Intercalation Calculus for Intuitionistic Propositional Logic

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Table of Contents

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INTRODUCTION

The rules in Gentzen's Natural Deduction calculi were formulated to reflect directly and, thus, in a natural way the reasoning used in carrying out mathematical proofs. These rules, dependent only on the syntactic form of assumptions and conclusion, are supposed to capture the meaning of sentential connectives: for each connective there is an introduction rule, which says how a formula with that connective as its principal sign can be obtained, and an elimination rule, which says how a formula which has that connective as its principal sign can be used to obtain other formulas. The "naturalness" of the rules makes the correctness of each step of a proof in a Natural Deduction calculus immediate. And yet, Natural Deduction calculi are generally considered not to be suitable for automated proof search (see for example [Fi1]), and have hardly influenced the developments of automated theorem proving, which have been based mostly on sequent calculi and resolution.

The idea of performing automated proof search in a more human-oriented way, motivated partly by pedagogical concerns, that is by the idea of using a computerized tutor to teach students how to do proofs in Logic, has led to the development of the *Intercalation Calculus*. This calculus was proposed in 1987 by Sieg, who also established its basic properties [Si1]. From a derivation in the Intercalation Calculus one can easily and in a unique way obtain a Natural Deduction derivation with the same conclusion and assumptions. Since the Intercalation Calculus is complete, it is possible to use it as a tool for obtaining Natural Deduction derivations, that is, one can search for a Natural Deduction derivation in the framework of the Intercalation Calculus.

The reason for searching for Natural Deduction proofs via the Intercalation Calculus is this: the latter calculus allows one to build up a *search space* that "codes" all possible normal derivations from given assumptions to a given conclusion. Derivations in the Intercalation Calculus have the Subformula Property: this guarantees (in the propositional case) the finiteness of the search space for a proof of a given conclusion from given assumptions. The finiteness of the search space allows to use the Intercalation Calculus as the basis for defining and implementing an algorithm which performs automated proof search. In the implementation certain heuristics are used to limit the search space further: these are discussed in [SiSc].

The algorithm has been implemented in a program called the Carnegie Mellon Proof Tutor, developed by J. Pressler, R. Scheines, W. Sieg and C. Walton, which is used in Logic courses at Carnegie Mellon University.
Of independent proof-theoretic interest are the following facts: the Natural Deduction derivations that are associated with derivations in the Intercalation Calculus are ***normal***. Thus, the Completeness Theorem for the Intercalation Calculus yields a novel semantic proof of the Normal Form Theorem for the Natural Deduction Calculus.

The Intercalation Calculus has been proposed and analyzed for classical logic, but it is possible to define versions of the calculus for different logics (current research tries to extend both the theoretical algorithm and the implementation to first order logic). In this report we define a version of the Intercalation Calculus for intuitionistic (propositional) logic, and prove its fundamental properties; in particular, its completeness. We note that to obtain the completeness it has been necessary to change one of the rules with respect to the original formulation of the classical Intercalation Calculus, namely the $\rightarrow$-elimination rule. The first chapter contains a review of the Natural Deduction Calculus and the Intercalation Calculus for classical propositional logic, with an outline of the proofs of the fundamental results about the latter. For the presentation of the Intercalation Calculus and its basic properties we follow [Si1].
In the second chapter we review Natural Deduction and Kripke semantics for intuitionistic propositional logic, define the intuitionistic Intercalation Calculus and prove its properties.
The third chapter is devoted to a discussion of the heuristics used in the current implementation for the classical case, and possible heuristics for the intuitionistic case.

CHAPTER 1

§1. Natural Deduction for classical propositional logic.

We assume that the reader has some familiarity with various logical systems. However, we will briefly review the Natural Deduction (ND) calculus for classical propositional logic.

We start with a language $\mathcal{L}$ containing a countable set of propositional variables, the sentential connectives $\&, \vee, \rightarrow$, the constant $\perp$ for falsehood and parentheses.
The set of formulas of $\mathcal{L}$ is defined inductively by:
i) $\perp$ is a formula;
ii) any propositional variable p is a formula;
iii) if φ, ψ are formulas, then so are $(\varphi \& \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$.

In the sequel, Greek letters $\varphi, \psi \ldots$, as well as capital Latin letters G,H... will denote formulas of $\mathcal{L}$, while Greek letters $\alpha, \beta \ldots$ will denote finite sequences of formulas of $\mathcal{L}$. The usual conventions about parentheses will be adopted (for example, the outer parentheses around a formula will often be dropped). We will write $\varphi \in \alpha$ to indicate that φ is one of the formulas in the sequence α. We also indicate with α, φ the extension of the sequence α by φ, and with $\alpha\beta$ the concatenation of the two sequences α, β.

Negation is defined by $\sim\varphi \equiv \varphi \rightarrow \perp$.

The set of subformulas of a formula φ is defined inductively by:
i) φ is a subformula of φ;
ii) if $\psi_1 \& \psi_2$, $\psi_1 \vee \psi_2$, or $\psi_1 \rightarrow \psi_2$ is a subformula of φ then so are ψ_1 and ψ_2.

The inference rules of the ND-calculus for classical propositional logic consist of an introduction rule (I-rule) and an elimination rule (E-rule) for each sentential connective. These rules may be indicated pictorially by the following figures:

&I) $\dfrac{\varphi \quad \psi}{\varphi \& \psi}$ &E) $\dfrac{\varphi \& \psi}{\varphi}$ $\dfrac{\varphi \& \psi}{\psi}$

$$\vee I)\ \frac{\varphi}{\varphi \vee \psi} \qquad \frac{\psi}{\varphi \vee \psi} \qquad \vee E)\ \frac{\varphi \vee \psi \quad \begin{matrix}[\varphi] \\ \chi\end{matrix} \quad \begin{matrix}[\psi] \\ \chi\end{matrix}}{\chi}$$

$$\rightarrow I)\ \frac{\begin{matrix}[\varphi] \\ \psi\end{matrix}}{\varphi \rightarrow \psi} \qquad \rightarrow E)\ \frac{\varphi \rightarrow \psi \quad \varphi}{\psi}$$

$$\sim I)\ \frac{\begin{matrix}[\varphi] \\ \bot\end{matrix}}{\sim\varphi} \qquad \sim E)\ \frac{\begin{matrix}[\sim\varphi] \\ \bot\end{matrix}}{\varphi}$$

where in each rule the formula below the horizontal line is the consequence and the formula(s) above it is (are) the premiss(es). We use square brackets [ ] to indicate the discharging of a premiss.

Note that the ~-introduction rule is a particular case of the →-introduction rule, and the ~-elimination rule implies the "ex falso quodlibet" rule:

$$\frac{\bot}{\varphi}$$

Given a tree of formulas built up according to the rules of the ND-calculus, we say it is a derivation of φ from α if and only if φ is the root of the tree and all non-discharged assumptions (leaves) are elements of α.

We state without proofs two fundamental results about the ND-calculus, namely the Normalization Theorem and the Subformula Property.
The crucial idea behind normalization is to avoid unnecessary detours in a derivation, that is to avoid introducing a connective by an I-rule and then eliminating it by an E-rule. In [Pr] a normalization theorem is proved for classical ND-calculus without the connective $\vee$ among the primitive symbols. The result we state is a consequence of a stronger normalization theorem which holds for the full calculus (see for example [Stå]), and implies the subformula property.
The following definitions are taken from [Pr].

Definition. A *segment* in a derivation is a sequence $\varphi_1,\ldots,\varphi_n$ of consecutive formula occurrences such that:
1) φ_1 is not the consequence of an application of $\vee$E;
2) φ_i, for each $i<n$, is a minor premiss of an application of $\vee$E;
3) φ_n is not a minor premiss of an application of $\vee$E.

Definition. A *maximum segment* is a segment that begins with the consequence of an application of an I-rule and ends with a major premiss of an E-rule.

Definition. An application of $\vee$E is said to be *redundant* if it has a minor premiss at which no assumption is discharged.
Redundant applications of $\vee$E can obviously be eliminated.

Definition. A derivation is *normal* if it does not contain a maximum segment or a redundant application of $\vee$E.

Theorem (Normalization Theorem). Any ND-derivation D of φ from α can be effectively transformed (via canonical reduction steps) into a normal derivation of φ from α.

Theorem (Subformula Property). Every formula occurrence in a normal derivation has the shape of either a subformula of a formula in α,φ, or the negation of such a subformula, or $\perp$.

§2. Intercalation Calculus for classical propositional logic.

The idea behind the Intercalation Calculus is to close the gap between assumptions and conclusion by intercalating formulas using the available rules of the ND-calculus. That is, to look for a normal derivation by pursuing all possibilities of applying elimination rules *from above* (that is, only to the assumptions, and the formulas that have already been derived from them, to get closer to the conclusion) and of applying inverted introduction rules *from below* (that is, only to the current conclusion to get closer to the assumptions), as well as using the indirect rules for negation.
The rules for the Intercalation Calculus are reformulations of those for the ND-calculus; the distinctive character of the calculus is in the restricted way in which the rules are used. Indeed, the fact that elimination rules are applied only from above and introduction rules only from below guarantees normality of the derivations. The rules operate on triples of the form $\alpha;\beta?G$, where α is the sequence of the available assumptions, G is the current conclusion (or goal), and β is a sequence of formulas obtained by &-elimination and $\rightarrow$-elimination

from elements of α. We will write simply $\alpha;?G$ in case β is the empty sequence Λ. We consider the questions $\alpha;\beta?G$ and $\alpha';\beta'?G'$ (and the nodes these questions are associated with) identical just in case G=G' and the set of formulas in $\alpha\beta$ is equal to the set of formulas in $\alpha'\beta'$.

Here we list the rules for the (classical) Intercalation (propositional) Calculus. The rules corresponding to elimination rules are indicated by $\downarrow$, those corresponding to inverted introduction rules by $\uparrow$, and the indirect rules by $\perp$ (note that we take also the symbol ~ for negation as primitive). The symbol "$\Rightarrow$" shall be interpreted here as follows: if one wants to answer the question on the left of the $\Rightarrow$ positively, one can reduce the problem to the one of answering the question(s) on the right of the $\Rightarrow$ positively.

$\downarrow\&_i$: $\alpha;\beta?G,\ \varphi_1\&\varphi_2\in\alpha\beta,\ \varphi_i\notin\alpha\beta \Rightarrow \alpha;\beta,\varphi_i?G$ (i=1 or 2)

$\downarrow\vee$: $\alpha;\beta?G,\ \varphi_1\vee\varphi_2\in\alpha\beta,\ \varphi_1\notin\alpha\beta,\ \varphi_2\notin\alpha\beta \Rightarrow \alpha,\varphi_1;\beta?G$ **AND** $\alpha,\varphi_2;\beta?G$

$\downarrow\rightarrow$: $\alpha;\beta?G,\ \varphi_1\rightarrow\varphi_2\in\alpha\beta,\ \varphi_2\notin\alpha\beta \Rightarrow \alpha;\beta,\varphi_2?G$ **AND** $\alpha;\beta?\varphi_1$

$\uparrow\&$: $\alpha;\beta?\varphi_1\&\varphi_2 \Rightarrow \alpha;\beta?\varphi_1$ **AND** $\alpha;\beta?\varphi_2$

$\uparrow\vee$: $\alpha;\beta?\varphi_1\vee\varphi_2 \Rightarrow \alpha;\beta?\varphi_1$ **OR** $\alpha;\beta?\varphi_2$

$\uparrow\rightarrow$: $\alpha;\beta?\varphi_1\rightarrow\varphi_2 \Rightarrow \alpha,\varphi_1;\beta?\varphi_2$

$\perp_c$: $\alpha;\beta?\varphi,\ \varphi\neq\perp \Rightarrow \alpha,\sim\varphi;\beta?\perp$

$\perp_i$: $\alpha;\beta?\sim\varphi,\ \varphi\neq\perp \Rightarrow \alpha,\varphi;\beta?\perp$

$\perp_{\mathcal{F}}$: $\alpha;\beta?\perp,\ \varphi\in\mathcal{F} \Rightarrow \alpha;\beta?\varphi$ **AND** $\alpha;\beta?\sim\varphi$

In the last rule, $\mathcal{F}$ is the finite set of all subformulas of formulas occurring in the sequence α. It is, of course, the subformula property that inspires the choice of this $\mathcal{F}$, and the Completeness Theorem stated in §3 justifies it. The finiteness of $\mathcal{F}$ is crucial for the finiteness of the search space. We remark also that it is possible to discount double negations: that is, if $\sim\varphi\in\mathcal{F}$, we consider only φ and $\sim\varphi$, not also $\sim\varphi$ and $\sim\sim\varphi$, among the possible contradictory pairs.

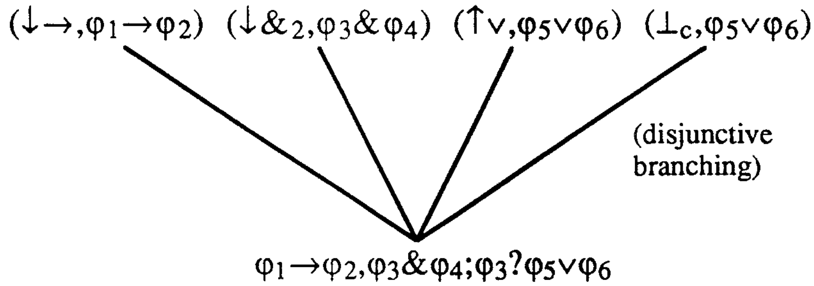
We remark that in the original version of the Intercalation Calculus the rule $\downarrow\rightarrow$ was formulated as $\alpha;\beta?G,\ \varphi_1\rightarrow\varphi_2\in\alpha\beta,\ \varphi_1\in\alpha\beta,\ \varphi_2\notin\alpha\beta \Rightarrow \alpha;\beta,\varphi_2?G$. In the classical case, the calculus obtained with this version is equivalent to the one we give. But the version turns out to be too weak in the intuitionistic case: in fact, using it the resulting calculus for intuitionistic logic is incomplete.

§3. The full intercalation tree.

Given a question $\sigma^*=\alpha^*;?G^*$, we want to define S, the *full intercalation tree* for σ^*. This tree will constitute the search space for a derivation of the goal G^* from the assumptions in α^*. Nodes in the tree will be of two kinds. Those of the first kind (regular nodes) will be labeled, and by abuse of notation identified, with questions of the form $\alpha;\beta?G$. Those of the second kind (special nodes) will be used to indicate the application of a rule, and will be labeled (and identified) with a pair (*, H), where * is the name of the rule and H is the formula to which the rule applies. For example, a special node $(\uparrow\&,\varphi_1\&\varphi_2)$ indicates the step from $\alpha;\beta?\varphi_1\&\varphi_2$ to $\alpha;\beta?\varphi_1$ **AND** $\alpha;\beta?\varphi_2$, while a special node $(\bot_c,\varphi)$ indicates the step from $\alpha;\beta?\varphi$ to $\alpha,\sim\varphi;\beta?\bot$.

We define by induction on n the set S_n of (regular) nodes of S of level n. Branchings from a regular node to special nodes will always be disjunctive, while branchings from a special node to regular nodes may be disjunctive or conjunctive. In a branching, the order of the branches (that is, which one is the leftmost, and so on) is arbitrary. We also define a subset S^t_n of S_n; S^t_n will be the set of terminal nodes (leaves) of level n. By convention, the special nodes are not counted in determining the level of a node. Again by convention, we say that no rule applies to a leaf. In the end, S may be defined as the union of all the S_n, plus all the special nodes encountered during the construction. We also define S^t as the union of all the S^t_n. The ordering $\preccurlyeq$ of S will be evident from the construction itself.

Stage 0. Let $S_0=\{\sigma^*\}$ (σ^* is the root of S). If $G\in\alpha$, or $\alpha=\Lambda$ and $G=\bot$, let $S^t_0=\{\sigma^*\}$. Otherwise let $S^t_0=\emptyset$.
Stage n+1: Assume S_n and S^t_n have been defined. If $S_n\text{-}S^t_n=\emptyset$, then let $S_{n+1}=S^t_{n+1}=\emptyset$. Otherwise, assume $S_n\text{-}S^t_n=\{\sigma_1,\ldots,\sigma_k\}$. We define a set S_{n+1} as follows.
Consider the node $\sigma_i=\alpha;\beta?G$ and assume that there are m_i different possible applications of the rules to σ_i. Extend the branch ending with σ_i by a disjunctive branching with m_i branches, each of which leads to a special node τ^i_j, $j\in\{1,\ldots,m_i\}$, labeled with one of the applicable rules. In the following example, there are four different possibilities of applying the rules.



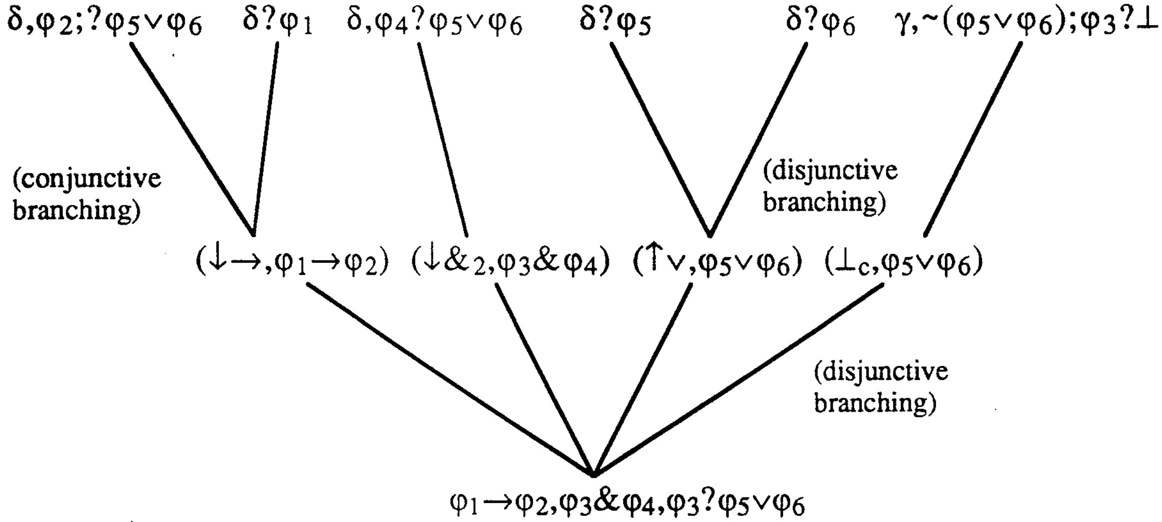
Now, for $i \in \{1,\dots,k\}$, $j \in \{1,\dots,m_i\}$:

1) if $\tau^i{}_j=(\downarrow\&_h,\varphi_1\&\varphi_2)$, h=1 or 2, then extend the branch ending with $\tau^i{}_j$ by a branch leading to $\rho=\alpha;\beta,\varphi_h?G$, and let $\rho \in S_{n+1}$.
2) if $\tau^i{}_j=(\downarrow\vee,\varphi_1\vee\varphi_2)$, then extend the branch ending with $\tau^i{}_j$ by a conjunctive branching leading to $\rho_1=\alpha,\varphi_1;\beta?G$ and $\rho_2=\alpha,\varphi_2;\beta?G$, and let $\rho_1,\rho_2 \in S_{n+1}$.
3) if $\tau^i{}_j=(\downarrow\rightarrow,\varphi_1\rightarrow\varphi_2)$, then extend the branch ending with $\tau^i{}_j$ by a conjunctive branching leading to $\rho_1=\alpha;\beta,\varphi_2?G$ and $\rho_2=\alpha;\beta?\varphi_1$, and let $\rho_1,\rho_2 \in S_{n+1}$.
4) if $\tau^i{}_j=(\uparrow\&,\varphi_1\&\varphi_2)$, that is, $G=\varphi_1\&\varphi_2$, then extend the branch ending with $\tau^i{}_j$ by a conjunctive branching leading to $\rho_1=\alpha;\beta?\varphi_1$ and $\rho_2=\alpha;\beta?\varphi_2$, and let $\rho_1,\rho_2 \in S_{n+1}$.
5) if $\tau^i{}_j=(\uparrow\vee,\varphi_1\vee\varphi_2)$, that is, $G=\varphi_1\vee\varphi_2$, then extend the branch ending with $\tau^i{}_j$ by a disjunctive branching leading to $\rho_1=\alpha;\beta?\varphi_1$ and $\rho_2=\alpha;\beta?\varphi_2$, and let $\rho_1,\rho_2 \in S_{n+1}$.
6) if $\tau^i{}_j=(\uparrow\rightarrow,\varphi_1\rightarrow\varphi_2)$, that is, $G=\varphi_1\rightarrow\varphi_2$, then extend the branch ending with $\tau^i{}_j$ by a branch leading to $\rho=\alpha,\varphi_1;\beta?\varphi_2$, and let $\rho \in S_{n+1}$.
7) if $\tau^i{}_j=(\bot_c,G)$, then extend the branch ending with $\tau^i{}_j$ by a branch leading to $\rho=\alpha,\sim G;\beta?\bot$, and let $\rho \in S_{n+1}$.
8) if $\tau^i{}_j=(\bot_i,\sim\varphi)$, that is, $G=\sim\varphi$, then extend the branch ending with $\tau^i{}_j$ by a branch leading to $\rho=\alpha,G;\beta?\bot$, and let $\rho \in S_{n+1}$.
9) if $\tau^i{}_j=(\bot_{\mathcal{F}},\varphi)$, that is, $G=\bot$, then extend the branch ending with $\tau^i{}_j$ by a conjunctive branching leading to $\rho_1=\alpha;\beta?\varphi$ and $\rho_2=\alpha;\beta?\sim\varphi$, and let $\rho_1,\rho_2 \in S_{n+1}$.

Assume $S_{n+1}=\{\rho_1,\dots,\rho_r\}$. The set $S^t{}_{n+1}$ of leaves of level n+1 consists of those nodes $\rho_i=\alpha;\beta?G$ such that either $G \in \alpha\beta$, or $\alpha=\Lambda$ and $G=\bot$, or if ρ_i is identical to a (regular) node occurring below it.

Thus the induction is complete.

So, in the example previously considered, after the first step we get (setting $\gamma=\varphi_1\rightarrow\varphi_2,\varphi_3\&\varphi_4$ and $\delta=\gamma;\varphi_3$) the following figure.



Due to the definition of the full intercalation tree, only finitely many different formulas can occur in it. Thus, we can formulate only finitely many different questions. Hence, the full intercalation tree is finite.

Define the height of S to be $h(S)=\max\{n: S_n \neq \emptyset\}$.

Now we define an evaluation of the leaves of S, that is we assign **T** (true) or **F** (false) to each terminal node $\rho=\alpha;\beta?G$, as follows.
If ρ is terminal on account of condition (i), that is, $G \in \alpha\beta$, the value of ρ is **T** (we have closed the gap between assumptions and conclusion). If ρ is terminal on account of condition (ii) (we are trying to prove an inconsistency from no assumptions) or (iii) (it is useless to continue the search, since whatever we can find above ρ is also above the identical node occurring below it), the value of ρ is **F**.
The evaluation of the leaves with **T** or **F** can be canonically extended to every node σ in S as follows:
(i) if σ has exactly one successor τ, the value of σ is that of τ;
(ii) if σ has exactly two successors and the branching is conjunctive, the value of σ is **T** if both successors have **T**, otherwise it is **F**;
(iii) if σ has two or more successors and the branching is disjunctive, the value of σ is **F** if all successors have **F**, otherwise it is **T**.
The idea behind this evaluation is that we have succeeded in closing the gap between assumptions and conclusion if the root node evaluates as **T**, while the search has failed if the root node evaluates as **F**.
We say that a full intercalation tree evaluates as **T** (respectively **F**) if its root node evaluates as **T** (respectively **F**).

§4. Completeness Theorem for the classical Intercalation Calculus.

Two important facts were shown about the (classical) Intercalation Calculus [Si1]. Assume S is the full intercalation tree for $\sigma=\alpha;?G$. Then:

Lemma 1. If σ evaluates as T, one can extract from S a ND-derivation of G from α.
Lemma 2. If σ evaluates as **F**, one can determine from S a countermodel for the question $\alpha;?G$, that is, an assignment of truth values to the propositional variables occurring in α,G which makes all formulas in α true and G false.

It should be noted that the ND-derivations extracted from S are normal and have the Subformula Property. Therefore, Lemma 1 and 2 together give a new, semantic proof of the Normal Form Theorem, that is, if there exist a ND-derivation of φ from α, then there is a normal derivation of φ from α.

In the next chapter we give a detailed proof of Lemma 1 for the intuitionistic case that is essentially identical with that for the classical case, the only difference being the rules for negation. Thus we give here just a brief outline. Given S and σ, we say that $\mathfrak{I}$ is an *intercalation derivation* (I-derivation) for σ if and only if $\mathfrak{I}$ is a subtree of S with root σ such that:
(i) the top nodes of $\mathfrak{I}$ evaluate as **T**;
(ii) to each regular node in $\mathfrak{I}$ (except leaves of S), exactly one rule is applied to obtain its successor(s).
Assuming that σ evaluates as T, the existence of an I-derivation for σ in S follows immediately. Then we associate to each I-derivation $\mathfrak{I}$ a ND-derivation with the same assumptions and conclusion, by induction on the height of $\mathfrak{I}$.

We also sketch the proof of Lemma 2. The idea is, assuming σ evaluates as **F**, to extract a special branch from S (the *canonical refutation branch*).[1] Now, let Γ be the set of formulas occurring on the left side of the question mark in the last node of that branch. Define a valuation ν on atomic formulas occurring in $\mathcal{F}$ by $\nu(p)=\mathbf{T}$ if $p\in\Gamma$, $\nu(p)=\mathbf{F}$ otherwise, and let ν' be the unique extension of ν to all formulas. We state the following facts without proofs.

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[1]The construction is presented in details in [Si2].

Fact 1 (closure lemma). For any $\varphi_1,\varphi_2 \in \mathcal{F}$:

(i) Either $\varphi_1 \in \Gamma$ or $\sim\varphi_1 \in \Gamma$, but not both.
(ii) $\sim\sim\varphi_1 \in \Gamma \Rightarrow \varphi_1 \in \Gamma$
(iii) $\varphi_1 \& \varphi_2 \in \Gamma \Rightarrow \varphi_1 \in \Gamma$ **AND** $\varphi_2 \in \Gamma$
$\sim(\varphi_1 \& \varphi_2) \in \Gamma \Rightarrow \sim\varphi_1 \in \Gamma$ **OR** $\sim\varphi_2 \in \Gamma$
(iv) $\varphi_1 \vee \varphi_2 \in \Gamma \Rightarrow \varphi_1 \in \Gamma$ **OR** $\varphi_2 \in \Gamma$
$\sim(\varphi_1 \vee \varphi_2) \in \Gamma \Rightarrow \sim\varphi_1 \in \Gamma$ **AND** $\sim\varphi_2 \in \Gamma$
(v) $\varphi_1 \rightarrow \varphi_2 \in \Gamma \Rightarrow \sim\varphi_1 \in \Gamma$ **OR** $\varphi_2 \in \Gamma$
$\sim(\varphi_1 \rightarrow \varphi_2) \in \Gamma \Rightarrow \varphi_1 \in \Gamma$ **AND** $\sim\varphi_2 \in \Gamma$

Fact 2. For every $\varphi \in \Gamma$, $v'(\varphi)=\mathbf{T}$.

From these two facts Lemma 2 follows immediately.

Putting together Lemma 1 and 2, we get the Completeness Theorem for the classical Intercalation Calculus, and the consequences mentioned above (cf. [Si1]):

Theorem. The full intercalation tree for $\alpha;?G$ either contains an I-derivation for $\alpha;?G$ or a branch that determines a counterexample to the inference from α to G.

Corollary 1. The ND-calculus with just normal derivations is complete.

Corollary 2 (Normal Form Theorem). For any ND-derivation there is a normal ND-derivation with the same assumptions and conclusion.

It should be mentioned that this semantic argument for normalization in the Natural Deduction calculus is parallel to the argument used by Schütte [Sch] to prove cut-elimination for the Sequent Calculus. In [Sch] a completeness proof for the Sequent Calculus without cut rule is given, and this yields the cut-elimination theorem for the Sequent Calculus.

It is easy to see that the rules of the Intercalation Calculus are such that the ND-derivations extracted from I-derivations satisfy a stricter subformula property. Namely, in an I-derivation for $\alpha;?G$ every formula is either a *positive subformula* of an assumption, or a subformula of the conclusion, or (the negation of) a negative subformula of $\alpha,\sim G$, or $\perp$, where the concept of positive (negative) subformula is defined as follows:
1) φ is a positive subformula of φ;

2) if $\psi_1 \& \psi_2$ or $\psi_1 \vee \psi_2$ are positive (negative) subformulas of φ, so are ψ_1 and ψ_2;
3) if $\psi_1 \rightarrow \psi_2$ or $\sim\psi_1$ is a positive (negative) subformula of φ, then ψ_1 is a negative (positive) and ψ_2 is a positive (negative) subformula of φ.
This leads to a further restriction of the set $\mathfrak{F}$ in the rule $\perp_{\mathfrak{F}}$: we can take $\mathfrak{F}$ to be the set of formulas φ such that $\sim\varphi$ is a positive subformula of an available assumption. This observation plays a significant role in further restricting the search space, and will be explored in chapter 3.

CHAPTER 2

§ 1. Natural Deduction and Kripke semantics for intuitionistic propositional logic.

The intuitionistic (propositional) ND-calculus is obtained from the classical one by replacing the ~E-rule with the weaker "ex falso quodlibet" rule. The Normalization Theorem and the Subformula Property holds also for the intuitionistic ND-calculus [Pr].

Here we want to review briefly the Kripke semantics for intuitionistic propositional logic.

<u>Definition</u>. A *Kripke model* (for intuitionistic propositional logic) is a triple $<W,R,\Sigma>$, where W is a non-empty set (whose elements are usually called *worlds*), R is a reflexive and transitive relation on W, and Σ is a function which takes elements of W to sets of propositional variables, such that, for any $u,v \in W$, $uRv \Rightarrow \Sigma(v) \supseteq \Sigma(u)$.

Given a Kripke model $<W,R,\Sigma>$, Σ can be canonically extended to a function Σ' which takes elements of W to sets of formulas and satisfying the following for any $u \in W$:
(i) $\varphi_1 \vee \varphi_2 \in \Sigma'(u) \Leftrightarrow \varphi_1 \in \Sigma'(u)$ or $\varphi_2 \in \Sigma'(u)$;
(ii) $\varphi_1 \& \varphi_2 \in \Sigma'(u) \Leftrightarrow \varphi_1 \in \Sigma'(u)$ and $\varphi_2 \in \Sigma'(u)$;
(iii) $\varphi_1 \rightarrow \varphi_2 \in \Sigma'(u) \Leftrightarrow$ for all $v \in W$ such that uRv, $(\varphi_1 \in \Sigma'(v) \Rightarrow \varphi_2 \in \Sigma'(v))$;
(iv) $\bot \notin \Sigma'(u)$.
This is accomplished by defining $\varphi \in \Sigma'(u)$ for all $u \in W$ simultaneously by induction on the length of φ. The statement "$\varphi \in \Sigma'(u)$" is interpreted as "u verifies φ" or "φ is true in world u".

Kripke [Kr] proved for intuitionistic logic the completeness theorem with respect to this semantics. That is, either there is an intuitionistic ND-derivation of φ from α, or there exist a Kripke model $<W,R,\Sigma>$ and a world $u \in W$ such that $\psi \in \Sigma'(u)$ for all $\psi \in \alpha$, but $\varphi \notin \Sigma'(u)$.

§ 2. The Intercalation Calculus for intuitionistic propositional logic and its properties.

The intuitionistic Intercalation Calculus is obtained from the classical one by replacing the $\bot_c$ rule with the following "ex falso quodlibet" rule:

$\bot_q$: $\alpha;\beta?\varphi \Rightarrow \alpha;\beta?\bot$

We consider $\sim$ now as a defined symbol in the Intercalation Calculus. Thus $\bot_i$ becomes a special case of $\uparrow\rightarrow$; moreover, $\bot_{\mathcal{F}}$ becomes superfluous, since it is a derived rule in the calculus obtained by dropping it (this will be a consequence of the Completeness Theorem, since in that proof $\bot_{\mathcal{F}}$ is never used).

The full intercalation tree is defined as in the classical case, with the obvious modifications (note that $\bot_q$ applies to each non-leaf). The evaluation of the nodes is defined just as in the classical case.

The rest of this chapter is devoted to the proofs of the fundamental properties of the intuitionistic Intercalation Calculus. As mentioned earlier, the proof of the first property is essentially identical with that in the classical case. It is, not surprisingly, the construction of a counterexample that requires novel considerations.

Theorem 1. Assume S is the intuitionistic full intercalation tree for $\sigma^*=\alpha^*;?G^*$. If σ^* evaluates as **T**, one can extract from S an intuitionistic ND-derivation of G^* from α^*.

Proof. It is clear from the definition of I-derivation that S contains an I-derivation for σ^*. In fact, since σ^* evaluates as **T**, at least one of its successors evaluates as T. One chooses the branch leading to such a node and then repeats the process, being careful to select just one branch in the case of a disjunctive branching and both branches in the case of a conjunctive branching, until leaves of S are reached. Obviously the result is an I-derivation for σ^*. The theorem follows immediately from the next Lemma, which asserts that to any I-derivation it is possible to associate an ND-derivation with the same assumptions and conclusion. QED

Main Lemma. To any intuitionistic I-derivation $\mathcal{I}$ for a node $\sigma=\alpha;\beta?G$, one can associate canonically an intuitionistic ND-derivation of G from $\alpha\beta$.

Proof. By induction on the height of $\mathcal{I}$.

$h(\mathcal{I})=1$: then $\mathcal{I}$ consists of just one node $\sigma=\alpha;\beta?G$ and $G\in\alpha\beta$. The associated ND-derivation is just the formula G.

$h(\mathcal{I})>1$: we distinguish cases according to the rule applied to $\alpha;\beta?G$ in $\mathcal{I}$, and assume that to any I-derivation $\mathcal{I}'$ of height less than $h(\mathcal{I})$ we have associated a ND-derivation with the appropriate assumptions and conclusion.

$\downarrow\&_i$: then there is a formula $\varphi_1\&\varphi_2 \in \alpha\beta$, and $\varphi_i \notin \alpha\beta$. The immediate subtree $\mathcal{T}'$ of $\mathcal{T}$ has root $\alpha;\beta,\varphi_i?G$. By induction hypothesis we have a ND-derivation D of G from $\alpha\beta,\varphi_i$. We express this fact with the following picture:

$$\begin{array}{c} (D) \\ | \\ | \\ G \end{array}$$

The ND-derivation associated with $\mathcal{T}$ is:

$$\begin{array}{c} \dfrac{\varphi_1\&\varphi_2}{\varphi_i}\ \&E) \\ (D) \\ | \\ | \\ G \end{array}$$

which means the derivation obtained from D by substituting each occurrence of φ_i as a non-discharged assumption with its immediate derivation from $\varphi_1\&\varphi_2$ via &E.

$\downarrow\vee$: then there is a formula $\varphi_1\vee\varphi_2 \in \alpha\beta$, $\varphi_1 \notin \alpha\beta$ and $\varphi_2 \notin \alpha\beta$. The immediate subtrees $\mathcal{T}'$ and $\mathcal{T}''$ of $\mathcal{T}$ have roots $\alpha,\varphi_1;\beta?G$ and $\alpha,\varphi_2;\beta?G$. By induction hypothesis we have ND-derivations D_1 and D_2 of G from $\alpha\beta,\varphi_1$ and $\alpha\beta,\varphi_2$ respectively:

$$\begin{array}{cc} (D_1) & (D_2) \\ | & | \\ | & | \\ G & G \end{array}$$

The ND-derivation associated with $\mathcal{T}$ is:

$$\dfrac{\varphi_1\vee\varphi_2 \qquad \begin{array}{c} [\varphi_1] \\ (D_1) \\ | \\ | \\ G \end{array} \qquad \begin{array}{c} [\varphi_2] \\ (D_2) \\ | \\ | \\ G \end{array}}{G}\ \vee E)$$

which means the derivation obtained by discharging in D_i (i=1,2) each occurrence of φ_i as a non-discharged assumption with the application of $\vee$E to $\varphi_1 \vee \varphi_2$.

$\downarrow\!\rightarrow$: then there is a formula $\varphi_1 \rightarrow \varphi_2 \in \alpha\beta$ and $\varphi_2 \notin \alpha\beta$. The immediate subtrees τ' and τ'' of τ have roots $\alpha;\beta,\varphi_2?G$ and $\alpha;\beta?\varphi_1$. By induction hypothesis we have ND-derivations D_1 and D_2 of G from $\alpha\beta,\varphi_2$ and of φ_1 from $\alpha\beta$:

$$\begin{array}{ccc} (D_1) & \quad & (D_2) \\ | & & | \\ | & & | \\ G & & \varphi_1 \end{array}$$

The ND-derivation associated with τ is:

$$\dfrac{\varphi_1 \rightarrow \varphi_2 \qquad \begin{array}{c} (D_2) \\ | \\ | \\ \varphi_1 \end{array}}{\begin{array}{c} \varphi_2 \\ (D_1) \\ | \\ | \\ G \end{array}} \rightarrow\!E)$$

which means the derivation obtained from D_1 by substituting each occurrence of φ_2 as a non-discharged assumption with its immediate derivation from $\varphi_1 \rightarrow \varphi_2$ and φ_1 (the latter is derived via D_2) via $\rightarrow$E.

$\uparrow\&$: then G has the shape $\varphi_1 \& \varphi_2$. The immediate subtrees τ' and τ'' of τ have roots $\alpha;\beta?\varphi_1$ and $\alpha;\beta?\varphi_2$. By induction hypothesis we have ND-derivations D_1 and D_2 of φ_1 and φ_2 respectively from $\alpha\beta$:

$$\begin{array}{ccc} (D_1) & \quad & (D_2) \\ | & & | \\ | & & | \\ \varphi_1 & & \varphi_2 \end{array}$$

The ND-derivation associated with τ is:

$$\dfrac{\begin{array}{c}(D_1)\\ \vdots\\ \varphi_1\end{array}\qquad\begin{array}{c}(D_2)\\ \vdots\\ \varphi_2\end{array}}{\varphi_1\&\varphi_2}\ \&I)$$

$\uparrow\vee$: then G has the shape $\varphi_1\vee\varphi_2$. The immediate subtrees τ' and τ'' of τ have roots $\alpha;\beta?\varphi_1$ and $\alpha;\beta?\varphi_2$. By induction hypothesis we have at least one ND-derivations D of φ_i from $\alpha\beta$ (i=1 or 2):

$$\begin{array}{c}(D)\\ \vdots\\ \varphi_i\end{array}$$

The ND-derivation associated with τ is:

$$\dfrac{\begin{array}{c}(D)\\ \vdots\\ \varphi_i\end{array}}{\varphi_1\vee\varphi_2}\ \vee I)$$

$\uparrow\rightarrow$: then G has the shape $\varphi_1\rightarrow\varphi_2$. The immediate subtree τ' of τ has root $\alpha,\varphi_1;\beta?\varphi_2$. By induction hypothesis we have a ND-derivation D of φ_2 from $\alpha\beta,\varphi_1$:

$$\begin{array}{c}(D)\\ \vdots\\ \varphi_2\end{array}$$

The ND-derivation associated with τ is:

$$\dfrac{\begin{array}{c}[\varphi_1]\\ (D)\\ \vdots\\ \varphi_2\end{array}}{\varphi_1\rightarrow\varphi_2}\ \rightarrow I)$$

which means the derivation obtained from D by discharging each occurrence of φ_1 as a non-discharged assumption with the application of $\rightarrow$I used to get $\varphi_1 \rightarrow \varphi_2$.

$\bot_q$: then the immediate subtree $\mathcal{T}'$ of $\mathcal{T}$ has root $\alpha;\beta?\bot$. By induction hypothesis we have a ND-derivations D of $\bot$ from $\alpha\beta$:

$$\begin{array}{c}(D)\\ |\\ |\\ \bot\end{array}$$

The ND-derivation associated with $\mathcal{T}$ is:

$$\dfrac{\begin{array}{c}(D)\\ |\\ |\\ \bot\end{array}}{G}\ \text{“ex falso quodlibet”}$$

Thus the Lemma is proved.

Now we approach the second result needed for proving the Completeness Theorem, namely our construction of a countermodel for a root node $\sigma^*=\alpha^*;?G^*$ which evaluates as **F**. For this construction novel considerations come in, as the selection of the canonical refutation branch in Sieg's proof for classical logic depends crucially on the classical rules for negation. Our construction parallels the construction of a countermodel in the completeness proof for Beth semantic tableaux [Be], as presented in [Fi2]. The argument proceeds in three steps. Let S be the full intercalation tree for σ^*. First, we construct inductively $\forall n \in \{0,1,\ldots,h(S)\}$ two sets S'_n, S^*_n of regular nodes of S of level n, proving simultaneously that $S'_n \supseteq S^*_n$ and all nodes in S'_n evaluate as **F**. Then we will set $S'=\bigcup_{1\leq n\leq h(S)} S'_n$, $S^*=\bigcup_{1\leq n\leq h(S)} S^*_n$. S^* is a set of particular nodes, that are roots of subtrees with some good closure properties, and will constitute the universe of a Kripke model. Finally, we will show that this Kripke model is a countermodel to $\alpha^*;?G^*$.

Stage 0. Let $S'_0=S^*_0=\{\sigma^*\}$.

Stage n+1: $(n+1\leq h(S))$. Assume S'_n and S^*_n have been defined, $S'_n \supseteq S^*_n$, and all nodes in S'_n evaluate as **F**. Assume $S'_n=\{\sigma_1,\ldots,\sigma_k\}$. For each

$i\in\{1,...,k\}$ define two sets S'_{n+1_i}, $S^*_{n+1_i}$ in the following way (note: the particular ordering of the cases is not important, except for cases 6 and 7; the reasons for which these cases must be the last ones will become clearer in the proof of claim 1; note that cases 6 and 7 are the only ones that determine a branching in S').

Case 1: $\uparrow\&$ applies to σ_i. Then σ_i has the form $\alpha;\beta?\varphi_1\&\varphi_2$ and above it there is a special node $(\uparrow\&,\varphi_1\&\varphi_2)$ from which a conjunctive branching leads to nodes $\alpha;\beta?\varphi_1$ and $\alpha;\beta?\varphi_2$ at least one of which evaluates as **F**. If $\alpha;\beta?\varphi_1$ evaluates as **F**, let $S'_{n+1_i}=\{\alpha;\beta?\varphi_1\}$ and $S^*_{n+1_i}=\emptyset$. Otherwise let $S'_{n+1_i}=\{\alpha;\beta?\varphi_2\}$ and $S^*_{n+1_i}=\emptyset$.

Case 2: the previous case does not apply, but $\downarrow\vee$ applies to σ_i. Then σ_i has the form $\alpha;\beta?G$, with at least a formula of the form $\varphi_1\vee\varphi_2$ in $\alpha\beta$, $\varphi_1\notin\alpha\beta$, $\varphi_2\notin\alpha\beta$. Pick the first such formula in the sequence. Above σ_i there is a special node $(\downarrow\vee,\varphi_1\vee\varphi_2)$ from which a conjunctive branching leads to nodes $\alpha,\varphi_1;\beta?G$ and $\alpha,\varphi_2;\beta?G$, at least one of which evaluates as F. If $\alpha,\varphi_1;\beta?G$ evaluates as **F**, let $S'_{n+1_i}=\{\alpha,\varphi_1;\beta?G\}$ and $S^*_{n+1_i}=\emptyset$. Otherwise let $S'_{n+1_i}=\{\alpha,\varphi_2;\beta?G\}$ and $S^*_{n+1_i}=\emptyset$.

Case 3: the previous cases do not apply, but $\downarrow\&_1$ applies to σ_i. Then σ_i has the form $\alpha;\beta?G$ with at least a formula of the form $\varphi_1\&\varphi_2$ in $\alpha;\beta$, $\varphi_1\notin\alpha\beta$. Pick the first such formula in the sequence. Above σ_i there is a branch leading, through a special node $(\downarrow\&_1,\varphi_1\&\varphi_2)$, to $\alpha;\beta,\varphi_1?G$, which evaluates as F. Let $S'_{n+1_i}=\{\alpha;\beta,\varphi_1?G\}$ and $S^*_{n+1_i}=\emptyset$.

Case 4: the previous cases do not apply, but $\downarrow\&_2$ applies to σ_i. Then the situation is exactly as in case 3 with $\alpha;\beta,\varphi_2?G$ in place of $\alpha;\beta,\varphi_1?G$. So let $S'_{n+1_i}=\{\alpha;\beta,\varphi_2?G\}$ and $S^*_{n+1_i}=\emptyset$.

Case 5: the previous cases do not apply, but $\downarrow\rightarrow$ applies to $\sigma_i=\alpha;\beta?G$ with a formula $\varphi_1\rightarrow\varphi_2\in\alpha\beta$, where $\varphi_2\notin\alpha\beta$, and it leads, through a special node $(\downarrow\rightarrow,\varphi_1\rightarrow\varphi_2)$ and a conjunctive branching, to nodes $\alpha;\beta,\varphi_2?G$ and $\alpha;\beta?\varphi_1$ such that $\alpha;\beta,\varphi_2?G$ evaluates as **F**. (Note that the other possibility will be treated in case 7a below.) Then pick the first such formula in the sequence $\alpha\beta$, let $S'_{n+1_i}=\{\alpha;\beta,\varphi_2?G\}$, and $S^*_{n+1_i}=\emptyset$.

Case 6: the previous cases do not apply, but $\uparrow\vee$ applies to σ_i. Then σ_i has the form $\alpha;\beta?\varphi_1\vee\varphi_2$, and above it there is a special node $(\uparrow\vee,\varphi_1\vee\varphi_2)$ from which a disjunctive branching leads to nodes $\alpha;\beta?\varphi_1$ and $\alpha;\beta?\varphi_2$, both of which evaluate as **F**. Let $S'_{n+1_i}=\{\alpha;\beta?\varphi_1, \alpha;\beta?\varphi_2\}$ and $S^*_{n+1_i}=\emptyset$.

Case 7: the previous cases do not apply to $\sigma_i=\alpha;\beta?G$. We consider two subcases.

a) Let $\varphi_1\rightarrow\psi_1,...,\varphi_p\rightarrow\psi_p$ be the list of all formulas of the form $\varphi_j\rightarrow\psi_j$ in $\alpha\beta$ (note that p may be 0, in which case the sets $\{1,..,p\}$ and the corresponding list are considered empty). For all $j\in\{1,..,p\}$, above σ_i there is a special node $(\downarrow\rightarrow,\varphi_j\rightarrow\psi_j)$, from which a conjunctive branching leads to nodes $\alpha;\beta,\psi_j?G$ and $\alpha;\beta?\varphi_j$, at least one of which evaluates as **F**. Indeed, $\alpha;\beta?\varphi_j$ evaluates as **F**, as $\alpha;\beta,\psi_j?G$ evaluates as **T**; otherwise case 5 would have applied. Let $X=\{\alpha;\beta?\varphi_1,..., \alpha;\beta?\varphi_p\}$.

b) If G has the form $\varphi\rightarrow\psi$, above σ_i there is a special node $(\uparrow\rightarrow,\varphi\rightarrow\psi)$ from which a branch leads to $\alpha,\varphi;\beta?\psi$, which evaluates as **F** . In this case, let $S^*_{n+1_i}=\{\alpha,\varphi;\beta?\psi\}$. Otherwise let $S^*_{n+1_i}=\emptyset$.

Finally, let $S'_{n+1_i}=S^*_{n+1_i}\cup X$.

Now, to complete the inductive step, define $S'_{n+1}=\bigcup_{1\le i\le k}S'_{n+1_i}$, $S^*_{n+1}=\bigcup_{1\le i\le k}S^*_{n+1_i}$.

Our second step is to define, for nodes $\sigma\in S^*$, certain subtrees $R(\sigma)$, and prove that these subtrees have good closure properties.

For each $\sigma\in S^*$, let $R(\sigma)=\{\tau\in S': \sigma\preccurlyeq\tau$, and for all $\sigma'\in S^*$ such that $\sigma\preccurlyeq\sigma'$, not $\sigma'\preccurlyeq\tau\}$. Note that this definition implies that if $\sigma,\sigma'\in S^*$ and $\sigma\neq\sigma'$, then $R(\sigma)$ and $R(\sigma')$ are disjoint.

Definition. If σ is a regular node $\alpha;\beta?G$, then $T(\sigma)$ is the set of formulas occurring in $\alpha\beta$ and $F(\sigma)$ is the set $\{G\}$. If Q is a set of regular nodes, then $T(Q)$ is the union of the sets $T(\sigma)$ for σ in Q, and $F(\sigma)$ is the union of the sets $F(\sigma)$ for σ in Q.

Remark 1. If σ,σ' are nodes in S and $\sigma\preccurlyeq\sigma'$, then $T(\sigma')\supseteq T(\sigma)$, since no rule takes away a formula on the left of the question mark.

Claim 1. If $\sigma,\sigma'\in S^*$ and $\sigma\preccurlyeq\sigma'$, then $T(R(\sigma'))\supseteq T(R(\sigma))$.

<u>Proof of claim 1</u>. It is enough to show that for any node $\alpha;\beta?G\in R(\sigma)$ and any formula $\varphi\in\alpha\beta$ we have $\varphi\in T(R(\sigma'))$. If $\alpha;\beta?G\preccurlyeq\sigma'$, this follows immediately from remark 1. If $\alpha;\beta?G$ and σ' are on different branches, then there must be a branching below these two nodes due to cases 6 or 7 of the construction of S'. Let $\sigma''=\alpha'';\beta''?G''$ be that branching point and assume $\varphi\notin\alpha''\beta''$.

The only cases through which φ can have been added to $\alpha''\beta''$ are cases 2-5 or 7b (note that 1, 6 and 7a do not change the sequence of formulas on the left side of the question mark). Thus, in the branch leading from σ'' to $\alpha;\beta?G$, the (regular) node immediately above σ''

has the form $\alpha'';\beta''?G''$. Now, if cases 2-5 would have been applied above $\alpha'';\beta''?G''$, then they could have been applied also to σ''. But this is impossible, since we know that case 6 or 7 has been applied to σ'', and the ordering we chose would have forced us to apply cases 2-5 *before* case 6 or 7. So case 7b must have been used to get φ, and a new node τ has been put in S* before the first occurrence of φ on the left side of the question mark. So $\alpha;\beta?G\notin R(\sigma)$, contradiction.
Therefore $\varphi\in\alpha''\beta''$, and the result follows immediately from remark 1. Thus claim 1 is proved.

The next claim is the intuitionistic analogue of the closure lemma stated for the classical case.

Claim 2 (closure lemma). For any $\sigma\in S^*$, the following holds:
a) $\varphi_1\&\varphi_2\in T(R(\sigma)) \Rightarrow \varphi_1\in T(R(\sigma))$ and $\varphi_2\in T(R(\sigma))$
b) $\varphi_1\vee\varphi_2\in F(R(\sigma)) \Rightarrow \varphi_1\in F(R(\sigma))$ and $\varphi_2\in F(R(\sigma))$
c) $\varphi_1\vee\varphi_2\in T(R(\sigma)) \Rightarrow \varphi_1\in T(R(\sigma))$ or $\varphi_2\in T(R(\sigma))$
d) $\varphi_1\&\varphi_2\in F(R(\sigma)) \Rightarrow \varphi_1\in F(R(\sigma))$ or $\varphi_2\in F(R(\sigma))$
e) $\varphi_1\rightarrow\varphi_2\in T(R(\sigma)) \Rightarrow \varphi_1\in F(R(\sigma))$ or $\varphi_2\in T(R(\sigma))$
f) $\varphi_1\rightarrow\varphi_2\in F(R(\sigma)) \Rightarrow \exists\sigma'\in S^*$ such that $\sigma\preccurlyeq\sigma'$, $\varphi_1\in T(R(\sigma'))$ and $\varphi_2\in F(R(\sigma'))$

<u>Proof of claim 2</u>. For a)-e), the key element in the proof is that negations and implications on the left of the question mark, all conjunctions, and all disjunctions are always dealt with before a new node is put in S*, or at the same time but on a different branch. Consider for example a): if there is a node $\alpha;\beta?G\in R(\sigma)$ with $\varphi_1\&\varphi_2\in\alpha\beta$, then this formula is dealt with in case 3 and 4 before any node could be added to S* in case 7b, hence $\varphi_1\in T(R(\sigma))$ and $\varphi_2\in T(R(\sigma))$. Similarly, b) follows from case 6, c) from case 2, d) from case 1, e) from cases 5 and 7a. For f), if there is a node $\alpha;\beta?\varphi_1\rightarrow\varphi_2\in R(\sigma)$, then in case 7b a new node $\sigma'\in S^*$ is defined, such that $\sigma\preccurlyeq\sigma'$, $\varphi_1\in T(R(\sigma'))$ and $\varphi_2\in F(R(\sigma'))$.
Thus claim 2 is proved.

Finally, our third step is to define a Kripke model $K=\langle W,\leq,\Sigma\rangle$ with the required properties. Let $W=S^*$, and $\leq$ be the partial ordering relation $\preccurlyeq$ of the tree restricted to W. For any $\sigma\in W$, let $\Sigma(\sigma)=\{p$: p is a propositional variable and there exists a node $\alpha;\beta?G\in R(\sigma)$ such that $p\in\alpha\beta\}$. That K is a Kripke model follows immediately from claim 1.

Claim 3. Let Σ' be the canonical extension of Σ. For any $\sigma\in W$:
1) $\varphi\in T(R(\sigma)) \Rightarrow \varphi\in\Sigma'(\sigma)$
2) $\varphi\in F(R(\sigma)) \Rightarrow \varphi\notin\Sigma'(\sigma)$

Proof of claim 3. The proof is by induction on the complexity of φ.

Case 0 : φ is atomic.

Consider first the case $\varphi \neq \bot$. Then 1) follows directly from the definition of Σ. For 2), suppose there is a node $\tau = \alpha;\beta?\varphi \in R(\sigma)$ and $\varphi \in \Sigma'(\sigma)$. Then by definition of Σ, Σ' there is a node $\tau' = \alpha';\beta'?G \in R(\sigma)$ such that $\varphi \in \alpha'\beta'$. If $\tau' \preccurlyeq \tau$, then by remark 1 $\varphi \in \alpha\beta$. But this means that τ evaluates as **T**, contrary to the fact that all nodes in S' evaluate as **F**. If $\tau \preccurlyeq \tau'$, it cannot be that $G = \varphi$ since otherwise τ' would evaluate as **T**. This means that the formula on the right side of the question mark has been modified in the construction, and since φ is atomic this may have happened only through case 7a with an application of $\downarrow\rightarrow$ where the rightmost branch was chosen. Let $\tau'' = \alpha'';\beta''?\varphi$ be the node to which $\downarrow\rightarrow$ has applied. Since $\downarrow\rightarrow$ does not change the sequence on the left side of the question mark, we can reason as in the proof of claim 1 and conclude $\varphi \in \alpha''\beta''$, which implies that τ'' evaluates as **T**, contradiction. If τ and τ' are on different branches, then the branching may have happened only through case 6 or 7a. Let $\tau'' = \alpha'';\beta''?G''$ be the branching point. Since $\uparrow\vee$ and $\downarrow\rightarrow$ do not change the sequence on the left side of the question mark, we can reason as in the proof of claim 1 and conclude that $\varphi \in \alpha''\beta''$. But this implies that $\varphi \in \alpha\beta$, since $\tau'' \preccurlyeq \tau$. It follows that τ evaluates as **T**, contradiction.

Now consider the case $\varphi = \bot$. Then 2) follows trivially from the definition of Σ'. For 1), suppose there is a node $\tau = \alpha;\beta?\psi \in R(\sigma)$ such that $\bot \in \alpha\beta$. We may always apply $\bot_q$, leading to the node $\alpha;\beta?\bot$. This node clearly evaluates as **T**, and thus τ evaluates as **T**, contradicting the fact that all nodes of S' evaluate as **F**. So $\bot \notin T(R(\sigma))$, from which 1) follows trivially.

Case 1 : φ is $\varphi_1 \& \varphi_2$. Then, for 1), $\varphi_1 \& \varphi_2 \in T(R(\sigma)) \Rightarrow \varphi_1 \in T(R(\sigma))$ and $\varphi_2 \in T(R(\sigma))$ (by claim 2.a) $\Rightarrow \varphi_1 \in \Sigma'(\sigma)$ and $\varphi_2 \in \Sigma'(\sigma)$ (by induction hypothesis) $\Rightarrow \varphi_1 \& \varphi_2 \in \Sigma'(\sigma)$ (by definition of Kripke model). For 2), $\varphi_1 \& \varphi_2 \in F(R(\sigma)) \Rightarrow \varphi_1 \in F(R(\sigma))$ or $\varphi_2 \in F(R(\sigma))$ (by claim 2.d) $\Rightarrow \varphi_1 \notin \Sigma'(\sigma)$ or $\varphi_2 \notin \Sigma'(\sigma)$ (by induction hypothesis) $\Rightarrow \varphi_1 \& \varphi_2 \notin \Sigma'(\sigma)$ (by definition of Kripke model).

Case 2 : φ is $\varphi_1 \vee \varphi_2$. Then, for 1), $\varphi_1 \vee \varphi_2 \in T(R(\sigma)) \Rightarrow \varphi_1 \in T(R(\sigma))$ or $\varphi_2 \in T(R(\sigma))$ (by claim 2.c) $\Rightarrow \varphi_1 \in \Sigma'(\sigma)$ or $\varphi_2 \in \Sigma'(\sigma)$ (by induction hypothesis) $\Rightarrow \varphi_1 \vee \varphi_2 \in \Sigma'(\sigma)$ (by definition of Kripke model). For 2), $\varphi_1 \vee \varphi_2 \in F(R(\sigma)) \Rightarrow \varphi_1 \in F(R(\sigma))$ and $\varphi_2 \in F(R(\sigma))$ (by claim 2.b) $\Rightarrow$ $\varphi_1 \notin \Sigma'(\sigma)$ and $\varphi_2 \notin \Sigma'(\sigma)$ (by induction hypothesis) $\Rightarrow \varphi_1 \vee \varphi_2 \notin \Sigma'(\sigma)$ (by definition of Kripke model).

Case 3 : φ is $\varphi_1 \rightarrow \varphi_2$. Then, for 1), $\varphi_1 \rightarrow \varphi_2 \in T(R(\sigma)) \Rightarrow \forall \sigma' \in W\ [\sigma \leq \sigma' \Rightarrow$ $\varphi_1 \rightarrow \varphi_2 \in T(R(\sigma'))]$ (by claim 1) $\Rightarrow \forall \sigma' \in W\ [\sigma \leq \sigma' \Rightarrow \varphi_1 \in F(R(\sigma'))$ or $\varphi_2 \in T(R(\sigma'))]$ (by claim 2.e) $\Rightarrow \forall \sigma' \in W\ [\sigma \leq \sigma' \Rightarrow \varphi_1 \notin \Sigma'(\sigma')$ or $\varphi_2 \in \Sigma'(\sigma')]$ (by induction hypothesis) $\Rightarrow \varphi_1 \rightarrow \varphi_2 \in \Sigma'(\sigma)$ (by definition of Kripke model). For 2), $\varphi_1 \rightarrow \varphi_2 \in F(R(\sigma)) \Rightarrow \exists\sigma' \in W$ [$\sigma \leq \sigma'$ and $\varphi_1 \in T(R(\sigma'))$ and $\varphi_2 \in F(R(\sigma'))$] (by claim 2.h) $\Rightarrow \exists\sigma' \in W$ [$\sigma \leq \sigma'$ and $\varphi_1 \in \Sigma'(\sigma')$ and $\varphi_2 \notin \Sigma'(\sigma')$] (by induction hypothesis) $\Rightarrow \varphi_1 \rightarrow \varphi_2 \notin \Sigma'(\sigma)$ (by definition of Kripke model).

Thus the induction is complete, and so is the proof of claim 3.

At this point we can state the result about the existence of a countermodel.

Theorem 2. Suppose the full intercalation tree S for $\sigma^*=\alpha^*;?G^*$ evaluates as F. Then it is possible to define from S a countermodel for σ^*, that is, a Kripke model one of whose worlds verifies all the formulas in the sequence α^* and does not verify G^*.

Proof. Construct K as before. Apply claim 3 to the root node $\sigma^*=\alpha^*;?G^*$. It follows that, for each $\varphi \in \alpha^*$, $\varphi \in \Sigma'(\sigma^*)$, and $G \notin \Sigma'(\sigma^*)$. Therefore K is a countermodel for σ^*. QED.

Theorem 1 and 2 immediately yield the Completeness Theorem for the intuitionistic Intercalation Calculus:

Theorem. The full intuitionistic intercalation tree for $\alpha;?G$ either contains an I-derivation for $\alpha;?G$ or allows the definition of a counterexample to the intuitionistic inference from α to G.

Again we remark that ND-derivations extracted from I-derivations are normal and have the Subformula Property: as remarked for the classical case, this depends crucially on the fact that $\downarrow$-rules are only applied from above and $\uparrow$-rules only from below. Thus the Completeness Theorem yields the following corollaries, exactly as in the classical case:

Corollary 1. The intuitionistic ND-calculus with just normal derivations is complete.

Corollary 2 (Normal Form Theorem). For any ND-derivation there is a normal ND-derivation with the same assumptions and conclusion.

CHAPTER 3

§1. Heuristics for search in the classical case.

For a good, efficient implementation of an algorithm for proof search based on the Intercalation Calculus one has to try to reduce the search space further, and to decide in an intelligent way which of the various possibilities should be pursued first.
In the current implementation (for the classical case) various heuristics, based on logical considerations, are used to achieve these goals.

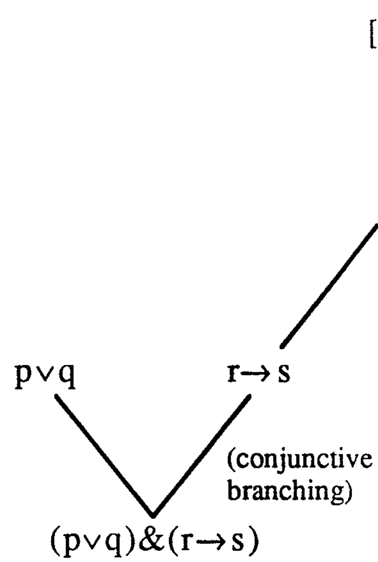
First of all, as we remarked at the end of Chapter 1, we may look for a proof in which only subformulas of the conclusion or positive subformulas of one of the assumptions (or negations of such formulas, or $\perp$) occur. Thus, in looking for a contradictory pair, we have to look only for formulas of the second kind.

Another important point is this: during the construction of the search tree, certain information can be stored in order to avoid answering the same question twice. That is, one keeps track of all the questions of the form $\alpha;\beta?G$ that have been answered negatively. Then, if on another branch a question $\alpha^*;\beta^*?G$ such that the set of formulas in $\alpha^*\beta^*$ is a subset of those in $\alpha\beta$ is met, one stops pursuing it, since it cannot be answered positively. In fact, though a failure in solving a question $\alpha;\beta?G$ is always caused by the fact that a repeated node has been met, and thus it may depend on nodes that are below $\alpha;\beta?G$ in the intercalation tree, the completeness proof for the Intercalation Calculus ensures us that if *all* strategies for $\alpha;\beta?G$ fail, then G is indeed not provable from $\alpha\beta$.
Furthermore, it is possible to keep track of all the questions of the form $\alpha;\beta?G$ that have been answered positively. Then, if on another branch a question $\alpha^*;\beta^*?G$ such that the set of formulas in $\alpha\beta$ is a subset of those in $\alpha^*\beta^*$ is met, one stops pursuing it, since a fortiori G is derivable from $\alpha^*\beta^*$: so one can just copy the derivation previously obtained without duplicating the steps leading to it.

There are essentially three different strategies to close the gap between assumptions and conclusion. One is trying to break the conclusion apart, in case it is a complex formula, making use of the $\uparrow$-rules, thus trying to get closer to the assumptions from below (inversion strategy). Another possibility is trying to extract the conclusion by breaking the assumptions apart using the $\downarrow$-rules, thus trying to get closer to the conclusion from above (extraction strategy): this strategy, however, may succeed only when the conclusion is a positive subformula of some assumption. The third possibility is using the indirect rules (indirect strategy).

Here is a very rough description of what the currently implemented algorithm does when it faces a question $\alpha;\beta?G$.

a) First of all, it looks for the formula G in the sequence $\alpha\beta$; if it is there, the question is answered positively. If it is not, it looks if G can be obtained from formulas in $\alpha\beta$ in just one step: if it can, then it applies the corresponding rule. Otherwise it forms strategies to obtain G, as follows.
b) It looks whether G is a positive subformula of some formula in $\alpha\beta$; if it is, it forms extraction strategies for G. In doing this, a certain number of open questions, that is other formulas that must be proved to obtain G, will be met (for example, if there is a formula $H\rightarrow G$ in $\alpha\beta$, H will be an open question).
c) Then it forms the inversion strategy for G. That is, it tries to break G into simpler formulas, and continues to break these formulas apart until either an atom, a negation or a disjunction is found, or one of the above cases a), b) applies. The figure below illustrates an example of the building of the inversion strategy: in this example, the goal is $(p\vee q)\&(r\rightarrow s)$.

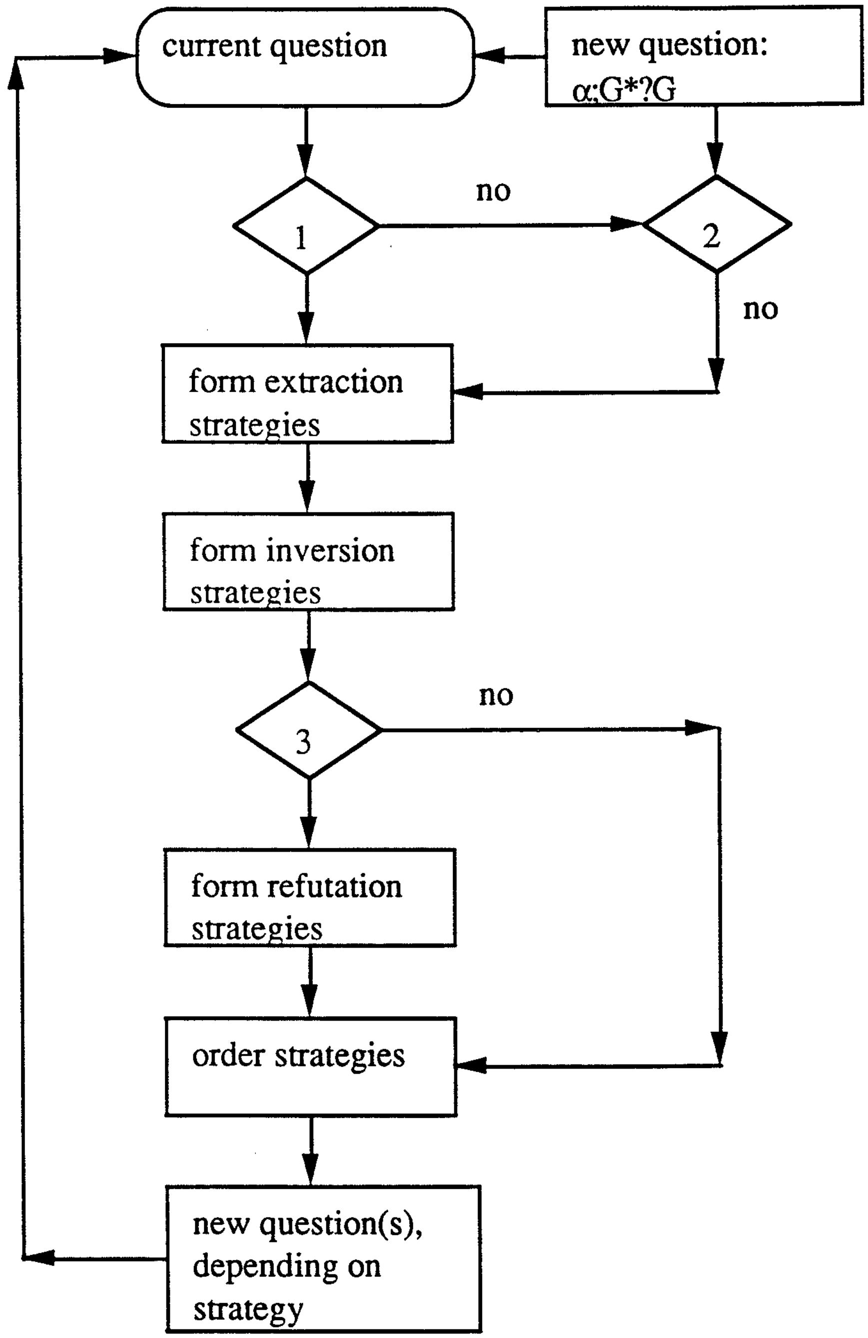


The leftmost branch stops when the disjunction $p\vee q$ is hit, the rightmost branch when the atom s is hit (with a new assumption r).
d) Then it ranks the possible extraction and inversion strategies, according to the context (that is, the shape of the conclusion and the assumptions, the number of open questions, how deep G is

embedded as a subformula in one of the assumptions, and so on), and it pursues the first strategy in this ranking. If the first strategy fails, the second is pursued and so on.
e) Finally, if everything else has failed, it forms the indirect strategies, using all possible contradictory pairs: then it ranks these strategies, too, and proceeds as before. If one of the indirect strategies eventually succeeds, it is always possible to check, whether the new introduced assumption was in fact used in the proof: if not, the refutation may be turned into a direct argument.

Certain heuristics provide exceptions to the algorithm described above. The most relevant is that, when for the first time on a branch the goal is an atom, a negation, or a disjunction, the indirect strategy is tried first, using the goal itself and its negation (or unnegated part) as the contradictory pair. This is motivated by the fact that in many common problems the indirect rules must indeed be used to prove an atom, negation, or disjunction: it will lead to non-optimal proofs in certain cases, but often it saves a lot of time, as for example in proving the law of the excluded middle $\varphi \vee \sim\varphi$. In fact (and this is also the reason to stop the building of the inversion strategy when a disjunction is met), while a conjunction can be proved from some assumptions *if and only if* both conjuncts can be proved from those assumptions, and an implication can be proved from some assumptions *if and only if* the consequent is provable from those assumptions plus the antecedent, it is not the case that to prove a disjunction from some assumptions one must necessarily prove one of the disjuncts from those assumptions.

Thus, in the currently implemented algorithm, the choice of the next question (when it has been determined that the branch with question α;?G has to be expanded) may be described with the flow diagram on next page (the diagram is taken from [SiSc]).



At 1 the algorithm determines whether it is in an indirect argument with respect to G. If not, at 2 it asks whether G is a negation, a disjunction or an atom. If it is, then it tries the indirect strategy (G* is ~G in the latter two cases, the unnegated part of G in the first case). At 3 it determines whether the set of extraction and inversion strategies is empty or not.

Remark. A good implementation of the ∨-elimination rule is not easy to achieve. Hence in the current implementation a different ∨-elimination rule is considered, namely the following:

$$\vee E')\quad \frac{\varphi\vee\psi \qquad \sim\varphi}{\psi} \qquad \frac{\varphi\vee\psi \qquad \sim\psi}{\varphi}$$

In the classical case the system resulting from this rule is equivalent to the one formulated before, but unfortunately in the intuitionistic case this rule turns out to be too weak.

§2. Possible heuristics for the intuitionistic case.

The treatment of the intuitionistic case is absolutely parallel to that for the classical case, but for proving atoms or disjunctions. Indeed, it is possible to form the extraction and inversion strategies just as in the classical case; of course, the indirect strategy is not available, and it shall be replaced by something else. The changes to be made to the algorithm concern essentially $\perp$ and $\vee$.

The only rule for $\perp$ we have now is the ex falso quodlibet. This rule has nothing to do with the shape of the goal (provided it is not $\perp$) and is needed to make sure that we can prove anything from an inconsistent set of assumptions. For these reasons it would probably be better to use the ex falso quodlibet only as a last resort, that is if all other strategies have failed. This could be the case, for example, if the goal G is an atom which is not a positive subformula of one of the assumptions: in fact then we cannot use the extraction nor the inversion strategy, and thus our only hope to get G is finding an inconsistency in the assumptions.

The issue about disjunction is more complex. First of all, the remark at the end of the last paragraph must be taken into account: in the intuitionistic case, we must implement the "traditional" ∨-elimination rule. The question now becomes, how should we use it ?

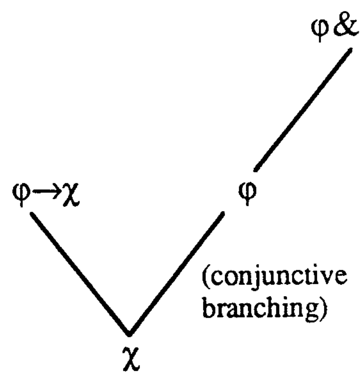
One thing to observe is that, though considerations similar to those made for the classical case apply, that is, for conjunctions and implications the inversion strategy shall be pursued as far as possible, in the intuitionistic case this is also true for disjunctions if all the assumptions are *Harrop formulas*.
For defining the concept of Harrop formula we need first to define inductively the concept of *strictly positive part* of a formula (cf. [Tr]):

1) φ is a strictly positive part of φ;
2) if $\psi_1\&\psi_2$ or $\psi_1\vee\psi_2$ are strictly positive parts of φ, so are ψ_1 and ψ_2;
3) if $\psi_1\to\psi_2$ is a strictly positive part of φ, then so is ψ_2.
A Harrop formula is a formula which does not contain a disjunction as a strictly positive part.
The reason why the inversion strategy shall be pursued as far as possible also in the case of a disjunction, if all the assumptions are Harrop formulas, is a theorem of Harrop [Ha], which states that, if φ is a Harrop formula, then a formula $\varphi\to(\psi_1\vee\psi_2)$ is provable in intuitionistic logic if and only if either $\varphi\to\psi_1$ or $\varphi\to\psi_2$ is provable in intuitionistic logic.

If at least one of the assumptions is not a Harrop formula, then there are other strategies to consider in the case of a disjunction: here, of course, the indirect strategy is not available, but it is possible to try the $\vee$-elimination rule. In this case, using it as first strategy makes sense because of considerations similar to those made for the indirect strategy in the classical case: many common problems actually have to use of the $\vee$-elimination rule (a typical example is the commutative law for $\vee$, that is $\varphi\vee\psi;?\psi\vee\varphi$), so this may be preferred to the $\vee$-introduction rule. In this case, too, it is always possible to check, whether the new assumptions introduced by $\vee$E have in fact been used in the proof, and if not, eliminate the redundant application of $\vee$E.
The problem with this is that, while it seems not to cost too much time to try the indirect strategy in the classical case (one just has to examine a few contradictory pairs), the amount of computation here could become difficult to deal with, especially if we have several disjunctions. Therefore it may be advisable to try the $\vee$-elimination strategy first only in certain special cases and with care.
The situation is further complicated by the fact that, in certain cases, trying the $\vee$-elimination strategy first would be better even if the goal is a conjunction or an implication, as for example if one wants to prove $(\varphi\&\psi)\vee(\varphi\&\psi);?\varphi\&\psi$.

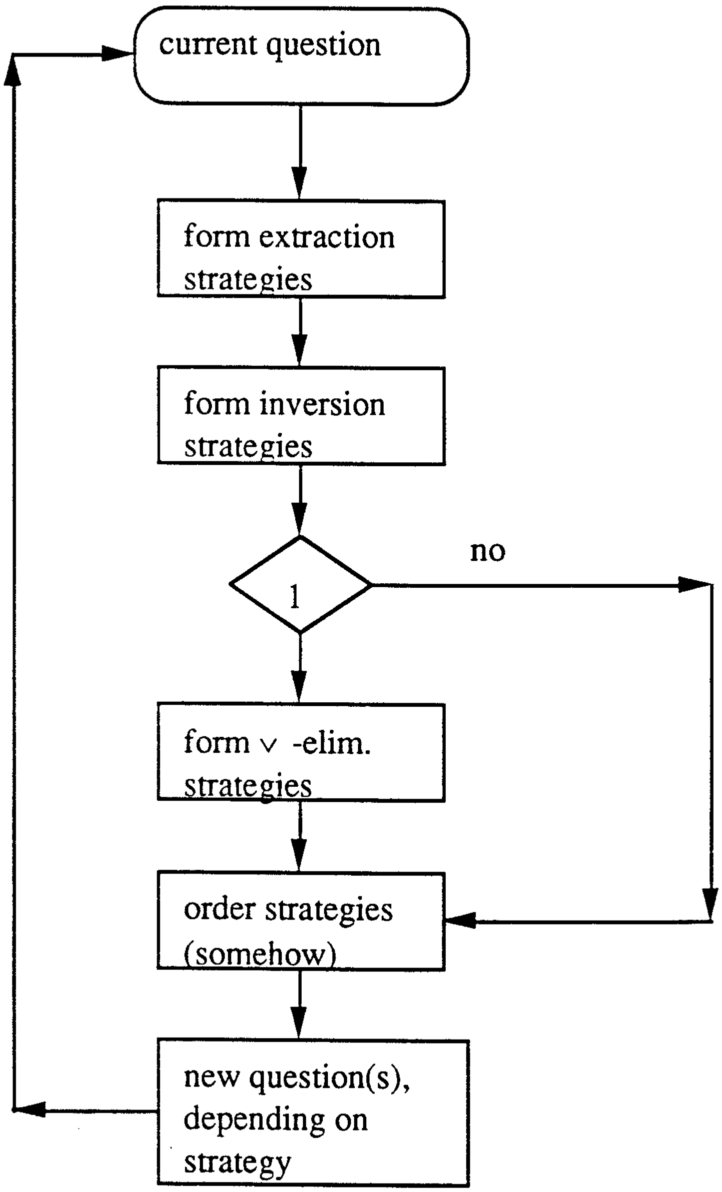
Thus, in order to decide whether to use $\vee$-elimination first or not, one should try to look for some connections between the goal and the disjunction in the assumptions (that is, look if one of the two is a positive subformula of the other, if they have propositional variables or even a disjunct in common, and so on). The problem is that $\vee$E may be necessary also in some cases in which the shape of the goal has really nothing to do with the shape of the disjunction in the assumptions, as for example in the question $\varphi\vee\psi,\varphi\to\chi,\psi\to\chi;?\chi$. So, if one has, for example, a question like $\varphi_1\vee\psi_1,\varphi_2\vee\psi_2,\varphi_1\to\chi,\psi_1\to\chi;?\chi$, one

should find a way to express in the algorithm the fact that the disjunction $\varphi_1 \vee \psi_1$ is helpful, while $\varphi_2 \vee \psi_2$ is not.
A step towards a solution may be to see, while forming extraction strategies for the goal, whether the open questions met have some relationships with a disjunction that occurs as a positive subformula of one of the assumptions. The ideal would be, of course, to find as an open question one of the disjuncts, since using $\vee$E this formula would become a new available assumption. Then one might take these considerations into account in ranking the strategies. Probably this would require a more sophisticated way of assigning scores to the strategies, since - as remarked before - taking a non-optimal direction might cause a big waste of time, apparently much worse than in the classical case. For example, consider the question $\varphi \& \psi, \varphi \rightarrow \chi, \psi \rightarrow \chi, \varphi \vee \theta, \psi \vee \theta; ?\chi$. Here the goal is a positive subformula of some of the assumptions. If one builds the extraction strategy, one meets the open question φ (or ψ), which is indeed a positive subformula of a disjunction in the assumptions. But trying to use $\vee$E would not be a good idea, because θ cannot be of any help in proving χ. It is clear that in such a case one should prefer the extraction strategy for φ (or ψ), using the assumption $\varphi \& \psi$. This would lead to a proof almost immediately, as the following figure shows:



The open question φ is obtained from $\varphi \& \psi$ by &E, and χ from $\varphi \rightarrow \chi$ and φ by $\rightarrow$E.

Here we try to describe the algorithm for the intuionistic case with a flow diagram, too; anyway, since the way of ordering strategies is still quite vague, such a diagram is probably less significant than that for the classical case:



At 1 the algorithm determines whether there are non-Harrop formulas among the assumptions.

§3. Complexity considerations.

In this last paragraph we want to consider the computational complexity of the set of classically and intuitionistically provable formulas. Statman [Sta] has shown that the set of intuitionistically provable formulas is PSPACE-complete, while the set of classically provable formulas is known to be in Co-NP. Thus it looks likely that the intuitionistic theorem predicate is by its nature more complex than that for classical logic: if not, in fact, Statman's result would yield NP=PSPACE.

The discussion in §2 of this chapter provides a heuristic explanation of this higher complexity, and shows where the difference between classical and intuitionistic proofs lie.
We have seen that conjunctions and implications are dealt with in the same way in both cases. Thus the differences appear when one has to prove a disjunction or an atom (remember that we considered negation as a special case of implication). In these cases, classical logic allows one to use the indirect strategy, which provides a new piece of information (the negation of the goal, which is taken as a new assumption) immediately available for trying to prove a contradiction. The syntactic form of the contradictory formulas one has to prove is always strongly related to that of the assumptions; in fact, as said before, the choice can be restricted to those contradictory pairs in which the negated formula is a positive subformula of one of the assumptions.
In intuitionistic logic this strategy is not available, and must be replaced by the use of the $\vee$-elimination rule (when there is a non-Harrop formula among the assumptions). This rule implies building up two subproofs: the goal itself must be proved separately from each of the two disjuncts, taken as new assumptions. These subproofs may still be very complex, since the syntactic form of the goal may still not be related (or, at least, not so strongly as in the classical case) to that of the assumptions.

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