Complete Orientation Rules for Patterns

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The *pattern* for directed acyclic graph G is the graph which has the identical adjacencies as G and which has an oriented edge $A \to B$ if and only if there is a vertex $C \notin ADJ(A)$ such that $A \to B$ and $C \to B$ in G. Let $pattern(G)$ denote the pattern for G. We assume that $\Pi = pattern(G)$ is a pattern for some graph directed acyclic graph G. Let $\mathcal{K}$ be a set of oriented edges consistent with the unshielded collider orientations in Π; this set can be thought of as background knowledge.

A graph G *extends* graph H if and only if (i) G and H have the same adjacencies and (ii) if $A \to B$ is in H then $A \to B$ is in H. The *maximally oriented graph* for Π with respect to $\mathcal{K}$ is the (possibly partially oriented) graph $max(\Pi, \mathcal{K})$ such that for each unoriented edge $A — B$ in $max(\Pi, \mathcal{K})$ it is the case that there exist directed acyclic graphs G_1 and G_2 which extend $max(\Pi, \mathcal{K})$ such that (i) $A \to B$ in G_1 and $B \to A$ in G_2, (ii) G_1 and G_2 have the same colliders as G and adjacencies; $pattern(G_1) = pattern(G_2) = \Pi$, and (iii) every edge in $\mathcal{K}$ is oriented correctly in $max(\Pi, \mathcal{K})$.

A brief explanation of the schematic rules in Figure 1. Each orientation rule consists of a pair of schematic patterns. A schematic pattern matches a pattern Π' if there exists a set of vertices $\mathbf{D}$ in Π' and a bijective mapping (f) from the vertices in the schematic pattern to $\mathbf{D}$ such that (i) pairs of vertices are adjacent in the schematic if and only if the corresponding pair of vertices are adjacent in Π' and (ii) if $A \to B$ in the schematic then the corresponding edge is oriented $f(A) \to f(B)$ in Π' (iii) if $A — B$ in the schematic then the corresponding edge is unoriented and (iv) if A and B are connected by a dashed line then either $A — B$, $A \to B$, or $B \to A$ can appear in Π'. If the schematic to the left of the $\Rightarrow$ matches pattern Π' then we can orient the edges in Π' according to the schematic to the right of the $\Rightarrow$. An orientation rule is *sound* if and only if any orientation other than the orientation indicated by the rule would lead to a new unshielded collider.

Figure 1: Orientation rules for patterns

Theorem 1 (Orientation Soundness) *The four orientation rules given in Figure 1 are sound.*

Proof — Rule R1; If the edge were oriented in the opposite direction there would be a new unshielded collider. Rule R2; If the edge were oriented in the opposite direction there would be a cycle. Rule R3; if the edge were oriented in the opposite direction then by two application of the rule R2 there would be a new unshielded collider. Rule R4; If the edge were oriented in the opposite direction then by two applications of rule R2 there would be a new unshielded collider.□

Lemma 2 *Let Π_0 be the result of applying the orientation rules R1, R2, and R3 to the pattern Π. In Π_0, if $A \rightarrow B$ and B— C then $A \rightarrow C$.*

Proof — We say that vertex X is an ancestor of vertex Y with respect to Π_0 if there is a path such that every edge is directed from X to Y in Π_0. The orientations in Π_0 induce a partial ordering on the vertices by the following rule; $X < Y$ if X is an ancestor of Y. With respect to this partial ordering, choose vertex B to be a minimal vertex such that there are edges $A \rightarrow B$ and B— C in Π_0. Note that $A \in ADJ(C)$ otherwise B— C would be oriented by rule R1. Furthermore, A— C must be unoriented; if A— C is oriented $A \rightarrow C$ then we are done and if the edge is oriented $C \rightarrow A$ then B— C oriented by rule R2.

Case 1 — Edge $A \rightarrow B$ orient by rule R1. Thus there is an edge $D \rightarrow A$ such that $D \notin ADJ(B)$. By construction B is a minimal vertex such that there are edges $A \rightarrow B$ and B— C in Π_0 but A meets this requirement and $A < B$. Contradiction.

Case 2 — Edge $A \rightarrow B$ oriented because it is part of an unshielded collider. In this case, there is an edge $D \rightarrow B$ such that $D \notin ADJ(A)$. If $D \notin ADJ(C)$ then B— C would be oriented by rule R1. If $D \in ADJ(C)$ and D— C is unoriented then B— C oriented by R3. Suppose that D— C is oriented. If $C \rightarrow D$ then B— C oriented by rule R2 else if $D \rightarrow C$ then A— C oriented $C \rightarrow A$ by rule R1 and B— C oriented by rule R2. Contradiction.

Case 3 — Edge $A \rightarrow B$ oriented by R3. Observe that there is an unshielded collider colliding at B. This case is sufficiently similar to case 2 that I will not give the proof.

Case 4 — Edge $A \rightarrow B$ oriented by R2. In this case there exists a vertex D such that $A \rightarrow D$ and $D \rightarrow B$ are in Π_0. $D \in ADJ(C)$ otherwise B— C oriented by R1. Edge D— C is oriented by construction ($D < B$). If $C \rightarrow D$ the B— C oriented by R2 else if $D \rightarrow C$ then $A \rightarrow C$ by R2.□

An undirected graph H is *chordal* if and only if every undirected cycle of length four or more has an edge between two nonconsecutive vertices on the cycle (i.e. has a chord). A total order ($<$) induces an orientation in an undirected graph H by the rule that if A— B is in H then orient the edge $A \rightarrow B$ if and only if $A < B$. If α is the total order of vertices and H is the undirected graph then H_α is the induced directed graph obtained by the rule given above. Clearly H_α is acyclic. We say that a total order α is a *consistent ordering* with respect to H if and only if H_α has no unshielded colliders.

Lemma 3 *Only chordal graphs have consistent orderings.*

Proof — Suppose that α is a consistent ordering with respect to non-chordal graph H. Let $\langle A_1, A_2, A_3, \ldots, A_n \rangle$ be a non-chordal cycle with $n \geq 4$ in undirected graph H. Let A_i be the largest vertex (with respect to the ordering α) in the cycle. If $i = n$ then $A_{i+1} = A_1$ and if $i = 1$ then $A_{i-1} = A_n$. In H_α, $A_{i-1} \to A_i$ and $A_{i+1} \to A_i$ and since the cycle is non-chordal we have $A_{i+1} \notin ADJ(A_{i-1})$. Contradiction.□

A *clique* in graph H is a set of vertices such that there is an edge in H between each pair of vertices in the set. A *maximal clique* is a set of vertices which are a clique and such that no superset of the set is a clique. Let $\mathcal{C}_H = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ denote the set of maximal cliques of graph H. Note that maximal cliques in $\mathcal{C}_H$ can overlap and that the union of all of the maximal cliques is the set of vertices in H. A *join tree* for H is a tree whose vertices are in $\mathcal{C}_H$ and such that (i) Each edge $\mathcal{C}_i — \mathcal{C}_j$ is labeled by the set $\mathcal{C}_i \cap \mathcal{C}_j$, and (ii) for every pair $\mathcal{C}_i$ and $\mathcal{C}_j$ $(i \neq j)$ and for every $A \in \mathcal{C}_i \cap \mathcal{C}_j$ each edge along the unique path between $\mathcal{C}_i$ and $\mathcal{C}_j$ includes label A. Now I state a useful result from Beeri et al. 1983.

Lemma 4 (Beeri et al.) *Graph H is chordal if and only if H has a join tree.*

A partial order π is a *tree order* for tree T if and only if for all A and B which are adjacent in T either $\pi(A, B)$ or $\pi(B, A)$. Conceptually a tree order is obtained by choosing one node as the root of the tree and ordering vertices based on their distance from the root; all tree orderings for a tree T can be obtained in this fashion by selecting each vertex as the root.

Let π_{T} be a tree ordering of the join tree T for graph H. π_{T} induces a partial ordering $\prec_{\pi_{\mathrm{T}}}$ on the vertices of H by the following rules; (i) if $\pi_{\mathrm{T}}(\mathcal{C}_i, \mathcal{C}_j)$ and $\mathcal{C}_i$ is not the minimum element of π_{T} then for all $A \in \mathcal{C}_i \backslash \mathcal{C}_j$ and $C \in \mathcal{C}_j \backslash \mathcal{C}_i$ and $B \in \mathcal{C}_j \cap \mathcal{C}_i$ order $A \prec_{\pi_{\mathrm{T}}} B$ and $B \prec_{\pi_{\mathrm{T}}} C$, (ii) if $\pi_{\mathrm{T}}(\mathcal{C}_i, \mathcal{C}_j)$ and $\mathcal{C}_i$ is the minimum element of π_{T} then for all $C \in \mathcal{C}_j \backslash \mathcal{C}_i$ and $B \in \mathcal{C}_j \cap \mathcal{C}_i$ order $B \prec_{\pi_{\mathrm{T}}} C$, (iii) if $A \prec_{\pi_{\mathrm{T}}} D$ and $D \prec_{\pi_{\mathrm{T}}} B$ then $A \prec_{\pi_{\mathrm{T}}} B$ (i.e. transitive closure of $\prec_{\pi_{\mathrm{T}}}$).

Let π be a tree ordering for join tree T of H. Note that the partial order $\prec_\pi$ on the vertices in H induced by the partial order π only orients edges which are involved in an unshielded triple; i.e. $A \prec_\pi B$ only if there is a C such that $\langle A, B, C \rangle$ or $\langle C, A, B \rangle$ is an unshielded triple. In fact all edges involved in unshielded colliders except those edges $A — B$ where both A and B are in the minimum vertex (the "root clique") of the join tree.

A partial order π_1 is an *extension* of a partial order π_2 if and only if for all A and B such that $\pi_2(A, B)$ it is the case that $\pi_1(A, B)$.

Lemma 5 *Let π be a tree ordering of a join tree T for H. Any extension of $\prec_\pi$ to a total ordering is a consistent ordering for H.*

Proof— Let α be a total ordering which extends $\prec_\pi$. No unshielded collider can occur inside a clique since all triples are shielded. Let $\langle A, B, C\rangle$ be an unshielded triple (i.e. A is adjacent to B, B is adjacent to C, and A is not adjacent to C). There exists an i and j such that $A \in \mathcal{C}_i \wedge A \notin \mathcal{C}_j \wedge C \notin \mathcal{C}_i \wedge C \in \mathcal{C}_j \wedge B \in \mathcal{C}_i \cap \mathcal{C}_j$; if not A and C would be adjacent. By the join tree property we know that there is a unique path p between $\mathcal{C}_j$ and $\mathcal{C}_i$ in T.

Case 1 — $\neg(\pi(\mathcal{C}_i, \mathcal{C}_j) \vee \pi(\mathcal{C}_j, \mathcal{C}_i))$. There must be a k such that $\mathcal{C}_k$ is on p such that $\pi(\mathcal{C}_k, \mathcal{C}_i) \wedge \pi(\mathcal{C}_k, \mathcal{C}_j)$. We know that $A \notin \mathcal{C}_k \vee C \notin \mathcal{C}_k$ otherwise $\langle A, B, C\rangle$ is not unshielded since $\mathcal{C}_k$ is a clique. Without loss of generality suppose that $C \notin \mathcal{C}_k$. We know that $B \in \mathcal{C}_k$ by the join tree property and since $\pi(\mathcal{C}_k, \mathcal{C}_j)$ it is the case that $B \prec_\pi C$ and thus $\langle A, B, C\rangle$ is not an unshielded collider in H_α.

Case 2 — $\pi(\mathcal{C}_i, \mathcal{C}_j)$ (other case is symmetric). In this case the $\langle A, B, C\rangle$ unshielded triple is oriented as a non-collider by any extension of $\prec_\pi$ to a total order since $B \prec_\pi C$.□

Lemma 6 (Orienting chordal graphs) *Let H be an undirected chordal graph. For all pairs of adjacent vertices A and B in H there exist total orderings α and γ which are consistent with respect to H and such that $A \rightarrow B$ is in H_α and $B \rightarrow A$ is in H_γ.*

Proof — For the case where H is disconnected apply the argument to each of the disconnected components.

Case 1 — For all i either $A \in \mathcal{C}_i \wedge B \in \mathcal{C}_i$ or $A \notin \mathcal{C}_i \wedge B \notin \mathcal{C}_i$. Let π be a tree ordering of a join tree for H. A and B are not comparable with respect to $\prec_\pi$. Thus by Lemma 5 we simply choose two extensions of $\prec_\pi$; one with $A \prec_\pi B$ and another with $B \prec_\pi A$.

Case 2 — There exists an i such that $A \notin \mathcal{C}_i \vee B \notin \mathcal{C}_i$ and $A \in \mathcal{C}_i \vee B \in \mathcal{C}_i$. Without loss of generality assume that $A \notin \mathcal{C}_i \wedge B \in \mathcal{C}_i$. Given that there is an edge between A and B there is a j such that $j \neq i$ and $A \in \mathcal{C}_j \wedge B \in \mathcal{C}_j$. Let π_1 be a tree ordering of a join tree for H with $\mathcal{C}_i$ is the root and let π_2 be a tree ordering of a join tree for H with $\mathcal{C}_j$ as the root. Then consider any extension of $\prec_{\pi_1}$ and $\prec_{\pi_2}$ to total orderings and apply Lemma 5. We are done since $B \prec_{\pi_1} A$ and $A \prec_{\pi_2} B$.□

Theorem 7 (Orientation completeness) *The result of applying rules R1, R2 and R3 to a pattern is a maximally oriented graph.*

Proof— Let Π_0 be the result of applying the orientation rules R1, R2, and R3 to the pattern Π. Given Lemma 2 no orientation of edges not oriented in Π_0 will create a cycle which includes an edge or edges oriented in Π_0 and no orientation of an edge not oriented in Π_0 can create an unshielded collider with an edge oriented in Π_0. Consider the undirected graph H, a subgraph of Π_0, obtained by removing all of the oriented edges in Π_0. We show that H is the union of disjoint chordal graphs. Suppose this is not the case. Then, by Lemma 3 all total ordering of the vertices leads to a new unshielded collider $\langle A, B, C\rangle$ in H. By Lemma 2, the triple $\langle A, B, C\rangle$ also forms an unshielded triple in Π_0, that is $A \notin ADJ(C)$ in Π_0. This is a contradiction since we assume that the graph Π and thus Π_0 have all unshielded colliders oriented and that there is an acyclic orientation of the graph Π with no new unshielded colliders. Finally, by applying Lemma 6 we have completed the theorem.□

Let H be a partially oriented chordal graph and let T be a join tree for H. Let $\Lambda_{ij} = \mathcal{C}_i \cap \mathcal{C}_j$. We define a relation γ_{T} on the nodes of T, the maximal cliques of H, from the orientations in H as follows; $\gamma_{\mathrm{T}}(\mathcal{C}_i, \mathcal{C}_j)$ if and only if (i) $\Lambda_{ij} \neq \emptyset$, (ii) for all $A \in \Lambda_{ij}$ and $B \in \mathcal{C}_j \backslash \Lambda_{ij}$ it is the case that $A \rightarrow B$ is in H and (iii) it is not the case that for all $A \in \Lambda_{ij}$ and $B \in \mathcal{C}_i \backslash \Lambda_{ij}$ $A \rightarrow B$ is in H. We define the partial order ϵ_{T} on the nodes of T as follows; (i) $\epsilon_{\mathrm{T}}(\mathcal{C}_i, \mathcal{C}_j)$ if $\gamma_{\mathrm{T}}(\mathcal{C}_i, \mathcal{C}_j)$ and (ii) $\epsilon_{\mathrm{T}}(\mathcal{C}_i, \mathcal{C}_k)$ if $\epsilon_{\mathrm{T}}(\mathcal{C}_i, \mathcal{C}_j) \wedge \epsilon_{\mathrm{T}}(\mathcal{C}_j, \mathcal{C}_k)$. That ϵ_{T} is a partial order follows from the fact that T is a tree and condition (iii) of the definition of γ.

Lemma 8 *Let T be a join tree for a partially oriented chordal graph H without any unshielded colliders and with orientations closed under rules R1, R2, R3, and R4. If there exists an unshielded triple* $\langle A, B, C\rangle$ *such that* $A \rightarrow B$ *in H then for all i and j such that* $A \in \mathcal{C}_i \wedge B \in \mathcal{C}_i \wedge C \notin \mathcal{C}_i$ *and* $A \notin \mathcal{C}_j \wedge B \in \mathcal{C}_j \wedge C \in \mathcal{C}_j$ *it is the case that* $\gamma_{\mathrm{T}}(\mathcal{C}_i, \mathcal{C}_j)$.

Proof— The proof is in two parts; Figure 2 helps to clarify the proof somewhat. Part (i) — Show that for all $C \in \mathcal{C}_j \backslash \Lambda_{ij}$ it is the case that $B \rightarrow C$ is in H. Simply apply R1 to each of the required edges.

Part (ii) — Show that for all $D \in \Lambda_{ij}$ and for all $C \in \mathcal{C}_j \backslash \Lambda_{ij}$ it is that case that $D \rightarrow C$ is in H. This follows by application of R4 to A, B, C, and D if A— D. If $D \rightarrow A$ then $D \rightarrow B$ by R2 and $D \rightarrow C$ by R2. If $A \rightarrow D$ then $D \rightarrow C$ by R1.□

Lemma 9 *Let T be a join tree for a partially oriented chordal graph H without any unshielded colliders and with orientations closed under rules R1, R2, R3, and R4. (i) If* $\epsilon_{\mathrm{T}}(\mathcal{C}_i, \mathcal{C}_j)$ *then for all k such that the (unique) path p between* $\mathcal{C}_i$ *and* $\mathcal{C}_k$ *in T is through j then* $\epsilon_{\mathrm{T}}(\mathcal{C}_i, \mathcal{C}_k)$ *and (ii) if* $\mathcal{C}_l$ *and* $\mathcal{C}_m$ *are adjacent on the path p then* $\gamma(\mathcal{C}_l, \mathcal{C}_m)$.

Proof — Part (i) is proved by induction on length of path between $\mathcal{C}_j$ and $\mathcal{C}_k$ in the join tree for T. The base case ($j = k$) is trivial and apply Lemma 8

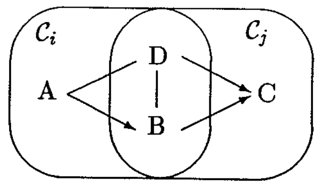

Figure 2: Schematic for Lemma 8

for induction step. Part (ii) follows in a similar fashion. Consider the minimal element C_l of $\epsilon_{\rm T}$ such that $\epsilon_{\rm T}(C_l, C_i)$ or $C_l = C_i$. Let C_m be an arbitrary clique such that $\epsilon_{\rm T}(C_l, C_m)$ and $\Lambda_{lm} \neq \emptyset$. It must be the case that $\gamma(C_l, C_m)$ otherwise it would not be the case that $\epsilon_{\rm T}(C_l, C_m)$. Then we simply apply Lemma 8 to extend the chain of γ between adjacent cliques.□

A partial order π over vertices is *compatible* with the orientations in graph H if and only if for no pair of vertices A and B such that $A \to B$ in H is it the case that $\pi(B, A)$.

Lemma 10 *Let T be a join tree for a partially oriented chordal graph H without any unshielded colliders and with orientations closed under rules R1, R2, R3, and R4. (i) there exists a tree ordering which extends $\epsilon_{\rm T}$, (ii) for all tree orderings π which extend $\epsilon_{\rm T}$ it is the case that $\prec_\pi$ is compatible with H.*

Proof—

(i) Since $\epsilon_{\rm T}$ is a partial order there is a minimal element. Choose any minimal element as the root of the tree order. By Lemma 9, a tree order constructed in such a manner extends $\epsilon_{\rm T}$.

(ii) Let π be a tree order which extends $\epsilon_{\rm T}$. Suppose that $\prec_\pi$ is *not* compatible with H. Then there exists a pair of vertices A and B such that $A \to B$ in H and $B \prec_\pi A$. Let C_j be a clique which contains both A and B. For $B \prec_\pi A$ to hold it must be the case that there is a i such that $\pi(C_i, C_j)$. By Lemma 8 $B \to A$. Contradiction.□

Theorem 11 (Orientation completeness with Background Knowledge) *The result of applying rules R1, R2, R3 and R4 (and orienting edges according to $\mathcal{K}$) to a pattern is a maximally oriented graph with respect to $\mathcal{K}$.*

Proof— Let Π_0 be the result of applying the orientation rules R1, R2, and R3 to the partially directed graph Π. Given Lemma 2 no orientation of edges not oriented in Π_0 will create a cycle which includes an edge or edges oriented in Π_0 and no orientation of an edge not oriented in Π_0 can create an unshielded collider with an edge oriented in Π_0. Consider the undirected graph H, a subgraph of Π_0, obtained by removing all of the oriented edges in Π_0. We show that H is a union of disconnected chordal graph(s). Suppose this is not the case. Then, by Lemma 3 all total ordering of the vertices leads to a new unshielded collider $\langle A, B, C\rangle$ in H. By Lemma 2, the triple $\langle A, B, C\rangle$ also forms an unshielded triple in Π_0, that is $A \notin ADJ(C)$ in Π_0. This is a contradiction since we assume that the graph Π and thus Π_0 have all unshielded colliders oriented and that there is an acyclic orientation of the graph Π with no new unshielded colliders. Let Π_1 be the result of orienting all of the edges in Π_0 that can be oriented with background knowledge and let Π_2 be the result of applying orientation rule R1, R2, R3, and R4 exhaustively to Π_1. Let A— B be unoriented in Π_2 and show that there exists consistent orderings α and γ such that $A \rightarrow B$ in H_α and $B \rightarrow A$ in H_γ.

Case 1 — For all i either $A \in \mathcal{C}_i \wedge B \in \mathcal{C}_i$ or $A \notin \mathcal{C}_i \wedge B \notin \mathcal{C}_i$. Let T be a join tree for H and let π be a tree ordering of T which extends ϵ_{T}; that one exists follows from Lemma 10. A and B are not comparable with respect to $\prec_\pi$ thus by Lemma 5 we simply choose two extensions (consistent with the ordering existing in Π_2) of $\prec_\pi$; one with $A \prec_\pi B$ and another with $B \prec_\pi A$. By Lemma 5 we are done.

Case 2 — There exists an i such that $A \notin \mathcal{C}_i \vee B \notin \mathcal{C}_i$ and $A \in \mathcal{C}_i \vee B \in \mathcal{C}_i$. Without loss of generality assume that $A \notin \mathcal{C}_i \wedge B \in \mathcal{C}_i$. Given that there is an edge between A and B there is a j such that $j \neq i$ and $A \in \mathcal{C}_j \wedge B \in \mathcal{C}_j$. Since the edge between A and B is unoriented we know that it is not the case that $\gamma(\mathcal{C}_i, \mathcal{C}_j)$ and thus it is not the case that $\epsilon_{\mathrm{T}}(\mathcal{C}_i, \mathcal{C}_j)$. Thus the tree order obtained from by letting $\mathcal{C}_j$ to be the root of the tree is compatible with H by Lemma 10 Let π be the tree ordering obtained by letting $\mathcal{C}_j$ to be the root of the tree. Note that for all pairs of vertices in the root clique of the tree ordering are not ordered in the partial order induced by the tree ordering. Let the total order $\prec_1$ be an extension of $\prec_\pi$ consistent with the orientations in Π_2 such that $A \prec_1 B$ and let the total order $\prec_2$ be an extension of $\prec_\pi$ consistent with the orientations in Π_2 such that $B \prec_1 A$. Apply Lemma 5 and we are done.□