Unidimensional Linear Latent Variable Models

by

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Abstract

Linear structural equation models with latent (unmeasured) variables are used widely in sociology, psychometrics, and political science. When such models have a unidimensional (pure) measurement model (Gerbing and Anderson 82, 88; Scheines 92) they imply constraints on the measured covariances which can be used to either confirm unidimensionality or find unidimensional submodels. Assuming unidimensionality, the causal relations among the latent variables can be partially determined by examing other (related) constraints on the measured covariances. In this paper I prove first that unidimensionality is detectible from constraints on only the measured covariances no matter what the structure among latent variables, and second that in a structural equation model with a unidimensional measurement model, for any three latents T_i , T_j, and T_k, $\rho_{Ti,Tj} = 0$ and $\rho_{Ti,Tj.Tk} = 0$ only if certain constraints hold on only the measured covariances.

1. Introduction

Linear structural equation models with latent variables are discussed in Bollen (89) and are used widely. When such models seek to model relations among the latent variables, they must specify measures for each latent so that some contact exists between theory and data. When multiple measures for each latent are given, such models imply testable constraints on the covariance matrix of measured variables. It is through these constraints that different structure among latent variables can be detected.

It is straightforward to represent, without loss of generality, a structural equation model with a directed graph. The graph contains a directed arrow from A to B just in case A is a direct cause of B, and moving from a graph to its corresponding system of equations involves simply specifying each effect as a linear combination of its immediate causes, including an independent error (Glymour, et.al, 87, Spirtes, Glymour, and Scheines, 93).[1] Structural equation models are typically divided into two parts: the "measurement model," and the "structural model." Roughly, the structural model involves only the causal connections among the latent variables, and the measurement model the rest, e.g. the connections between latent and measured variables. Consider the graph in figure 1, in which the T variables are latent, the Y variables are measured, and the ε and ζ variables are error terms.[2]

[1]One can extend the directed graph representation of structural equation models to include undirected edges, which represent unexplained correlations. In the formal analysis of the directed graph, these undirected edges are replaced with a new variable which is set to be a cause of both variables connected by the undirected edge.

[2]For purely illustrative purposes, one might imagine that this model applies to married, male Navy pilots. η_3 might express the pilots level of job satisfaction, η_4 how challenging he finds his career, η_1 how traditional a family the pilot comes from, and η_2 how supportive the pilot's spouse is toward his Navy career. The *Y* variables might be questionaire responses.

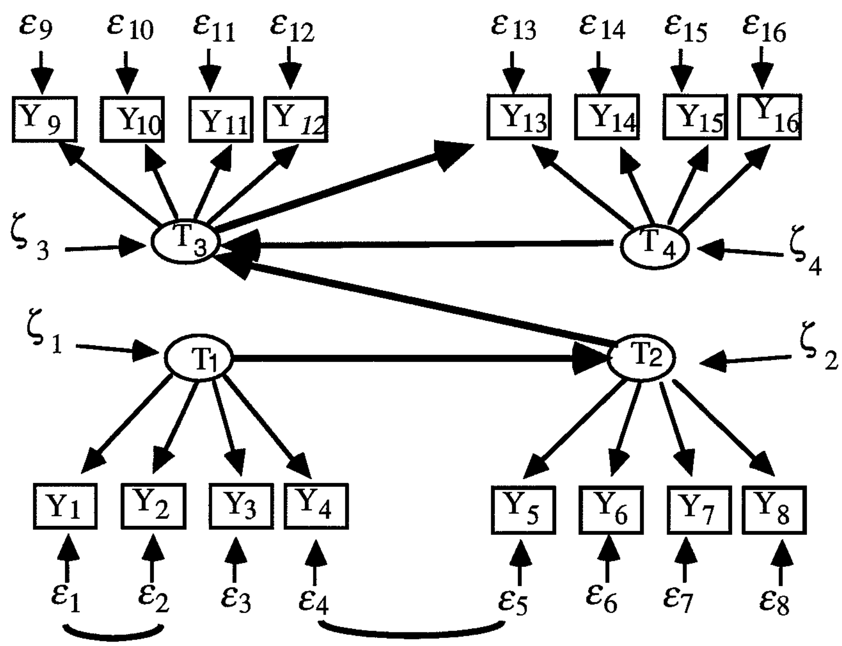


Figure 1

In this case the structural model is the maximal subgraph involving only T and ζ variables, and the measurement model its complement, eg. figure 2.

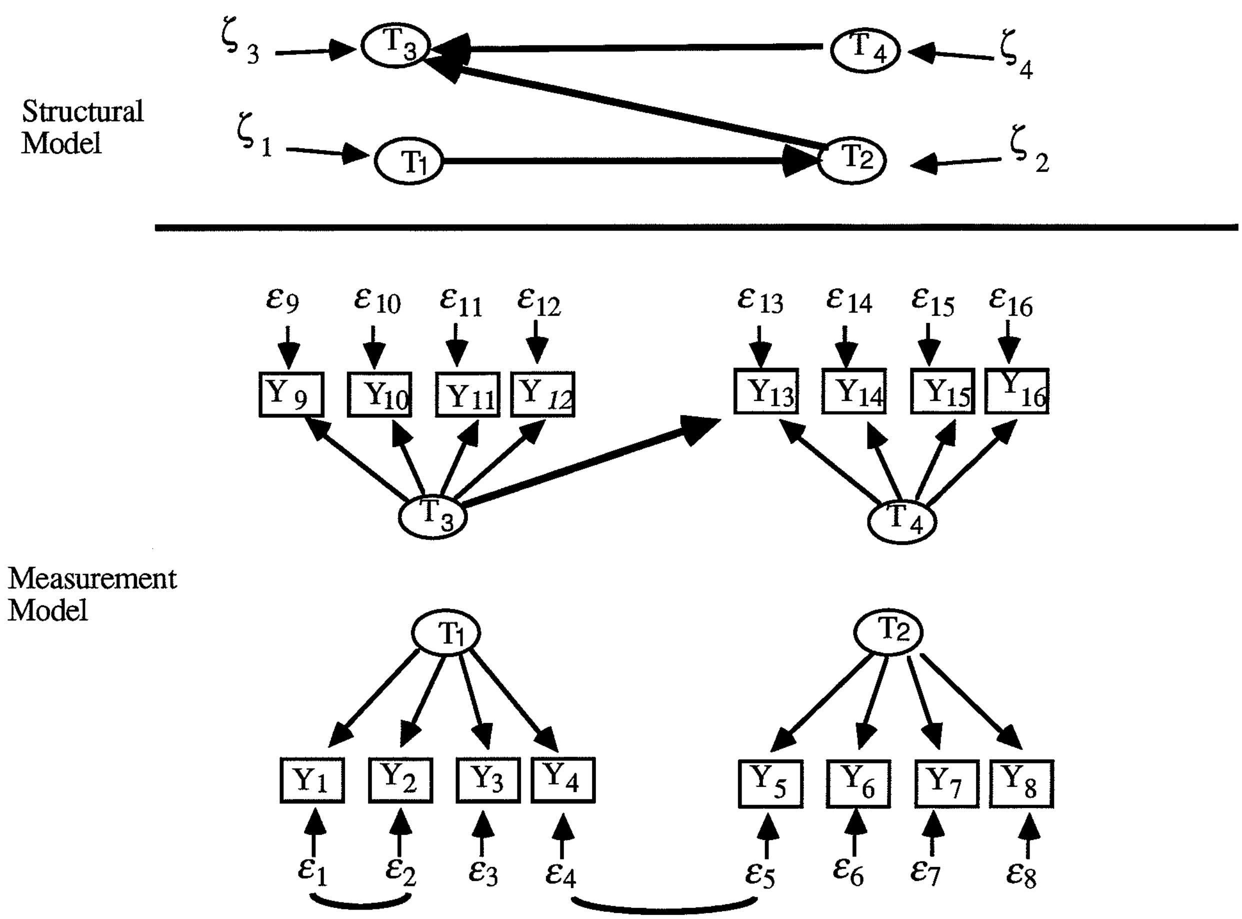


Figure 2

Intuitively, a measured indicator is pure, or unidimensional, if its only causal contact with the rest of the variables in the system is through its latent. So in the measurement model above, for example, all indicators are pure except for Y_1, Y_2, Y_4, Y_5, and Y_{13}. In what follows I make these notions precise and prove that unidimensionality is detectible, as is 0 and 1st-order d-separation among latent variables that have a pure measurement model.[3]

2. Unidimensional Measurement Models

As in (Spirtes, Glymour, and Scheines, 93),[4] a directed graph G with vertices **V** represents a causal structure S for a population of units when the vertices in **V** denote the variables in S, and

[3]D-separation is a graph theoretic relation given by Pearl (1988). For structural equation models, **X** and **Y** are d-separated by **Z** only if **X** and **Y** are independent given **Z** (Spirtes, Glymour, Scheines, 93). The order of the d-separation is the cardinality of the separting set.

[4]page 47.

there is a directed edge from A to B in G if and only if A is a direct cause of B relative to **V**. We call a directed acyclic graph that represents a causal structure a **causal graph.**

Let G be a causal graph over **T** ∪ **V** ∪ **C**. G is a **latent variable model** if

1) **T** is a set of latent variables, and

2) **V** is a set of measured variables such that each member of **V** is the direct effect of at least one member of **T** and V is the cause of no member of **T** ∪ **C**,

3) **C** is a set of latent variables disjoint from **T** such that each C ∈ **C** is either a common cause of some T ∈ **T** and some V ∈ **V**, or is a common cause of V_i, V_j ∈ **V**,

4) for each X ∈ **T** ∪ **V** ∪ **C**, X is a linear combination of its immediate causes in G and an error variable E_x such that for all i,j, E_i, E_j are independent and $Var(E_i) \neq 0$, and

5) **V** can be partitioned into **V**(T_i) such that for every T_i ∈ **T**, |**V**(T_i)| > 0, and for every V ∈ **V**(T_i), V is a direct effect of T_i ∈ **T**.

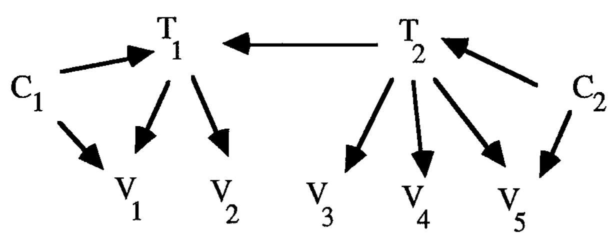
A measure V∈ **V**(T_i) is **almost pure** just in case

i) V is the cause of no variable in **V**/V, and either

ii) V is a direct effect of T_i only, or

iii) V is the direct effect of T_i, and there is a C ∈ **C**, such that C is a common cause of T_i and V only, and no other L ∈ **T** ∪ **C** is a cause of T_i.

A measure V∈ **V**(T_i) is **pure** just in case V is almost pure and is an effect of T_i only.



V1: Impure, V2- V4: Pure, V5: Almost Pure

V is **impure** if it is not almost pure. G is an **almost pure latent variable model** if it is a latent variable model and every V∈ **V** is pure or almost pure. G is a **pure latent variable model** if it is a latent variable model and every V∈ **V** is pure. G is a **unidimensional** latent variable model if it is either pure or almost pure.

There is a **trek** between X and Y in a graph G iff there is i) a directed path from X to Y, ii) a directed path from Y to X, or iii) a directed path from some Z≠X,Y to both X and Y. A measure can be impure for four reasons, which are exhaustive but not exclusive:

(i) If $V \in \mathbf{V}(T_i)$ and there is a trek between V and $T_j \neq T_i$ that does not contain T_i or any member of $\mathbf{V} \setminus V$ then we say V is **latent-measured impure**.

(ii) If $V, X \in \mathbf{V}(T_i)$ and there is a trek between V and X that does not contain any member of **T** then V and X are **intra-construct impure**.

(iii) If $V \in \mathbf{V}(T_i)$ and $Z \in \mathbf{V}(T_j)$ $i \neq j$, and there is a trek between V and Z that does not contain any member of **T** then we say that V_1 and V_2 are **cross-construct impure**.

(iv) If there is a $C \in \mathbf{C}$ that is the cause of both T_i and some $V \in \mathbf{V}(T_i)$, and there exists some other $R \in \mathbf{T} \cup \mathbf{C}$ that is a cause of T_i, then we say V is **nuisance impure**.

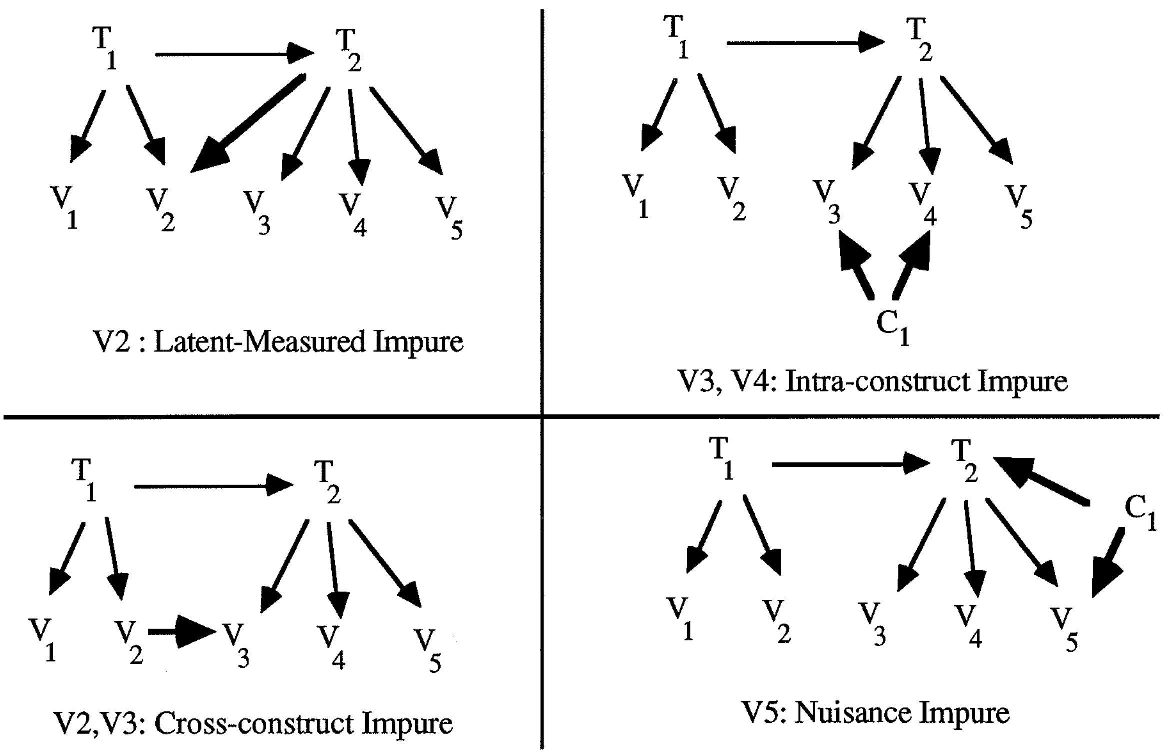


Figure 3

Theorem 1: If G is a latent variable model, then every $V \in \mathbf{V}$ is either almost pure, latent-measured impure, intra-construct impure, cross-construct impure, or nuisance impure.

Proof: By assumption G contains edges from each latent T_i to each $V \in V(T_i)$. Let B be the subgraph of G that contains only these edges. The proof is an induction on the number of edges that need to be added to B to get to G.

Basis case: All indicators are pure in B, so this case is trivial.

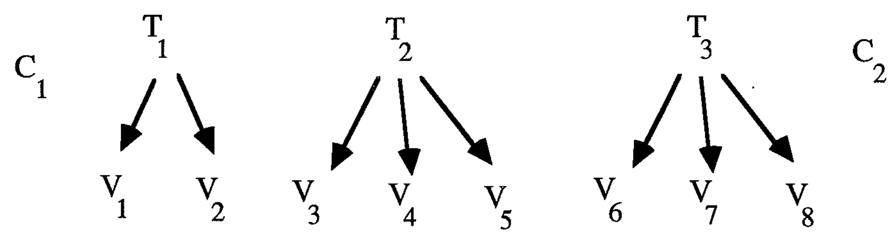


Figure 4: Beginning graph B

Induction: We assume that n-1 edges have been added to B to form B_{n-1}, and that all $V \in \mathbf{V}$ in B_{n-1} are either almost pure, latent-measured impure, intra-construct impure, cross-construct impure, or nuisance impure. We need to show that an additional edge E_n will not cause any V in B_n to fall outside of these categories.

First suppose that the additional edge E_n is out of some $T_i \in \mathbf{T}$. Then there are three cases:

1) E_n is from T_i to T_j,
2) E_n is from T_i to some $V \notin \mathbf{V}(T_i)$, or
3) E_n is from T_i to some $C \in \mathbf{C}$

If E_n is from T_i to T_j, then the only possible change in status is $V \in V(T_j)$ such that there is a C that is a cause of V and of T_j. But V would then be nuisance-impure. If E_n is from T_i to some $V \notin \mathbf{V}(T_i)$, then if V changes at all it becomes latent-measured impure. Suppose finally that E_n is from T_i to some $C \in \mathbf{C}$. This will change nothing save through some other connection involving C. Edges from C to some $T_j \neq T_i$ will produce in effect an edge from T_i to T_j, and thus produce nothing new. Edges from C to some $V \in \mathbf{V}(T_i)$ create a redundant path from T_i to V and are indistinguishable from a single edge from T_i to V (figure 5).

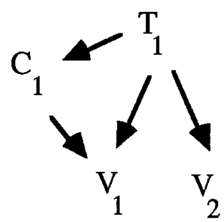


Figure 5

Edges from C to some V $\notin$ **V**(T_i) make V latent-measured impure in virtue of T_i -> C -> V. Edges into C (figure 6) will create no new treks from E_n because these edges and E_n will collide at C.

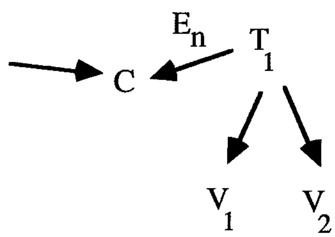


Figure 6

Edges from C to some other $C_2 \in$ **C** will only change things in combination with edges connected to C_2, which reduces to the cases just considered.

Next consider an edge E_n into some $T_i \in$ **T**. By assumption no edge is allowed from any V $\in$ **V** to T_i, so the only possibilities are edges from $T_j \neq T_i$ or from some C $\in$ **C**. Edges from from $T_j \neq T_i$ will change no V. If E_n is from C into T_i, then again E_n will only change something in combination with an edge involving C, and this case is similar to the one involving C above.

Next consider an edge E_n out of some C $\in$ **C**. If E_n is from C to T_i then we are covered by the preceding two paragraphs. Two edges, one from C to some $V_1 \in$ **V**(T_i) and one from C to some $V_2 \in$ **V**(T_j), i $\neq$ j make both V_1 and V_2 cross-construct impure. If i = j then both V_1 and V_2 are intra-construct impure. Edges from $C_1 \in$ **C** to $C_2 \in$ **C** will produce impurities only in combination with other edges connected to C_1 and C_2, and the argument here is the same as two paragraphs back. Edges into some C $\in$ **C** can only be from X $\in$ **T** $\cup$ **C**. In either case they are already covered above.

Next consider an edge E_n out of some V ∈ **V**. No edge out of V may go to any X ∈ **T** ∪ **C**. If E_n is from V_1 ∈ **V**(T_i) to V_2 ∈ **V**(T_j), i ≠ j, then both V_1 and V_2 will be cross-construct impure. If i = j, then both V_1 and V_2 are intra-construct impure.

Finally, consider an edges E_n into some V ∈ **V**. If the edge is from some other V_2 ∈ **V**, then both V and V2 are impure, either intra-construct or cross-construct. If the edge is from C ∈ **C**, then only other edges out of C can make indicators impure, and all those cases are covered above. If V ∈ **V**(T_i) and the edge into V is from Tj ∉ **V**(T_i), then V is latent-measured impure. **Q.E.D.**

3. Unidimensionality and Tetrad Equations

Based on the partition of V, we can use different types of tetrad equations to detect impure indicators. Let τ_{wxyz} stand for the **tetrad equation** $\rho_{wx} * \rho_{yz} = \rho_{wy} * \rho_{xz}$.

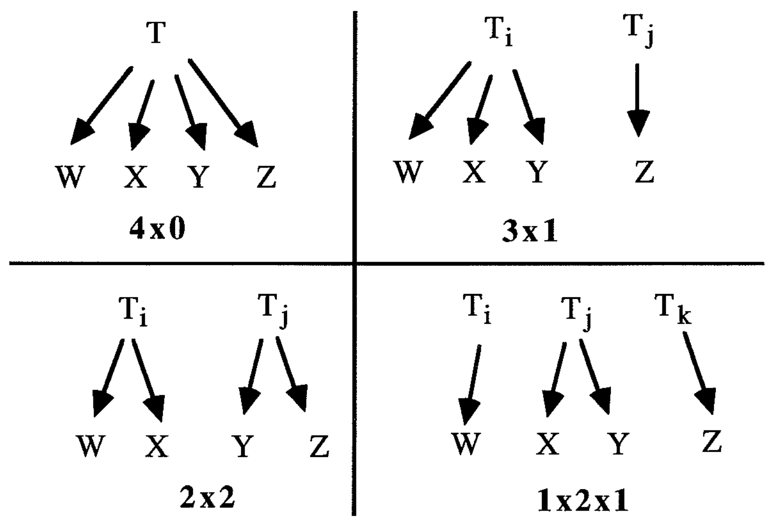


Figure 7

If W,X,Y,Z ∈ **V**(T_i), then τ_{wxyz}, τ_{wxzy}, and τ_{wyzx} are **4x0 tetrad equations**. If W,X,Y∈ **V**(T_i) and Z ∈ **V**(T_j), i ≠ j, then τ_{wxyz}, τ_{wxzy}, and τ_{wyzx} are **3x1 tetrad equations.** If W,X∈ **V**(T_i) and Y,Z ∈ **V**(T_j), i ≠ j, then τ_{wyzx} is a **2x2 tetrad equation.** If W∈ **V**(T_i), X,Y ∈ **V**(T_j), and Z ∈ **V**(T_k), i ≠ j ≠ k, then τ_{wyzx}, τ_{wxzy}, and τ_{wyzx} are **1x2x1 tetrad equations.**

A latent variable model G is parameterized by <ϕ, D>, where ϕ is a vector of the linear coefficients and D the distribution over the exogenous variables. A latent variable model G **linearly implies** a tetrad equation τ_{wxzy} if G implies τ_{wxzy} for every value of <ϕ, D>.

Theorem 2: If G is a latent variable model which linearly implies every 3x1 tetrad equation among **V**, then for every $T_i \in$ **T**, and every V $\in$ **V**(T_i) such that |**V**(T_i)| $\geq$ 3, V is almost pure or for every $T_j \in$ **T**/T_i, T_i and T_j are independent.

Proof: By reductio. Suppose that G is a latent variable model which linearly implies every 3x1 tetrad equation, and that there is some V $\in$ **V**(T_i) such that |**V**(T_i)| $\geq$ 3, V is not almost pure and there is some $T_j \in$ **T**/T_i, such that T_i and T_j are dependent.

By theorem 1, V is either i) latent-measured impure, ii) intra-construct impure, iii) cross-construct impure, or iv) nuisance impure. In each case I will show a contradiction.

<u>Latent-Measured</u>

First suppose that V is latent-measured impure. Then there is a trek between V and $T_j \neq T_i$ that does not contain T_i or any member of **V**\V. By hypothesis |**V**(T_i)| $\geq$ 3, so let V, X, Y $\in$ **V**(T_i), and Z $\in$ **V**(T_j).

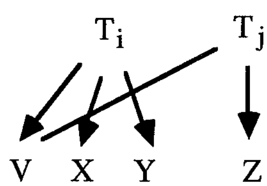


Figure 8

By assumption the 3x1 tetrad equation τ_{vxzy}: $\rho_{vx} * \rho_{zy} = \rho_{vz} * \rho_{xy}$ is linearly implied by G. By the Tetrad Representation Theorem (Spirtes 89; Spirtes, Glymour, and Scheines, 1993) τ_{vxzy} is linearly implied by G if and only if there exists in G either a XZ(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point or a VY(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point. Such a choke point exists trivially if one of V-X, Z-Y and one of V-Z, X-Y are not even trek connected. X-Y are trek connected because of the X-T_i-Y trek, and V-Z are trek connected because of the V-T_j-Z trek. If there is a non-trivial XZ(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point or a non-trivial VY(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point, it must be T_i, because T_i is the only variable on the path from the source of the T(Y,X) trek: X-T_i-Y to either X or Y. The trek between V and T_j can be extended into a trek between V and Z because the Tj

-> Z edge is out of T_j, but by the fact that V is latent-measured impure T_i is not anywhere on this trek. But both the XZ(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point and the VY(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point must include a variable that is on all treks between V and Z, thus T_i cannot be a choke point and we have a contradiction.

Intra-Construct

Next suppose that V is intra-construct impure. Then there is some X such that V, X $\in$ $\mathbf{V}(T_i)$ and there is a trek between V and X that does not contain T_i. $|\mathbf{V}(T_i)| \geq 3$, so let V, X, Y $\in$ $\mathbf{V}(T_i)$. By hypothesis there is a T_j such that T_j,T_i are dependent. Let Z $\in$ $\mathbf{V}(T_j)$.

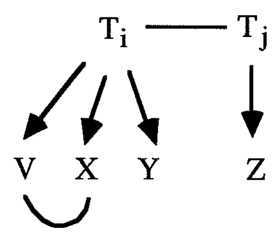


Figure 9

Again by hypothesis the 3x1 tetrad equation τ_{vxzy}: $\rho_{vx} * \rho_{zy} = \rho_{vz} * \rho_{xy}$ is linearly implied by G, and again either there is a trivial XZ(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) or VY(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point, or T_i must be the choke point. Since T_j,T_i are dependent and thus trek-connected, none of V-X, Z-Y, V-Z, X-Y are trek disconnected, so T_i must be the choke point. But there is a trek between V and X that does not contain T_i, so T_i cannot be the choke point.

Cross-Construct

Next suppose that V is cross-construct impure. Thus V $\in$ $\mathbf{V}(T_i)$ and Z $\in$ $\mathbf{V}(T_j)$ $i \neq j$, and there is a trek between V and Z that does not contain any member of **T**. $|\mathbf{V}(T_i)| \geq 3$, so let V, X, Y $\in$ $\mathbf{V}(T_i)$.

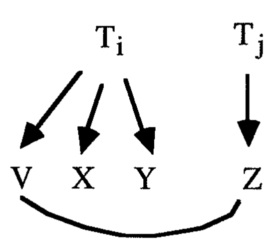


Figure 10

Again by hypothesis the 3x1 tetrad equation τ_{vxzy}: $\rho_{vx} * \rho_{zy} = \rho_{vz} * \rho_{xy}$ is linearly implied by G, and again either there is a trivial XZ(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) or VY(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point, or T_i must be the choke point. Since there is a trek between V and Z, neither of V-Z or X-Y are trek disconnected, so T_i must be the choke point. But there is a trek between V and Z that does not contain T_i, so T_i cannot be the choke point.

Nuisance

Finally, suppose that V is nuisance impure. Thus there is a C ∈ **C** that is the cause of both T_i and some V ∈ **V**(T_i), and there exists some other R ∈ **T** ∪ **C** that is a cause of T_i.

Suppose R = T_j ∈ **T**. Then let V, X, Y ∈ **V**(T_i) and Z ∈ **V**(T_j), and T_j is a cause of T_i.

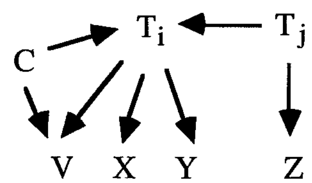
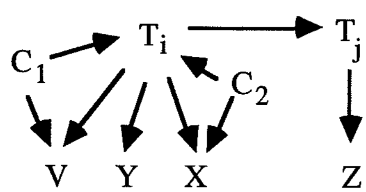


Figure 11

By hypothesis the 3x1 tetrad equation τ_{vxzy}: $\rho_{vx} * \rho_{zy} = \rho_{vz} * \rho_{xy}$ is linearly implied by G, and again either there is a trivial XZ(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) or VY(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point, or T_i must be the choke point. Since there is a trek between T_i and T_j, the choke point cannot be trivial, so it must be T_i. T_i is not a XZ(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point because T_j is the only variable on the path from the source of the V-T_i-T_j-Z trek to Z. T_i is not a VY(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point because C is the only variable on the path from the source of the V-C-T_i-X trek to V.

Suppose R = C_2 ∈ **C**. There are two cases. Either 1) C_2 is also a cause of some X ∈ **V**(T_i), or 2) C_2 is the cause of some Z ∈ **V**(T_j). In case 1 (figure 12), let T_j be the variable such that T_i, T_j are dependent by hypothesis, and let Z ∈ **V**(T_j). If the T_i-T_j trek is into T_i, then this case reduces to the last, so the only case that remains is T_i -> T_j.

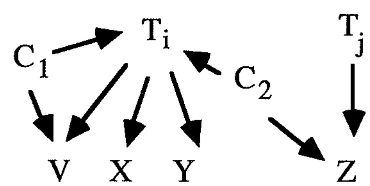


(1)

Figure 12

Astonishingly, we can assume that the 3x1 tetrad equation τ_{vxzy}: $\rho_{vx} * \rho_{zy} = \rho_{vz} * \rho_{xy}$ is linearly implied by G, and again either there is a trivial XZ(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) or VY(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point, or T_i must be the choke point. Since there is a trek between T_i and T_j, the choke point cannot be trivial, so it must be T_i. T_i cannot be a XZ(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point, because C_2 is the only variable on the path from the source of the Y-T_i-C_2-X trek to X. T_i cannot be a VY(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point, because C_1 is the only variable on the path from the source of the V-C_1-T_i-Z trek to V.

In case 2 (figure 13), C_2 is the cause of some $Z \in V(T_j)$.



(2)

Figure 13

The 3x1 tetrad equation τ_{vxzy}: $\rho_{vx} * \rho_{zy} = \rho_{vz} * \rho_{xy}$ is linearly implied by G, and again either there is a trivial XZ(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) or VY(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point, or T_i must be the choke point. C_2 is a common cause of every pair involving V,X,Y and Z, so the choke point cannot be trivial and it must be T_i. T_i cannot be a XZ(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point, because C_2 is the only variable on the path from the source of the V-T_i-C_2-Z trek to Z. T_i cannot be a VY(T(V,Z), T(Y,X), T(V,X), T(Z,Y)) choke point, because C_1 is the only variable on the path from the source of the V-C_1-T_i-X trek to V.

Q.E.D.

4. Structure Among the Latents

For convenience I restate the definitions of almost pure and pure.

A measure V∈ $\mathbf{V}(T_i)$ is **almost pure** just in case

i) V is the cause of no variable in V/V, and either

ii) V is a direct effect of T_i only, or

iii) V is the direct effect of T_i, and there is a C ∈ **C**, such that C is a common cause of T_i and V only, and no other L ∈ **T** ∪ **C** is a cause of T_i.

A measure V∈ $\mathbf{V}(T_i)$ is **pure** just in case V is almost pure and is an effect of T_i only.

Theorem 3: If G is an almost pure latent variable model in which $|\mathbf{V}(T_i)| \geq 2$ for every I,J ∈ $\mathbf{V}(T_1)$, L ∈ $\mathbf{V}(T_3)$, I,K ∈ $\mathbf{V}(T_2)$, then latents T_1 and T_3 are d-separated given T_2 if and only if G linearly implies $\rho_{JI}\rho_{LK} = \rho_{JL}\rho_{KI} = \rho_{JK}\rho_{IL}$.

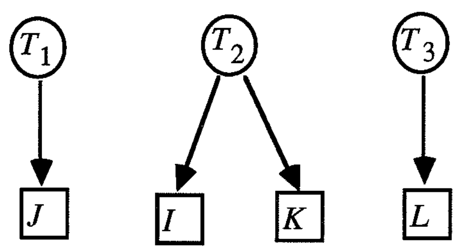


Figure 14

Lemma 3.1: If G is an almost pure measurement model, and X ∈ $\mathbf{V}(T_i)$ then T_i is a non-collider on every undirected path connecting X to any Y ∈ **T** ∪ V\{X} in G.

Proof: If G is an almost pure measurement model, then X is pure or almost pure. In either case X is an effect only. If X is pure, then it is an effect of T_i only so T_i is a non-collider on any undirected path involving X.

If X is almost pure then it is still an effect only but there is a C_1 ∈ **C** that is a direct cause of both X and T_i. There cannot be an edge between C_1 and any T_j ∈ T\$T_i$ or between C_1 and any Y ∈ V\X, because the C_1 -> X edge could always be concatenated to such an edge to form a trek that would make X impure. There cannot be an edge connecting C_1 to some other C_2 ∈ **C**, because in that case C_2 would also be the cause of some T_j ∈ T\$T_i$ or to some X' ∈ V\X, in which case again X would be impure because of the trek between X and T_j that is not through T_i. So all paths connecting X to any $V \in \mathbf{T} \cup \mathbf{V} \setminus \{X\}$ must go through T_i. If C_1 is a cause of T_i then nothing else can be a cause of T_i, so all other edges involving T_i are out of T_i, and therefore there is *no* path on which T_i is a collider if any of its indicators are almost pure but not pure. **Q.E.D.**

Corrolary 3.1.1: If G is an almost pure measurement model, and $X \in \mathbf{V}(T_i)$, then T_i is on every trek connecting X and $Y \in \mathbf{T} \cup \mathbf{V} \setminus \{X\}$ in G.

Lemma 3.2: If G is an almost pure latent variable model in which $|\mathbf{V}(T_i)| \geq 2$ for every i, $J \in \mathbf{V}(T_1)$, $L \in \mathbf{V}(T_3)$, $I,K \in \mathbf{V}(T_2)$, then latents T_1 and T_3 are d-separated given T_2 only if G linearly implies $\rho_{JI}\rho_{LK} = \rho_{JL}\rho_{KI} = \rho_{JK}\rho_{IL}$.

Proof. Because I and K are almost pure indicators of T_2 in G, by lemma 3.1 T_2 d-separates I-K. By similar reasoning T_2 d-separates J-I, J-K, L-I, and L-K. By lemma 3.1, T_1 and T_2 are non-colliders on every undirected path connecting J and L. Since T_1 and T_3 are d-separated given T_2, then J and L are d-separated given T_2. In general X and Z are d-separated given Y if and and only if G linearly implies $\rho_{XZ.Y} = 0$. Hence G linearly implies $\rho_{JI} = 0$, and $\rho_{JI} = \rho_{JT_2} * \rho_{IT_2}$. Similarly:

$$\rho_{JK} = \rho_{JT_2} * \rho_{KT_2},$$
$$\rho_{JL} = \rho_{JT_2} * \rho_{LT_2},$$
$$\rho_{IK} = \rho_{IT_2} * \rho_{KT_2},$$
$$\rho_{IL} = \rho_{IT_2} * \rho_{LT_2},$$
$$\rho_{KL} = \rho_{KT_2} * \rho_{LT_2}.$$

Hence G linearly implies:

$$\rho_{JI} * \rho_{LK} = \rho_{JT_2} * \rho_{IT_2} * \rho_{KT_2} * \rho_{LT_2},$$
$$\rho_{JK} * \rho_{IL} = \rho_{JT_2} * \rho_{KT_2} * \rho_{IT_2} * \rho_{LT_2},$$
$$\rho_{JL} * \rho_{IK} = \rho_{JT_2} * \rho_{LT_2} * \rho_{IT_2} * \rho_{KT_2},$$

and thus $\rho_{JI} * \rho_{LK} = \rho_{JK} * \rho_{IL} = \rho_{JL} * \rho_{IK}$. **Q.E.D.**

Lemma 3.3: If G is an almost pure latent variable model in which $|\mathbf{V}(T_i)| \geq 2$ for every i, $J \in \mathbf{V}(T_1)$, $L \in \mathbf{V}(T_3)$, $I,K \in \mathbf{V}(T_2)$, then latents T_1 and T_3 are d-separated given T_2 if G linearly implies $\rho_{JI}*\rho_{LK} = \rho_{JL}*\rho_{KI} = \rho_{JK}*\rho_{IL}$.

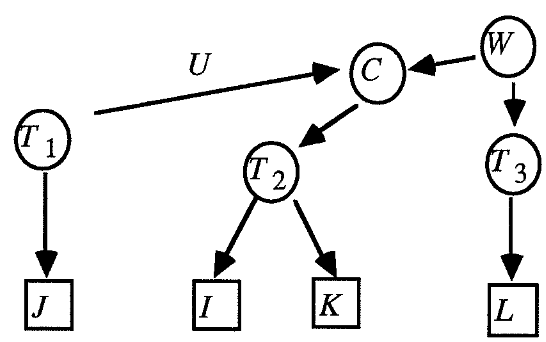
Proof. Suppose that G linearly implies $\rho_{JI}*\rho_{LK} = \rho_{JL}*\rho_{KI} = \rho_{JK}*\rho_{IL}$ but T_1 and T_3 are not d-separated given T_2. By the Tetrad Representation Theorem,[5] if G linearly implies $\rho_{JI}*\rho_{LK} = \rho_{JL}*\rho_{KI}$ then either there is an IL(T(I,J),T(L,K),T(L,J),T(I,K)) choke point, or there is a JK(T(I,J),T(L,K),T(L,J),T(I,K)) choke point.

Let T(I,K) be the trek consisting of the edges from T_2 to I and T_2 to K. Suppose first that there is an IL(T(I,J),T(L,K),T(L,J),T(I,K)) choke point. The choke point is either I or T_2 because those are the only vertices in I(T(I,K)). I is not the choke point because it does not lie on any trek between L and K. Hence T_2 is the choke point. Similarly, if there is a JK(T(I,J),T(L,K),T(L,J),T(I,K)) choke point it is T_2. Hence, in either case T_2 is a choke point.

There are two ways that T_1 and T_3 might fail to be d-separated given T_2. Either there is a trek between T_1 and T_3 that does not contain T_2, or there is some undirected path U between T_1 and T_3 such that T_2 is a descendent of every collider on U, and T_2 is not a non-collider on U.

First assume that there is some trek between T_1 and T_3 that does not contain T_2. Then there is a trek between J and L that does not contain T_2. But then T_2 is not a choke point, contrary to what we have just proved. Now assume that there is some undirected path U between T_1 and T_3 such that T_2 is a descendent of every collider on U, and T_2 is not a non-collider on U. In that case U d-connects T_1 and T_3 given T_2. Again there are two cases.

Suppose first that T_2 is an IL(T(I,J), T(L,K), T(L,J), T(I,K)) choke point. Let C be the collider on the undirected path U that is closest to T_3.



U(T_3,C) does not contain any colliders on U except C because C is the closest collider to T_3 on U; hence U(T_3,C) is a trek between T_3 and C. There is a vertex W on U(T_3,C) that is the

[5]See (Spirtes, Glymour, and Scheines 93), chapter 6.

source of a trek between T_3 and C. W $\neq$ C because W is not a collider on U, but C is. Hence U(W,T_3) contains no colliders on U. It follows that U(W,T_3) does not contain T_2, because T_2 is not a non-collider on U. Hence there is a trek T(K,L) between K and L whose K branch consists of the concatenation of U(W,C), a directed path from C to T_2, and the edge from T_2 to K, and whose L branch consists of the concatenation of U(W,T_3) and the edge from T_3 to L. Because neither U(W,T_3) nor the edge from T_3 to L contains T_2, T_2 is not in L(T(K,L)), and hence is not an IL(T(I,J),T(L,K), T(L,J),T(I,K)) choke point, contrary to our hypothesis.

A similar argument shows that if there is some undirected path U between T_1 and T_3 such that T_2 is a descendent of every collider on U and T_2 is not a non-collider on U, then there is no JK(T(I,J), T(L,K), T(L,J), T(I,K)) choke point. Therefore T_1 and T_3 are d-separated given T_2. **Q.E.D.**

Theorem 3 follows immediately from lemmas 3.2 and 3.3.

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