alpha : (I, x) \to (J, y)$ of $\int A$ is a morphism $\alpha : I \to J$ of $\mathcal{E}$ with $A(\alpha)(x) = y$. There is an evident forgetful functor $\pi : \int A \to \mathcal{E}$. The colimit in (7) is understood to be the colimit of the composite functor $F\pi$,

$$\varinjlim_{\int A} F(I) = \varinjlim\Big(\int A \xrightarrow{\pi} \mathcal{E} \xrightarrow{F} \mathbf{Sets}^{\mathrm{op}}\Big) \tag{8}$$

Since A is left exact, $\int A$ is a filtered category. Thus the category

$$a^*(\mathcal{E}//) = \varinjlim_{\int A} \mathcal{E}//I \tag{9}$$

is a filtered colimit (in **Cat**) of a diagram of topoi and logical morphisms, which implies that $a^*(\mathcal{E}//)$ is itself a topos, as the reader can easily check.
To show that $a^*(\mathcal{E}//)$ is hyperlocal, first observe the following. Since $A : \mathcal{E} \to \mathbf{Sets}$ preserves covers, if $(\alpha_n : C_n \to I)_n$ is a cover of the object I in $\mathcal{E}$, then the canonical map

$$(A\alpha_n) : \coprod_n AC_n \to AI \tag{10}$$

is a surjection in **Sets**. Thus given $(I, x) \in \int A$, so $x \in A(I)$, for some n there is an element $y \in AC_n$ with $\alpha_n(y) = x$. In sum:

(11) For any $(I, x) \in \int A$ and any cover $(\alpha_n : C_n \to I)_n$, for some n there is a map $\alpha_n : (C_n, y) \to (I, x)$ in $\int A$.

Now, the following two statements are clearly true.

(12) For any object I of $\mathcal{E}$ and any subobjects p and q of 1 in $\mathcal{E}//I$ with $p \vee q = 1$, there exists a cover $(\alpha_n : C_n \to I)_n$ such that, for each n, $\alpha_n^* p = 1$ or $\alpha_n^* q = 1$ in $\mathcal{E}//C_n$.

(13) For any object I of $\mathcal{E}$ and object X of $\mathcal{E}//I$ with $X \twoheadrightarrow 1$ epi, there exists a cover $(\alpha_n : C_n \to I)_n$ such that, for each n, there exists a morphism $1 \to \alpha_n^* X$ in $\mathcal{E}//C_n$.

Combining these with (11) then yields:

(14) For any object $(I, x) \in \int A$ and any subobjects p and q of 1 in $\mathcal{E}//I$ with $p \vee q = 1$, there is a map $\alpha : (C, y) \to (I, x)$ in $\int A$ such that $\alpha^* p = 1$ or $\alpha^* q = 1$ in $\mathcal{E}//C$.

(15) For any object $(I, x) \in \int A$ and any object X of $\mathcal{E}//I$ with $X \twoheadrightarrow 1$ epi, there is a map $\alpha : (C, y) \to (I, x)$ in $\int A$ and a morphism $1 \to \alpha^* X$ in $\mathcal{E}//C$.

One shows that $a^*(\mathcal{E}//)$ is local using (14) and that 1 is projective in $a^*(\mathcal{E}//)$ using (15). Since the arguments are similar, let us simply show the former: If p and q are subobjects of 1 in $a^*(\mathcal{E}//)$ with $p \vee q = 1$. Then there are objects $(I_p, x_p), (I_q, x_q) \in \int A$ and subobjects $p' \rightarrowtail 1$ in $\mathcal{E}//I_p$ and $q' \rightarrowtail 1$ in $\mathcal{E}//I_q$ projecting to p and q respectively in the colimit $a^*(\mathcal{E}//)$. Since $\int A$ is filtered, there exist an object (I, x) and morphisms $(I, x) \to (I_p, x_p)$ and $(I, x) \to (I_q, x_q)$ in $\int A$. Restricting p' and q' along these morphisms gives subobjects $p'', q'' \rightarrowtail 1$ in $\mathcal{E}//I$, still projecting to p and q respectively. Since $p \vee q = 1$ in the colimit, there is some $h : (J, y) \to (I, x)$ in $\int A$ such that the restriction $h^*(p'' \vee q'') = 1$ in $\mathcal{E}//J$. So also $h^* p'' \vee h^* q'' = h^*(p'' \vee q'') = 1$. Applying (14) gives a morphism $\alpha : (C, z) \to (J, y)$ in $\int A$ such that $\alpha^* h^* p'' = 1$ or $\alpha^* h^* q'' = 1$ in $\mathcal{E}//C$. Since $\alpha^* h^* p''$ also projects to p and $\alpha^* h^* q''$ to q, either $p = 1$ or $q = 1$ in $a^*(\mathcal{E}//)$. So $a^*(\mathcal{E}//)$ is local. $\square$

To prove the sheaf representation theorem, we shall make use of the following covering theorem for topoi, due to C. Butz & I. Moerdijk (see [5, 6]).

Butz-Moerdijk covering theorem. *Let $\mathcal{G}$ be a Grothendieck topos with enough points. There exists a topological space X and a connected, locally connected geometric morphism $\phi : \mathrm{Sh}(X) \to \mathcal{G}$.*

We remind the reader that a Grothendieck topos $\mathcal{G}$ is said to have *enough points* if the geometric morphisms $a : \mathbf{Sets} \to \mathcal{G}$ are jointly surjective (cf. e.g. [19, IX.11]); and furthermore that a geometric morphism $\gamma : \mathcal{G}' \to \mathcal{G}$ between Grothendieck topoi is called *connected* if its inverse image $\gamma^* : \mathcal{G} \to \mathcal{G}'$ is full and faithful, and *locally connected* if γ^* has a **Sets**-indexed left adjoint (cf. [22]). We make no use here of the local connectedness of the covering map $\phi : \mathrm{Sh}(X) \to \mathcal{G}$.

Theorem 9 (Sheaf representation for topoi). *Any small topos is equivalent to the topos of global sections of a sheaf of hyperlocal topoi on a topological space.*

Proof: As before, we have the small topos $\mathcal{E}$, the sheaf of categories $\mathcal{E}//$ on $\mathcal{E}$ with $\mathcal{E}// \simeq \mathcal{E}/$, and by the foregoing lemma every stalk of $\mathcal{E}//$ is a hyperlocal topos. Since the topology on $\mathcal{E}$ is generated by *finite* epimorphic families, the Grothendieck topos $\mathrm{Sh}(\mathcal{E})$ is coherent and so has enough points by Deligne's theorem [19, IX]. By the above covering theorem, there is thus a topological space $X_{\mathcal{E}}$ and a connected geometric morphism

$$\phi : \mathrm{Sh}(X_{\mathcal{E}}) \to \mathrm{Sh}(\mathcal{E}). \tag{16}$$

Applying the inverse image $\phi^* : \mathrm{Sh}(\mathcal{E}) \to \mathrm{Sh}(X_{\mathcal{E}})$ to $\mathcal{E}//$ gives a sheaf of categories

$$\widetilde{\mathcal{E}} =_{df} \phi^*(\mathcal{E}//) \tag{17}$$

on $X_{\mathcal{E}}$, which—we claim—is still a sheaf of hyperlocal topoi, and has "the same" category of global sections as $\mathcal{E}//$, namely $\mathcal{E}$.

First, since ϕ^* is full and faithful, the unit η of the adjunction $\phi^* \dashv \phi_*$ is a natural isomorphism,

$$\eta : 1_{\mathrm{Sh}(\mathcal{E})} \xrightarrow{\sim} \phi_*\phi^*.$$

The components of η at $\mathcal{E}//$ are therefore an isomorphism of (sheaves of) categories

$$\mathcal{E}// \cong \phi_*\phi^*(\mathcal{E}//). \tag{18}$$

Let $y : \mathcal{E} \to \mathrm{Sh}(\mathcal{E})$ by the sheafified Yoneda embedding. For the category

$\mathrm{Sh}(X_{\mathcal{E}})(1, \widetilde{\mathcal{E}})$ of global sections of $\widetilde{\mathcal{E}}$, we have:

$$\begin{aligned}
\mathrm{Sh}(X_{\mathcal{E}})(1, \widetilde{\mathcal{E}}) &\cong \mathrm{Sh}(X_{\mathcal{E}})(\phi^*1, \widetilde{\mathcal{E}}) & \\
&\cong \mathrm{Sh}(\mathcal{E})(1, \phi_*\widetilde{\mathcal{E}}) & \phi^* \dashv \phi_* \\
&\cong \mathrm{Sh}(\mathcal{E})(1, \phi_*\phi^*(\mathcal{E}//)) & \text{by (17)} \\
&\cong \mathrm{Sh}(\mathcal{E})(1, \mathcal{E}//) & \text{by (18)} \\
&\cong \mathrm{Sh}(\mathcal{E})(y1, \mathcal{E}//) & \\
&\cong \mathcal{E}//1 & \text{by Yoneda} \\
&\simeq \mathcal{E}/1 & \\
&\cong \mathcal{E}. &
\end{aligned}$$

Now consider the stalks of $\widetilde{\mathcal{E}}$. A point $p \in X_{\mathcal{E}}$ determines a unique (up to isomorphism) geometric morphism $p : \mathbf{Sets} \to \mathrm{Sh}(X_{\mathcal{E}})$ with inverse image

$$p^*(F) \cong F_p \qquad \text{(the stalk of } F \text{ at } p) \tag{19}$$

for each sheaf F on $X_{\mathcal{E}}$. Composing with the covering map $\phi : \mathrm{Sh}(X_{\mathcal{E}}) \to \mathrm{Sh}(\mathcal{E})$ we obtain a point

$$\phi p : \mathbf{Sets} \xrightarrow{p} \mathrm{Sh}(X_{\mathcal{E}}) \xrightarrow{\phi} \mathrm{Sh}(\mathcal{E}) \tag{20}$$

of $\mathrm{Sh}(\mathcal{E})$. For the stalk $\widetilde{\mathcal{E}}_p$ of $\widetilde{\mathcal{E}}$ at a point $p \in X_{\mathcal{E}}$ we then have

$$\begin{aligned}
\widetilde{\mathcal{E}}_p &\cong p^*(\widetilde{\mathcal{E}}) & \text{by (19),} \\
&= p^*(\phi^*(\mathcal{E}//)) & \text{by (17),} \\
&\cong (\phi p)^*(\mathcal{E}//), &
\end{aligned}$$

the last of which is hyperlocal, since it is a stalk of $\mathcal{E}//$ at the point ϕp of (20). Thus every stalk of $\widetilde{\mathcal{E}}$ is indeed a hyperlocal topos, completing the proof. □

We refer the reader to [3, appendix] for an explicit description of the space $X_{\mathcal{E}}$ and the covering map $\phi : \mathrm{Sh}(X_{\mathcal{E}}) \to \mathrm{Sh}(\mathcal{E})$ in the current situation.

Now let us turn to the special case of boolean topoi. The easy proof of the following lemma is left to the reader.

Lemma 10. *A topos is well-pointed just if it is hyperlocal and boolean.*

Let $\mathcal{B}$ be a boolean topos and take the sheaf $\widetilde{\mathcal{B}}$ on the space $X_{\mathcal{B}}$, as in the sheaf representation theorem. Given any point $x \in X_{\mathcal{B}}$, there is then a canonical logical morphism

$$\pi_x : \mathcal{B} \to \widetilde{\mathcal{B}}_x \tag{21}$$

since by (9) the stalk $\widetilde{\mathcal{B}}_x$ is a (filtered) colimit of slices of $\mathcal{B}$. Thus every stalk $\widetilde{\mathcal{B}}_x$ of $\widetilde{\mathcal{B}}$ is also boolean, since it has a logical morphism from a boolean topos. By lemma 10, then, every stalk of $\widetilde{\mathcal{B}}$ is in fact well-pointed. Whence:

Corollary 11 (Sheaf representation for boolean topoi). *Any small boolean topos is equivalent to the topos of global sections of a sheaf of well-pointed topoi on a topological space.*

Remark 12. (i) A somewhat stronger statement of corollary 11 can be given: If $\widetilde{\mathcal{E}}$ is the sheaf representation of a topos $\mathcal{E}$, then $\mathcal{E}$ is boolean if and only if $\widetilde{\mathcal{E}}$ is a sheaf of well-pointed topoi. For the "if" part, observe that $\mathcal{E}$ is boolean if it has a *faithful* logical morphism $\mathcal{E} \to \mathcal{B}$ to some boolean topos $\mathcal{B}$. The statement therefore follows from (lemma 10 and) the following.

(ii) If $\widetilde{\mathcal{E}}$ is the sheaf representation of a topos $\mathcal{E}$, then the canonical logical morphism

$$\langle \pi_x \rangle_{x \in X_{\mathcal{E}}} : \mathcal{E} \to \prod_{x \in X_{\mathcal{E}}} \widetilde{\mathcal{E}}_x \tag{22}$$

is faithful. Here each of the maps $\pi_x : \mathcal{E} \to \widetilde{\mathcal{E}}_x$ is as in (21) above. The functor (22) is faithful simply because $\mathcal{E} \simeq \Gamma(\widetilde{\mathcal{E}})$, and for any sheaf F the canonical map $\Gamma(F) \to \prod_{x \in X} F_x$ is injective.

3 Logical completeness

In this section we assume some familiarity with topos semantics for higher-order logic, e.g. as in [2]. By way of review, recall that a (higher-order, logical) theory consists of a finite list of basic type symbols, basic constant symbols, and (possibly higher-order) sentences in these parameters. Let $\mathbb{T}$ be a theory. For any topos $\mathcal{E}$, there is a category $\mathbf{Mod}_{\mathbb{T}}(\mathcal{E})$ of $\mathbb{T}$-models in $\mathcal{E}$ and their isomorphisms; furthermore, any logical morphism $f : \mathcal{E} \to \mathcal{F}$ of topoi induces an evident functor $\mathbf{Mod}_{\mathbb{T}}(f) : \mathbf{Mod}_{\mathbb{T}}(\mathcal{E}) \to \mathbf{Mod}_{\mathbb{T}}(\mathcal{F})$ by taking images. Moreover, there exists a (higher-order) classifying topos $\mathcal{S}[\mathbb{T}]$, determined uniquely up to equivalence by the natural (in $\mathcal{E}$) equivalence of categories

$$\mathbf{Log}(\mathcal{S}[\mathbb{T}], \mathcal{E}) \simeq \mathbf{Mod}_{\mathbb{T}}(\mathcal{E}), \tag{23}$$

where $\mathbf{Log}(\mathcal{S}[\mathbb{T}], \mathcal{E})$ is the category of logical morphisms $\mathcal{S}[\mathbb{T}] \to \mathcal{E}$ and natural isomorphisms between them.

As objects of the classifying topos $\mathcal{S}[\mathbb{T}]$ one can take equivalence classes of closed terms of the form $\{x|\varphi\}$ in the language of $\mathbb{T}$, identified under provable equality $\mathbb{T} \vdash \{x|\varphi\} = \{y|\psi\}$ (similarly, morphisms are suitable equivalence classes of provably functional relations). In particular, the Heyting algebra $\mathrm{Sub}_{\mathcal{S}[\mathbb{T}]}(1)$ of subobjects of 1 in $\mathcal{S}[\mathbb{T}]$ is just the Lindenbaum-Tarski algebra of $\mathbb{T}$-sentences. Thus the universal $\mathbb{T}$-model $U_{\mathbb{T}}$ in $\mathcal{S}[\mathbb{T}]$, which is associated to the identity morphism $\mathcal{S}[\mathbb{T}] \to \mathcal{S}[\mathbb{T}]$ under (23), has the property that for any $\mathbb{T}$-sentence σ:

(24) $$U_{\mathbb{T}} \models \sigma \quad \text{iff} \quad \mathbb{T} \vdash \sigma.$$

Observe that (23) and (24) together entail the soundness and completeness of higher-order, intuitionistic deduction $\vdash$ with respect to topos semantics: $M \models \sigma$ for every $\mathbb{T}$-model M just if $\mathbb{T} \vdash \sigma$. Finally, a theory $\mathbb{T}$ is called *classical* if $\mathbb{T} \vdash \forall_p . p \vee \neg p$, which is the case just if the classifying topos $\mathcal{S}[\mathbb{T}]$ is boolean.

Now let us say that a collection $\mathbf{E}$ of topoi *suffices* for a collection $\mathbf{T}$ of theories if, for any theory $\mathbb{T} \in \mathbf{T}$ and any $\mathbb{T}$-sentence σ, $M \models \sigma$ for every $\mathbb{T}$-model M in every topos $\mathcal{E} \in \mathbf{E}$ implies $\mathbb{T} \vdash \sigma$. The idea, of course, is that $\mathbf{E}$ provides complete semantics for the theories $\mathbf{T}$. For example, the completeness of topos semantics just mentioned says that (small) topoi suffice for theories in intuitionistic logic, and (small) boolean topoi for classical theories. In these terms, by the sheaf representation theorems of the previous section, one then has the following.

Theorem 13 (Strong completeness). *Hyperlocal topoi suffice for theories in intuitionistic logic, and well-pointed topoi for classical theories.*

Proof: Let $\mathbb{T}$ be a theory and $\mathcal{S}[\mathbb{T}]$ its classifying topos. Identifying a $\mathbb{T}$-sentence σ with the subobject of 1 in $\mathcal{S}[\mathbb{T}]$ that it determines, for any logical morphism $f \colon \mathcal{S}[\mathbb{T}] \to \mathcal{E}$ to a topos $\mathcal{E}$, we have:

(25) $$f(U_{\mathbb{T}}) \models \sigma \quad \text{iff} \quad f\sigma = 1,$$

where $f(U_{\mathbb{T}}) = \mathbf{Mod}_{\mathbb{T}}(f)(U_{\mathbb{T}}) \in \mathbf{Mod}_{\mathbb{T}}(\mathcal{E})$ is associated to f via (23).
Now let $\widetilde{\mathcal{S}[\mathbb{T}]}$ be a sheaf representation of $\mathcal{S}[\mathbb{T}]$ on a space X, and consider the faithful logical morphism

$$\langle \pi_x \rangle_{x \in X} : \mathcal{S}[\mathbb{T}] \to \prod_{x \in X} \widetilde{\mathcal{S}[\mathbb{T}]}_x$$

of remark 12(ii) above. For each point $x \in X$, there is a $\mathbb{T}$-model $\pi_x(U_{\mathbb{T}}) \in \mathbf{Mod}_{\mathbb{T}}(\widetilde{\mathcal{S}[\mathbb{T}]}_x)$ in the (hyperlocal) stalk of $\widetilde{\mathcal{S}[\mathbb{T}]}$ at x. If the $\mathbb{T}$-sentence σ is such that $\mathcal{H} \models \sigma$ for any hyperlocal topos $\mathcal{H}$, then for each $x \in X$, $\pi_x(U_{\mathbb{T}}) \models \sigma$, and so $\pi_x\sigma = 1$ by (25). But then $\sigma = 1$ in $\mathcal{S}[\mathbb{T}]$, since $\langle \pi_x \rangle_{x \in X}$ is faithful. So $U_{\mathbb{T}} \models \sigma$ by (25) again, whence $\mathbb{T} \vdash \sigma$ by (24). Thus hyperlocal topoi suffice.
If $\mathbb{T}$ is classical, $\mathcal{S}[\mathbb{T}]$ is boolean and so each stalk $\widetilde{\mathcal{S}[\mathbb{T}]}_x$ is well-pointed. The result then follows similarly. □

The existence of a logical embedding of any boolean topos into a product of well-pointed topoi was established already in [7]. It is of interest to note that well-pointed topoi arise independently, both as models of Lawvere's categorical set theory [18], and as models of a certain well-known fragment Z^- of Zermelo-Fraenkel set theory, called variously bounded (or weak) Zermelo set theory or Mac Lane set theory ([19, 11, 21] and the references there). The classical part of the foregoing strong completeness theorem can therefore also be stated in terms of models (of theories) in models of Z^-: a sentence in the language of a classical theory $\mathbb{T}$ is provable if it is true in every $\mathbb{T}$-model in every model of Z^-. There is also a more or less obvious proof-theoretic statement of this situation.

We conclude by indicating how to pass from the strong compleness theorem to the classical higher-order completeness theorem using "non-standard" models in the single topos **Sets**, in the style of Henkin [10]. First, observe that any well-pointed topos $\mathcal{W}$ has a canonical faithful functor into **Sets**, namely the global sections functor

$$\Gamma = \mathcal{W}(1, -) \colon \mathcal{W} \to \mathbf{Sets}.$$

Lemma 14. *For any well-pointed topos $\mathcal{W}$, the global sections functor Γ has the following properties:*

(i) Γ is left-exact and continuous for the finite epi topology;

(ii) Γ preserves finite coproducts and the internal first-order logic of $\mathcal{W}$;

(iii) for any objects Y and Z of $\mathcal{W}$ there is a canonical inclusion

$$\Gamma(Z^Y) \subset \Gamma Z^{\Gamma Y}.$$

Proof: The proof of statement (i) is straightforward, and (ii) results from

(i) and the fact that $\mathcal{W}$ is boolean (cf. [3] for details). For (iii), we have

$$\begin{aligned}
\Gamma(Z^Y) &= \mathcal{W}(1, Z^Y), \\
&\cong \mathcal{W}(Y, Z), \\
&\subset \mathbf{Sets}(\Gamma Y, \Gamma Z) && \Gamma \text{ is faithful,} \\
&= \Gamma Z^{\Gamma Y}.
\end{aligned}$$

$\square$

Note that by (ii) and (iii) one also has a canonical inclusion

$$\Gamma(PX) \cong \Gamma(2^X) \subset \Gamma 2^{\Gamma Y} \cong 2^{\Gamma Y} \cong P(\Gamma X) \tag{26}$$

for any object X, its power object PX, and the powerset $P(\Gamma X)$.

Now, given a model M of a classical theory $\mathbb{T}$ in a well-pointed topos $\mathcal{W}$, the image of M under Γ is a Henkin model of $\mathbb{T}$ (a general model in the sense of Henkin [10]; cf. [1] for a recent treatment). More precisely, recall that such a Henkin model $\mathfrak{M}$ consists of sets $X_{\mathfrak{M}}, \ldots$ (interpreting the basic types of $\mathbb{T}$), plus subsets $(PZ)_{\mathfrak{M}} \subseteq P(Z_{\mathfrak{M}})$ for each type Z (interpreting the power types of $\mathbb{T}$), plus distinguished elements $c_{\mathfrak{M}}, \ldots$ of these sets (interpreting the basic constants of $\mathbb{T}$), and satisfying suitable closure conditions ensuring that there are enough sets to interpret the logical operations ($x \cap y \in (PZ)_{\mathfrak{M}}$ if $x, y \in (PZ)_{\mathfrak{M}}$, and so on). Given a model M in $\mathcal{W}$, in virtue of (26) we then have the Henkin model ΓM with

$$\begin{aligned}
Z_{\Gamma M} &= \Gamma(Z_M) \\
c_{\Gamma M} &= \Gamma(c_M)
\end{aligned}$$

for each type Z and each basic constant c. Moreover, since Γ is faithful,

$$\Gamma M \models \sigma \quad \text{just if} \quad M \models \sigma$$

for any $\mathbb{T}$-sentence σ.

Combining this last equivalence with theorem 13 plainly yields the classical higher-order completeness theorem with respect to Henkin models, which was our objective.

Just to wrap things up, take a stalk $\widetilde{\mathcal{S}[\mathbb{T}]}_x$ of the sheaf representation of a classifying topos for a classical theory $\mathbb{T}$, and consider the composite functor

$$\mathcal{S}[\mathbb{T}] \xrightarrow{\pi_x} \widetilde{\mathcal{S}[\mathbb{T}]}_x \xrightarrow{\Gamma} \mathbf{Sets}, \tag{27}$$

where the logical morphism $\pi_x : \mathcal{S}[\mathbb{T}] \to \widetilde{\mathcal{S}[\mathbb{T}]}_x$ is as in remark 12(ii) and Γ is the global sections functor as before. By (i) of lemma 14, this composite is then left exact and continuous. And indeed, every left exact, continuous functor $A : \mathcal{S}[\mathbb{T}] \to \mathbf{Sets}$ arises in this way as the global sections of a stalk of $\widetilde{\mathcal{S}[\mathbb{T}]}$. For we can take $a : \mathbf{Sets} \to \mathrm{Sh}(\mathcal{S}[\mathbb{T}])$ to be the associated point of $\mathrm{Sh}(\mathcal{S}[\mathbb{T}])$ and we then have the stalk $a^*\widetilde{\mathcal{S}[\mathbb{T}]}$. The displayed composite (27) is then (isomorphic to) A just if the stalks agree, inasmuchas $\widetilde{\mathcal{S}[\mathbb{T}]}_x \simeq a^*\widetilde{\mathcal{S}[\mathbb{T}]}$. In particular, then, we see that the image of the universal model $U_{\mathbb{T}}$ under *any* left exact, continuous functor $A : \mathcal{S}[\mathbb{T}] \to \mathbf{Sets}$ is (the global sections of) a standard model in a well-pointed topos, and is thus a Henkin model.

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