"Church's Thesis", "Consistency", "Formalization", "Proof Theory": DICTIONARY ENTRIES

by

Wilfried Sieg

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Pittsburgh, Pennsylvania 15213-3890

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Wilfried Sieg
Department of Philosophy
Carnegie Mellon University
Pittsburgh, PA 15213

Church's Theorem: see Church's Thesis.

Church's Thesis: Church proposed at a meeting of the American Mathematical Society in April 1935, "that the notion of an effectively calculable function of positive integers should be identified with that of a recursive function ...". This proposal has been called Church's Thesis ever since Kleene used that name in his *Introduction to Metamathe-matics* (1952). The informal notion of an effectively calculable function (effective procedure, or algorithm) had been used in mathematics and logic, when indicating that a class of problems is solvable in a "mechanical fashion", by following fixed elementary rules. Underlying epistemological concerns came to the fore, when modern logic moved in the late 19-th century from axiomatic to formal presentations of theories. Hilbert suggested in 1904 to take such formally presented theories as objects of mathematical study, and metamathematics has been pursued vigorously and syste-matically since the Twenties, see FORMALIZATION, PROOF THEORY. In its pursuit concrete issues arose that required for their resolution a delimitation of the class of effective procedures. Hilbert's important "Entscheidungsproblem", the decision problem for predicate logic, was one such issue. It was solved negatively by Church and Turing -- relative to the precise notion of recursiveness, repectively machine-computabilty; the result was obtained independently by Church and Turing, but is usually called Church's Theorem. A second significant issue was the general formulation of the INCOMPLETENESS THEOREMS as applying to *all* formal theories (satisfying the usual representability and derivability conditions), not just to specific formal systems like that of *Principia Mathematica*.

According to Kleene, Church proposed in 1933 the identification of effective calculability with λ-definability. That proposal was not published at the time, but in 1934 Church mentioned it in conversation to Gödel who judged it to be "thoroughly unsatisfactory". In his Princeton Lectures of 1934 Gödel defined the concept of a recursive function, but he was not convinced that all effectively calculable functions would fall under it. The proof of the equivalence between λ-definability and recursiveness (by Church and Kleene) led to Church's first published formulation of the thesis as quoted above. The thesis was reiterated in Church's *An unsolvable problem of elementary number theory* (1936). Turing introduced in *On computable numbers, with an application to the Entscheidungsproblem* (1936) a notion of computability by machines and maintained that it captures effective calculability exactly; see TURING MACHINEs. Post's paper *Finite combinatory processes. Formulation 1* (1936) contains a model of computation that is strikingly similar to Turing's. However, Post did not provide any analysis; he suggested considering the identification of effective calculability with his concept as a working-hypothesis that should be verified by investigating ever wider formulations and reducing them to his basic formulation. (The classical papers of Gödel, Church, Turing, Post, and Kleene are all reprinted in *The Undecidable*, Davis (ed.), 1965.)

In his 1936-paper Church gave one central reason for the proposed identification, namely that other plausible explications of the informal notion lead to mathematical concepts weaker than or equivalent to recursiveness. Two paradigmatic explications, *calculability* of a function *via algorithms* or *in a logic*, were considered by Church. In either case, the *steps* taken in determining function values have to be effective; and if the effectiveness of steps is, as Church put it, *interpreted to mean* recursiveness, then the function is recursive. The fundamental, interpretative difficulty in Church's "step-by-step-argument" (was turned into one of the "recursiveness conditions" Hilbert and Bernays (1939) used in their characterization of functions that can be evaluated according to rules and) was bypassed by Turing. Analyzing human mechanical computations Turing was led to finiteness conditions that are motivated by the human computor's sensory limitations, but ultimately based on memory limitations. Then he showed that any function calculable by a computor satisfying these conditions is also computable by one of his machines. Both Church and Gödel found Turing's analysis convincing; indeed, Church wrote in a 1937-review of Turing's paper that Turing's notion makes "the identification with effectiveness in the ordinary (not explicitly defined) sense evident immediately".

This reflective work of partly philosophical and partly mathematical character provides one of the fundamental notions in mathematical logic. Indeed, its proper understanding is crucial for (judging) the philosophical significance of central metamathematical results - like Gödel's Incompleteness Theorems or Church's Theorem. The work is also crucial for computer science, artificial intelligence, and cognitive psychology as it provides also there a basic theoretical notion. For example, Church's Thesis is *the* cornerstone for Newell and Simon's delimitation of the class of physical symbol systems, i.e. universal machines with a particular architecture; see Newell's *Physical symbol systems* (1980). Newell views the delimitation "as the most fundamental contribution of artificial intelligence and computer science to the joint enterprise of cognitive science". In a turn that had been taken by Turing in *Intelligent Machinery* (1948) and *Computing Machinery and Intelligence* (1950), Newell points out the basic role physical symbol systems take on in the study of the human mind: "... the hypothesis is that humans are instances of physical symbol systems, and, by virtue of this, mind enters into the physical universe. ... this hypothesis sets the terms on which we search for a scientific theory of mind."

Consistency: Consistency is viewed in traditional, Aristotelian logic as a semantic notion: two or more statements are called consistent, if they are simultaneously true under some interpretation. Compare, for example, W.S. Jevons, *Elementary Lessons in Logic*, 1870. In modern logic there is a syntactic definition that fits also complex, e.g. mathematical, theories since Frege's *Begriffsschrift*, 1879. A set of statements is called consistent with respect to a certain logical calculus, if no formula (P & -P) is derivable from those statements by the rules of the calculus; i.e., the theory is free from contradictions. If these definitions are equivalent for a logic, we have a significant fact, as the equivalence amounts to the completeness of its system of rules. The first such COMPLETENESS THEOREM was obtained for sentential or propositional logic by Paul Bernays in 1918 (in his Habilitationsschrift that was partially published as *Axiomatische Untersuchung des Aussagen-Kalküls der "Principia Mathematica"*, 1926) and, independently, by Emil Post, *Introduction to a general theory of elementary propositions*, 1921; the completeness of predicate logic was proved by Gödel, *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, 1930. The crucial step in such proofs shows that syntactic consistency implies semantic consistency.

Cantor applied the notion of consistency to sets. In his well-known letter to Dedekind (1899) he distinguished between an inconsistent and a consistent multiplicity; the former is such "that the assumption that all of its elements 'are together' leads to a contradiction", wheras the elements of the latter "can be thought of without contradiction as 'being together'". Cantor had conveyed these distinctions and their motivation by letter to Hilbert in 1897; see, W. Purkert and H.J. Ilgauds, *Georg Cantor*, 1987. Hilbert pointed out explicitly in 1904 that Cantor had not given a rigorous criterion for distinguishing between consistent and inconsistent multiplicities. Already in his *Über den Zahlbegriff* (1899) Hilbert had suggested to remedy the situation by giving consistency proofs for suitable axiomatic systems; e.g., to give the proof of the "existence of the totality of real numbers or - in the terminology of G. Cantor - the proof of the fact that the system of real numbers is a consistent (complete) set" by establishing the consistency of an axiomatic characterization of the reals, in modern terminology: of the theory of complete, ordered fields. And he claimed, somewhat indeterminately, that this could be done "by a suitable modification of familiar methods."

Since 1904 Hilbert pursued a new way of giving consistency proofs. This novel way of proceeding, still aiming for the same goal, was to make use of the FORMALIZATION of the theory at hand. However, in the formulation of HILBERT's PROGRAM during the 1920's the point of consistency proofs was no longer to guarantee the existence of suitable sets, but rather to establish the instrumental usefulness of strong mathematical theories **T**, like axiomatic set theory, relative to finitist mathematics; cp. also PROOF THEORY. That focus rested on the observation that the statement formulating the syntactic consistency of **T** is equivalent to the reflection principle Pr(a,'s') -> s; here Pr is the finitist proof predicate for **T**, s a finitistically meaningful statement, and 's' its translation into the language of **T**. If one could establish finitistically the consistency of **T**, one could be sure - on finitist grounds - that **T** is a reliable instrument for the proof of finitist statements.

There are many examples of significant relative consistency proofs: (i) Non-Euclidean geometry relative to Euclidean, Euclidean geometry relative to analysis; (ii) set theory with the axiom of choice relative to set theory (without the axiom of choice), set theory with the negation of the axiom of choice relative to set theory; (iii) classical arithmetic relative to intuitionistic arithmetic, subsystems of classical analysis relative to intuitionistic theories of constructive ordinals. The mathematical significance of relative consistency proofs is often brought out by sharpening them to establish conservative extension results; the latter may then ensure, e.g., that the theories have the same class of provably total functions. The initial motivation for such arguments are, however, very frequently philosophical: one wants to guarantee the coherence of the original theory on an epistemologically distinguished basis.

Formalism: see proof theory, Hilbert's Program.

Formalization: Formalizations of theories must satisfy requirements that are sharper than those imposed on the structure of theories by the axiomatic-deductive method; that method can be traced back to Euclid's *Elements*. The crucial additional requirement is the regimentation of inferential steps in proofs: not only axioms have to be given in advance, but the rules representing argumentative steps have also to be taken from a predetermined list. To avoid a regress in the definition of proof and to achieve intersubjectivity on a minimal basis, the rules are to be "formal" or "mechanical" and must take into account only the form of statements. Thus, to exclude any ambiguity, a precise and effectively described language is needed to formalize particular theories. The general kind of requirements was clear to Aristotle and explicit in Leibniz; but it was only FREGE who presented in his *Begriffsschrift* in addition to an expressively rich language with relations and quantifiers an adequate logical calculus. Indeed, Frege's calculus, when restricted to the language of predicate logic, turned out to be semantically complete. Frege provided for the first time the means necessary to formalize mathematical proofs.

Frege pursued a clear philosophical aim, namely, to recognize the "epistemological nature" of theorems. In the introduction to his *Grundgesetze der Arithmetik*, 1893, Frege wrote: "By insisting that the chains of inference do not have any gaps we succeed in bringing to light every axiom, assumption, hypothesis or whatever else you want to call it on which a proof rests; in this way we obtain a basis for judging the epistemological nature of the theorem." - The Fregean frame was used in the later development of mathematical logic, in particular, in PROOF THEORY. Gödel established through his INCOMPLETENESS THEOREMS fundamental limits of formalizations of particular theories, like the system of *Principia Mathematica* or axiomatic set theories. The general notion of formal theory emerged from the subsequent investigations of Church and Turing clarifying the concept of "mechanical procedure" or "algorithm"; see CHURCH's THESIS. Only then was it possible to state and prove the Incompleteness Theorems for all formal theories satisfying certain very basic representability and derivability conditions. Gödel emphasized repeatedly that these results do not establish "any bounds for the powers of human reason, but rather for the potentialities of pure formalism in mathematics"; see, Gödel, *Collected Works*, Volume I, 1986, page 370.

Proof theory: Proof theory is a branch of mathematical logic founded by David Hilbert in the 1920's to pursue HILBERT's PROGRAM. The foundational problems underlying that program had been formulated already around the turn of the century, for example, in Hilbert's famous address to the International Congress of Mathematicians in Paris (1900). They were closely connected with investigations on the foundations of analysis carried out by Cantor and Dedekind; but they were also related to their conflict with Kronecker on the nature of mathematics and to the difficulties of a completely unrestricted notion of set or multiplicity. At that time, the central issue was for Hilbert the CONSISTENCY of sets in Cantor's sense. He suggested that the existence of consistent sets (multiplicities), e.g., that of real numbers, could be secured by proving the consistency of a suitable, characterizing axiomatic system; but there were only the vaguest indications on how to do that. In a radical departure from standard practice and his earlier hints Hilbert proposed four years later a novel way of attacking the consistency problem for theories in *Über die Grundlagen der Logik und der Arithmetik*, 1904. This approach would require, first, to give a strict FORMALIZATION of logic together with mathematics; then, one would have to consider the finite syntactic configurations constituting the joint formalism as mathematical objects and show - by mathematical arguments - that contradictory formulas cannot be derived.

Though Hilbert lectured on issues concerning the foundations of mathematics during the subsequent years, the technical development and philosophical clarification of (the aims of) proof theory began only around 1920. That involved, first of all, a detailed description of logical calculi and the careful development of parts of mathematics in suitable systems. A record of the former is found in Hilbert and Ackermann, *Grundzüge der theoretischen Logik*, 1928, of the latter in Supplement IV of Hilbert and Bernays, *Grundlagen der Mathematik II*, 1939. This presupposes the clear distinction between metamathematics and mathematics introduced by Hilbert. For the purposes of the consistency program metamathematics was now taken to be a very weak part of arithmetic, so-called finitist mathematics, believed to correspond to the part of mathematics that was accepted by constructivists like Kronecker and Brouwer. Additional metamathematical issues concerned the completeness and decidability of theories. The crucial technical tool for the pursuit of the consistency problem was Hilbert's ε-calculus.

The metamathematical problems attracted the collaboration of young and quite brilliant mathematicians (with philosophical interests); among them were Paul Bernays, Wilhelm Ackermann, Johan von Neumann, Jacques Herbrand, Gerhard Gentzen, and Kurt Schütte. The results obtained in the twenties were disappointing when measured against the hopes and ambitions: Ackermann, von Neumann, and Herbrand established essentially the consistency of arithmetic with a very restricted principle of induction. That limits of finitist considerations for consistency proofs had been reached became clear in 1931 through Gödel's INCOMPLETENESS THEOREMS. Also, special cases of the decision problem for predicate logic (Hilbert's "Entscheidungsproblem") had been solved; its general solvability was made rather implausible by some of Gödel's results in his 1931-paper. The actual proof of unsolvability had to wait until 1936 for a conceptual clarification of "mechanical procedure" or "algorithm"; that was achieved through the work of CHURCH and TURING.

The further development of proof theory is roughly characterized by two complementary tendencies: (i) the extension of the metamathematical frame relative to which "constructive" consistency proofs can be obtained, and (ii) the refined formalization of parts of mathematics in theories much weaker than set theory or even full second order arithmetic. The former tendency started with work of Gödel and Gentzen in 1933 establishing the consistency of full classical arithmetic relative to intuitionistic arithmetic; it led in the seventies and eighties to consistency proofs of strong subsystems of second order arithmetic relative to intuitionistic theories of constructive ordinals. The latter tendency reaches back to Weyl's book *Das Kontinuum* (1918) and culminated in the seventies by showing that the classical results of mathematical analysis can be formally obtained in conservative extensions of first order arithmetic. For the metamathematical work Gentzen's introduction of sequent calculi and the use of transfinite induction along constructive ordinals turned out be very important, as well as Gödel's primitive recursive functionals of finite type. The methods and results of proof theory are playing, not surprisingly, a significant role in computer science.

Work in proof theory has been motivated by issues in the foundations of mathematics with the explicit goal of achieving epistemological reductions of strong theories for mathematical practice (like set theory or second order arithmetic) to weak, philosophically distinguished theories (like primitive recursive arithmetic). As the formalization of mathematics in strong theories is crucial for the metamathematical approach, and as the programmatic goal can be seen as a way of circumventing the philosophical issues surrounding strong theories, e.g., the nature of infinite sets in the case of set theory, Hilbert's philosophical position is often equated with formalism - in the sense of Frege in his *Über die Grundlagen der Geometrie* (1903-1906) and also of Brouwer's inaugural address *Intuitionism and Formalism* (1912). Though such a view is not completely unsupported by some of

Hilbert's polemical remarks during the twenties, on balance, his philosophical views developed into a sophisticated INSTRUMENTALISM, if that label is taken in Ernest Nagel's judicious sense (*The Structure of Science*, 1961). Hilbert's is an instrumentalism emphasizing the contentual motivation of mathematical theories; that is clearly expressed in the first chapter of Hilbert and Bernays' *Grundlagen der Mathematik I* (1934). A sustained philosophical analysis of proof theoretic research in the context of broader issues in the philosophy of mathematics was provided by Bernays; his penetrating essays stretch over five decades and have been collected in *Abhandlungen zur Philosophie der Mathematik* (1976).