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On the decision-theoretical origins of minimal change and entrenchment

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Abstract. We present a decision-theoretically motivated notion of contraction which, we claim, encodes the principles of minimal change and entrenchment. Contraction is seen as an operation whose goal is to minimize loses of informational value. The operation is also compatible with the principle that in contracting A one should preserve the sentences better entrenched than A (when the belief set contains A). Even when the principle of minimal change and the latter motivation for entrenchment figure prominently among the basic intuitions in the works of, among others, Quine (Quine, 1973), Levi (Levi, 1980) (Levi, 1991), Harman (Harman, 1988), and Gardenfors (Gärdenfors, 1988), formal accounts of belief change (AGM, KM - see (Gärdenfors, 1988) and (Katsuno *et al.*, 1991)) have abandoned both principles (see (Rott, 2000)). We argue for the principles and we show how to construct a contraction operation which obeys both. An axiom system is proposed. We also prove that the decision-theoretic notion of contraction can be completely characterized in terms of the given axioms. Proving this type of completeness result is a well-known open problem in the field, whose solution requires employing both decision-theoretical techniques and logical methods recently used in belief change.

Keywords: Belief Revision, Contraction, Decision Theory, Entrenchment, Severe Withdrawal

1. Introduction

Students of belief change have recognized the need to give some sort of account of how to *contract* a belief state represented by a deductively closed theory or corpus K to a deductively closed subset K' when some specific sentence is to be removed. Contraction from K removing A may be achieved in many ways so that the inquirer is called upon to make a decision. There is a broad and often deceptive unanimity that the choice made should "Keep loss at a minimum" as the *Principle of Economy* as formulated by Rott and Pagnucco ((Rott and Pagnucco, 1999), 502) stipulates. The unanimity unravels when the issue of evaluating "loss" is addressed. Nonetheless, the principle of economy recommends optimizing by minimizing loss of something of value.

An alternative type of principle for recommending a contraction removing A urges retaining all and only those elements of K that are better *entrenched* than A. Such a *Principle of Entrenchment* is formulated as a satisficing principle rather than a maximizing (or minimizing) one. Instead of urging the choice of a "best" contraction, one may urge the choice of a contraction consisting of sentences in K that are "good enough" to be retained. Once more there has been controversy concerning how entrenchment is to be evaluated.[1]

To obtain a viable account of contraction based on both principles as authors like Gardenfors (Gärdenfors, 1988) sought to do, not only must the methods of evaluating loss and of assessing entrenchment be specified but they must be specified in a manner that assures that the maximizing Principle of Economy and the satisficing Principle of Entrenchment recommend the same contraction.

In this discussion, we shall focus on the Principle of Economy. The Principle of Entrenchment will receive brief attention towards the end of the paper.

The Principle of Economy is a schema. The loss to be minimized needs to be specified. The approach to contraction introduced in the classic paper by Alchourrón, Gardenfors and Makinson (AGM, (Alchourrón *et al.*, 1985) considered loss of information incurred in contraction. Potential state K_2 carries more information than K_1 if and only if the set of sentences in K_1 is a subset of the set of sentences in K_2 so that K_2 can be said to be logically stronger than K_1. Hence if two contractions K_1 and K_2 of K removing A are compared with respect to the loss of information incurred where K_2 is more informative than K_1, the loss incurred by shifting from K to K_2 is clearly less than that incurred by shifting from K to K_1.

When the Principle of Economy is construed as recommending the minimization of loss of information in this subset sense, it has become known as a *Principle of Conservatism* as in Harman, (Harman, 1988).

[1] Levi ((Levi, 1980), (Levi, 1991)) uses a notion of *degrees of incorrigibility* rather than degrees of entrenchment. According to this view, the recommended contraction of K removing A is the set of incorrigible sentences in K which in turn is the set of sentences in K better entrenched than A. Gardenfors (Gärdenfors, 1988) uses a different condition relating contraction with entrenchment; but he presupposes the controversial Recovery Postulate (Recovery stipulates that removing an item of information from K by contraction and then adding it back to the contracted belief state yields the initial point of view K.). In the presence of the Recovery Postulate, his approach corresponds to the account of contraction given in terms of incorregibility.

Conservatism played, without doubt, a motivating role in the early stages of research in the AGM tradition. It led to some precise formulations of contraction, like the so-called *maxichoice contraction* (Gärdenfors, (Gärdenfors, 1988)) and it paved the way towards the more sophisticated account of contraction defended in (Alchourrón et al., (Alchourrón *et al.*, 1985)): *partial meet contraction.*

Maxichoice contractions propose to achieve the contraction of a theory by a sentence A by selecting some maximal subset of K that does not imply A. This account is directly motivated by Conservatism, but it produces an unintuitive account of revision. In fact, if we denote the (maxichoice) contraction of K with $\neg A$, by $K/\neg A$, it seems reasonable to represent the revision of K with A as the logical closure of the set $\{K/\neg A \cup \{A\}\}$. But then all revisions of theories will be represented by maximal and consistent theories, an undesirable result.

AGM departed from the Principle of Conservatism by rejecting the recommendation of maxichoice contractions as mandatory in all cases. The central idea in (AGM, (Alchourrón *et al.*, 1985)) was to make a selection of the 'best' elements in the set of all maximal non-A-implying subsets of K; and then take the intersection of this selection. This is what is usually called a *partial-meet contraction.* Recommending partial meet contraction is generally paradigmatic of the AGM approach.

It is clear that partial-meet contractions do not follow Conservativism. As Rott and Pagnucco ((Rott and Pagnucco, 1999), 503) have recently observed: "The Principle of Economy has been severely compromised in the AGM framework." In a more recent article, Rott called the principles of Economy and Entrenchment "dogmas" "...*not* because almost all researchers kept to these principles (quite the opposite is true) but because so many authoritative voices *proclaimed* them to be the philosophical or methodological rationale for their theories (Rott, 2000).

As a matter of fact, the philosophical motivation for the AGM approach remains unclear. Attempts to clarify the main guiding principles allegedly used in formulating AGM contraction have ended up in many cases in the proposal of deviant notions ((Rott and Pagnucco, 1999) is a perfect example of this kind of attempt). Perhaps the influence of the AGM approach is due to the fact that the AGM trio were pioneers in providing exact axiomatic formulations of their proposals. And in formal epistemology, like in other fields, the use of the axiomatic method promotes progress by systematization of ideas. But the mere use of the axiomatic method does not guarantee conceptual clarity.

Usually the AGM approach has been criticized in a piecemeal fashion, by pointing out to the eventual lack of intuitiveness of some of the postulates it proposes. For example, the so-called axiom of *recovery* has been copiously criticized, in part by the members of AGM. But this process of criticism itself requires a certain initial level of conceptual clarity as to what contraction is supposed to achieve. The conceptual soundness of particular axioms can only be judged in terms of a set of basic principles regulating contractions. We want to propose in this article that the two principles formulated above are indeed these guiding principles. The principle of Economy is basically correct, except for the fact that the main idea in contraction is not to minimize losses of information, but to minimize *informational value* (Levi, 1980). As we shall see later, the troubles encountered by the principle of entrenchment are removed by this suggestion as well.

The principle of Cognitive Economy
Keep loss of informational value to a minimum in contraction.

We propose to use this instance of the principle of Economy together with the principle of Entrenchment as the main foundational guidance for articulating the notion of contraction. Some questions seem pertinent even at this early stage in the analysis. For example, what is the connection between information measured by set inclusion and informational value? Let V be a real value index on theories. Then we will appeal to the following principle:

Weak Monotony
If $X \subset Y$, then $V(X) \leq V(Y)$

In this discussion, we consider comparisons of theories with respect to informational value that constitute a weak ordering of the set of theories while satisfying Weak Monotony. According to this principle a theory Y can be a strict superset of another theory X, i.e. a theory Y can carry strictly more information than another theory X, but the informational value of the two theories can be the same. The extra information might not matter. Of course, one gets more specialized instances of the principle of Cognitive Economy by telling a more detailed story about informational value. Here we will propose a specific notion of informational value and tackle the problem of axiomatizing the resulting notion.

The idea of using informational value as the central notion in belief change is not new. Isaac Levi proposed it in his early writings (Levi, 1980) and he has refined it progressively in more recent writings (Levi, 1991), (Levi, 200+). The axiomatic account of AGM clearly does not fit this approach. Our task here is to go from first principles to axiomatization, by finding the axioms that *completely* characterize the contractions obeying appropriate instances of the principles of Economy and Entrenchment.

Of course these axioms will not coincide with axioms for AGM contraction or with other versions of contraction that have been proposed. We will focus on a salient notion of informational value, but our account can be extended parametrically to give foundations to a relatively large class of contraction and revision functions. We will therefore point occassionally below to other permissible notions of informational value and to their respective axiomatic encodings.

We agree with Rott (and Pagnucco) that many authors in the field have proclaimed allegiances to instances of the two principles while at the same time developing theories that do not obey those principles. The AGM tradition is only one instance of this mismatch between foundations and axiomatic proposals. We, nevertheless, do not agree that the principles should be abandoned as dogmas. We think that, when appropriately formulated, they are sound. The main task of this paper is to characterize these sound principles by a complete set of axioms.

We will proceed as follows. First we will define an operator $\div$ of informational value encoding decision theoretically the principle of Cognitive Economy. That is to say $K \div A$ is a theory removing A from K with minimum loss of informational value. Then we will propose a set of axioms characterizing what we call mild contractions (following the notation in (Levi, 200+) – the same notion is called 'severe withdrawal' in (Rott and Pagnucco, 1999)). We will prove both soundness and completeness for these postulates. I.e. we will show that $\div$ obeys the postulates of mild contractions; and we will show that any mild contraction operator can be represented as an operator of informational value. Once this is done we will show that mild contractions fit the principle of Entrenchment, while AGM contractions do not. We will conclude by discussing the status of some axioms of mild contractions and their role in our proposal.

2. Operators of informational value

We will assume a classical propositional language **L** as a representational tool. We assume that **L** contains the classical connectives. The underlying logic will be identified with its Tarskian consequence operator Cn: $2^{\mathbf{L}} \to 2^{\mathbf{L}}$ which is assumed to obey for all subsets X and Y of **L**:

(Inclusion) $X \subseteq Cn(X)$

(Monotony) If $X \subseteq Y$, then $Cn(X) \subseteq Cn(Y)$

(Iteration) $Cn(X) = Cn(Cn(X))$

(Superclassicality) If A can be derived from X by classical logic, then $A \in Cn(X)$

We also assume that Cn obeys the deduction theorem and that Cn is compact. A *theory* is any set K such that $K = Cn(K)$. Theories can be used advantageously in order to represent *commitments to full belief* of agents. So, our investigation here will focus on the logic of theory change.

We understand the problem of how to contract by removing A from a theory K to be a decision problem where one is called upon to choose a contraction removing A from K *from among all the contraction strategies removing A from K available in the context.*

Let K be a theory (representing the current commitments for full belief) and let LK be a *minimal theory* such that $LK \subseteq K$. The *basic partition* Π is a set of expansions of LK, not necessarily all of them and not necessarily all (or some of) the maximal and consistent ones. A necessary constraint on the admissibility of Π is that should be formed by expanding LK with sentences that are relevant answers to questions under investigation and that the expansions are restricted to expansions by adding to LK elements of a set of sentences such that LK entails that exactly one of them is true and each element of the set is consistent with LK.

The *ultimate partition* is the subset Π_K of partition cells of Π whose intersection is exactly K. In addition Π - Π_K is the *dual ultimate partition* Δ.

Call $\mathcal{M}$ the set of maximal and consistent theories definable in $\mathbf{L}$. For every $A \in \mathbf{L}$, $[A] = \{w \in \mathcal{M} : A \in w\}$. By the same token for every theory T definable in $\mathbf{L}$, $[T] = \{w \in \mathcal{M}\text{: } K \subseteq w\}$. When T is a theory obtained by intersecting a set of cells of Δ, we will use the notation $|T|$ to denote the set of partition cells (of the basic partition) whose intersection determines T. Also if the theory T in question is finitely axiomatizable via a sentence $A \in L$, $|A| = |T|$. Finally let $L \subseteq \mathbf{L} = \{A \in \mathbf{L}\text{: } |A| \neq \emptyset \text{ and } |\neg A| \neq \emptyset\}$.

Every *potential contraction removing $A \in L$ from K* is the intersection with K of a nonempty subset R of $\neg A$-entailing cells of Δ and a subset R^* of A-entailing cells of Δ that may or may not be empty. A *maxichoice contraction* of K relative to Δ is the intersection of K with a single element of Δ. A *maxichoice contraction of K removing $A \in L$*

relative to Δ is the intersection of K with a single element of Δ that entails $\neg A$. A *saturatable contraction* of K removing $A \in L$ relative to Δ is the intersection of a maxichoice contraction of K removing A relative to Δ with the intersection of a set of elements of Δ none of which entail $\neg A$.

DEFINITION 2.1. *Let $S(K, A)$ be the family of A-saturatable sets of K. I.e. if K is a theory, $X \in S(K, A)$ if and only if $X \subseteq K$, X is closed, and $Cn(X \cup \{\neg A\})$ is an element of the partition Δ.*

$\Phi = \{X : X = \cap Y$, with $Y \in 2^{\Delta} \cup [K]\}$. With these preliminary elements we can now introduce now a *measure of informational value* $V : \Phi \rightarrow [0,1]$. V is not just any value function. As the terminology indicates V is supposed to deliver a measure of the value of *information.* As such we assume that it inherits some basic properties of classical measures of information which are probability-based. A classical manner of utilizing probability in order to measure the content of information is to utilize the measure Cont(.) = 1 - Prob(.) - see for example (Levi, 1980) for an account of how this measure can be used in order to construct a decision-theoretically motivated theory of *expansion.*

There are two basic properties that probability-based measures of information satisfy. First they *respect entailment* in the following sense:

(Weak Monotony) For any two sets X, Y, that are elements of Φ, such that $X \subset Y$, $V(X) \leq V(Y)$.

The second important postulate is the following one:

(Extended Weak Monotonicity) Let $X, Y \subseteq \Phi$. If S is incompatible with both X and Y, and if $V(X) \leq V(Y)$, then $V(X \cap S) \leq V(Y \cap S)$.

Unfortunately one cannot preserve all the properties of Cont in characterizing a notion of information value useful in contraction. The trouble with Cont is that it cannot rationalize (in terms of optimality) moving to a position of suspense when there is a tie in optimality. In fact, the Cont-value of the intersection of two optimal saturatable contractions need not and, in general, will not carry maximum Cont-value. So we propose to preserve the first two postulates while adding a third that permits rationalizing suspense among optimal options as optimal. In order to present this third postulate we need an additional piece of notation. Any saturatable contraction S in $S(K, A)$ has the canonical form $K \cap T_A \cap m_{\neg A}$, where T_A is an intersection of A-cells of Δ and where $m_{\neg A}$ is a single $\neg A$-cell of Δ.

Then we can say that two saturatable contractions removing A from K are *A-equivalent* if and only if they are constituted as intersections of K with different $\neg A$-cells in Δ and the same subset T_A of the subset all of whose members entail A. A saturatable contraction S removing A is A-equivalent to an intersection of a set T of saturatable contractions removing A (including S) if S is constituted as the intersection of K, a set T_A of A-entailing cells and a $\neg A$-cell in Δ, and $[(\cap T) \cap A] = T_A$.

(Weak Intersection Equality) For every subset T of of potential contractions removing A from K each element of which is of equal informational value and such that all elements in T are A-equivalent to their intersection, for every $X \in T$, $V(\cap T) = V(X)$.

Given a set of optimal saturatable contractions removing A from K relative to Δ, the previous principle guarantees that its intersection is also an optimal saturatable contraction. Consider now the following important property entailed by these requirements.

(Weak Min) If a finite $T \subset S(K, A)$, $V(\cap T) = min(V(X) : X \in T)$.

OBSERVATION 2.1. *Weak monotony, extended weak monotony and weak intersection equality imply Weak Min.*

The three postulates that we just introduced are the *core postulates* of the notion of *damped* (Levi, 200+) informational value used in contraction (as opposed to the notion of undamped informational value characterized by the first two postulates - which is central in decision-theoretical characterizations of expansion). We will assume as well here the following stronger property.

(Strong Intersection Equality) For every subset T of Φ each element of which is of equal informational value and for every $X \in T$, $V(\cap T)$ $= V(X)$.

Strong intersection equality combined with weak positive monotonicity and extended weak positive monotonicity imply the following:

(Min) If X and Y are potential contractions from K in Φ, $V(X \cap Y)$ $= min(V(X), V(Y))$.

It is obvious that there are other forms of contraction parametrically obtainable by relaxing some of the principles that entail Min (in particular Strong Intersection Equality). Nevertheless, the form of contraction we are studying here is salient, we would like to argue, given its compatibility with the Principle of Entrenchment.

3. Shells of informational value

The assumption of the core postulates and the stronger Min condition allows us to construct the following notion of *rank*.

DEFINITION 3.1. *Let $I = range(V)$ be a set of indices. For $x \in I$ let R^x be the non-empty set X of partition cells in Δ such that for every $Y \subseteq X$, $V((\cap Y) \cap K) = x$.*

Intuitively R^x groups the partition cells of Δ such that the intersection of any subset of them with K has value x. By Min the intersection of any subset of them with K, has also value x. We can extend here the notion of rank, by adjudicating ranks to sets $P \subseteq 2^{\Delta}$.

$\rho^+(P) = \max(y\text{: } R^y \cap P \neq \emptyset)$

So, for P $\subseteq 2^{\Delta}$, such that there is $A \in L$, with $|A| = P$, we have that $\rho^+(|A|) = y$, where R^y is the set of partition cells of Δ of largest rank intersecting $|A|$. Of course, we have then that $\rho^+(\{w\}) = y$ when $w \in R^y$ and for every $Y \subseteq R^y$, $\rho^+(Y) = y$.

We can now introduce the notion of *m-shell of informational value.* The idea of a m-shell is to group together all the ranks R^x where x is greater or equal than the index m.

DEFINITION 3.2. *The* x-shell of informational value $S^x = \cup_{i \in I}^{i \geq x} R^i$. *The* system of shells of informational value *(SS) $\mathcal{S}$ is defined as: $\mathcal{S} = \{S^x : \cup S^x = \Delta\}$*

It should be obvious that shells of a shell system (SS) are nested. Notice in addition that for any cell $w \in \Delta$ we do not necessarily have $V(w) = \rho^+(w)$. For, by definition, $\rho^+(w) =$ V$(K \cap w)$. The only constraint imposed by WM in this case is that $\rho^+(w) \leq V(\{w\})$. So every cell in Δ has a *value-level* which might not coincide with its rank.

A SS for a value function V determines a grading on Δ. So, none of the maximals in Π_K appear in the SS. But of course there are some constraints relating the value of K and the value of the sets in the SS. One important constraint (given by WM) is that $V(K) \geq i$, where S^i

is the innermost shell of $\mathcal{S}$. Therefore $V(K)$ is greater than the value of any rank in $\mathcal{S}$.

With the help of the previous definitions we can now characterize our operator of informational value as an operation defined in systems of shells of informational value. We only need an additional definition. Let's consider $\mathbf{L} \subseteq L$ such that $\mathbf{L} = \{A \in L$: there are cells $C_1, ...C_1$ in Π such that $[\cap_{i=1,n} C_i] = [A]\}$. Of course, for every $A \in \mathbf{L}$ there are cells $C_1, ...C_1$ in Π such that $|A| = \{C_1, ...C_1\}$. Let a sentence A be *rejected in* K if and only if $\neg A \in K$. Notice that as long as a sentence $\neg A \notin \cup(LK)$ a sentence A rejected in K should also belong to $\mathbf{L}$, in such a way that $|A|$ is well defined for it.

DEFINITION 3.3. *Let $A \in \mathbf{L}$ be a sentence rejected in K. Then S_A is the union of $|K|$ with the set $X \in \mathcal{S}$ such that $X \cap |A| \neq \emptyset$ and for any other $Y \in \mathcal{S}$, such that $Y \cap |A| \neq \emptyset$, $X \subseteq Y$.*

S_A just picks the union of $|K|$ with the innermost shell in the SS $\mathcal{S}$ for V containing A-partition-cells of Δ. Now we can define some salient operators of informational value.

DEFINITION 3.4. *$\div$ is an operator of informational value for a closed set K if and only if there is a selection function γ such that for all A in $\mathbf{L}$: (i) if $A \in K$, then $K \div A = \cap\gamma(S(K, A))$, where $\gamma(S(K, A)) = \{X \in S(K, A)$: $V(Y) \leq V(X)$ for all $Y \in S(K, A)\}$ and (ii) $K \div A = Cn(K)$ otherwise.*

When the value function V is constrained by WM, the resulting operator is called a *basic* operator of informational value. When it obeys all core postulates the resulting operator is called a *core* operator of informational value. Finally when V is constrained by all cores postulates plus Min, the resulting operator is called the *standard operator of informational value.* From now on we will mainly work with standard operators of informational value and we will use the notation '$\div$' to refer to them. Specific references and clarifications will be made otherwise.

OBSERVATION 3.1. $|K \div \neg A| = S_A$

Given a value function V defined on Φ it is possible to define the following useful relation:

DEFINITION 3.5. $P \leq_V Q$ *if and only if* $V(P) < V(Q)$

In particular given a theory of reference K and a value function this relation orders all the potential contractions for the theory K. Moreover, it is immediate how to retrieve a relation $\leq_V$ from the system of shells for V and K. This can be done as follows:

OBSERVATION 3.2. *If P, Q are potential contractions of K then $P \leq_V Q$ if and only if there is S^x and S^y, such that $S^x \subseteq S^y$, R^x is the minimum rank intersecting $|P|$ and R^y is the minimum rank intersecting $|Q|$.*

This property flows from Min. Notice that if $P \leq_V Q$ this is so independently of the ranks of $|P|$ and $|Q|$ in the SS for V and K. Propositions in 2^{Δ} are ordered by $\leq_V$ in virtue of an index different than its rank. In fact, if $P \subseteq 2^{\Delta}$ we can define the following index of informational value ρ^-:

$$\rho^-(P) = \min(y\colon R^y \cap P \neq \emptyset)$$

Notice that for any $P \subseteq 2^{\Delta}$ we have that $V((\cap P) \cap K) = \rho^-(P)$. Nevertheless $V((\cap P)$ need not coincide with $\rho^-(P)$ – the theory $\cap P$ could have some value lower than $\rho^-(P)$. The index ρ^- has some obvious properties. For example: $\rho^-(P \cup Q) = \min(\rho^-(P), \rho^-(Q))$. And the index of informational value can be combined with ranks to give a simple definition of contraction. For any $A \in \mathbf{L}$ rejected in K:

COROLLARY 3.1. $|K \div \neg A| = \cup\{P \subseteq 2^{\Delta}\colon \rho^-(P) = \rho^+(|A|)\} \cup |K|$.

In words, in order to construct $|K \div \neg A|$ we take the union of $|K|$ with all the propositions in 2^{Δ} such that their index of informational value equals the 'upper' rank of $|A|$. It is quite obvious that S_A is one of these propositions. We can now go back to some additional properties of $\leq_V$:

OBSERVATION 3.3. *(d1) Either $[K \div A] \leq_V [K \div B]$, or $[K \div B] \leq_V [K \div A]$*

Which, in turn, means that we can easily establish a pretty strong property of informational value contractions, namely that: (d1) Either $[K \div A] \subseteq [K \div B]$, or $[K \div B] \subseteq [K \div A]$. This property will be used later on in the proof of our main result.

Shells of informational value are structures which, at first sight at least, might be easy to conflate with Spohn's *ranking systems* (Spohn, 1988), (Spohn, 2002). A ranking function κ is a function from $\mathcal{M}$ to the set of extended non-negative integers $\mathcal{N}^+ = \mathcal{N} \cup \{\infty\}$, such that

$\kappa(w) = 0$, for some $w \in \mathcal{M}$. For each proposition $P \subseteq \mathcal{M}$ the *rank* $\kappa(P)$ of P is defined by $\kappa(P) = \min \{\kappa(w): w \in P\}$ and $\kappa(\emptyset) = \{\infty\}$.

Spohn proposes to interpret ranks as *grades of disbelief.* $\kappa(P) = 0$ says that P is not disbelieved at all. It does not say that P is believed; this is rather expressed by $\kappa(P^c) > 0$, i.e., that non-P is disbelieved (to some degree). The set $C_\kappa = \{w: \kappa(w) = 0\}$ is called the *core* of κ and C_κ is the strongest proposition believed (to be true) in κ. So, if A^c is believed to be true in κ, one way of representing the contraction of A^c from C_κ is to take the union of C_κ with the set of least disbelieved A points, i.e. $\{w: \kappa(w) = \kappa(A)\}$. This is a simple way of defining an AGM contraction in this setting.

Ranking systems are different, both formally and conceptually from shell systems. Notice first that since we are working with finite partitions we can define shells also with range over $\mathcal{N}$, but in our case the domain is restricted to $\Phi = \{X : X = \cap Y, \text{ with } Y \in 2^\Delta\}$. Moreover in the case of rankings one proceeds by assigning first natural numbers to points (maximal and consistent theories in this case) and then ranks are assigned to propositions in an unproblematic manner. In our case an assignment of values to maximal and consistent theories does not fully determine the ranks of contractions for a theory of reference K even when the cells of the basic partition are constituted only by maximal and consistent theories in L . In fact, notice that in this limit case we can also define both $\kappa^-(P) = \min \{V(w): w \in P\}$ and $\kappa^+(P) = \max \{V(w): w \in P\}$. The second notion is not usually defined in Spohn's systems. But even if we were to use it notice that, given any proposition P in 2^Δ, nothing guarantees that $\kappa^+(P) = \rho^+(P)$ or that $\kappa^-(P) = \rho^-(P)$. As we explained before, the partition cells w in Δ receive a rank $\rho^-(\{w\}) = \rho^+(\{w\}) = x$, for R^x such that $w \in R^x$. But this rank need not coincide with w's value-level (measured by $\kappa^+(\{w\})$ or $\kappa^-(\{w\})$).[2]

In our framework the value-level of propositions is, of course, quite useful. It puts a constraint on permissible rankings ρ and it is crucial for determining iterated contractions (and revisions). But the value-level of partition cells does not fully determine ranks (ρ) of sets P in

[2] The 'upper' rank κ^+ and the 'lower' rank κ^- can be used in order to determine an epistemic ordering solely on the basis of point-value utility. The procedure, suggested by John Collins in (Collins, 2002), consists (roughly) in stipulating that proposition P is preferred to proposition Q if and only if the upper rank of P is greater than the upper rank of Q and the lower rank of P is no worse than the lower rank of Q. Or, alternatively that the lower rank of P is better than the lower rank of Q and the upper rank of P is no worse than the upper rank of Q. The procedure allows for incomparability of preference. This view of preference, nevertheless, does not satisfy postulates that are typical of probability-based notions of utility, like Weak Monotony, and therefore is quite different from the one presented here.

2^{Δ}, and the 'upper' and 'lower' point-ranks κ do not play a significant role in our proposal. Moreover even if we were to restrict our attention exclusively to ranks in our sense to the detriment of value levels we would need to use *both* the 'upper' and 'lower' ranks ρ^+ and ρ^-. So ranking systems and systems of shells are quite different. Both induce an indexed grading, but they induce gradings over different domains and the algorithm for assigned grades to propositions is different in each account. Spohn's ranking functions assign ranks to propositions identical to the degree of disbelief of its least disbelieved points, while in our account the rank of a proposition P (relative to K and Δ) is identical to the degree of informational value carried by the P-maxichoice contractions of K of maximal value. Notice that this notion of 'upper' rank has no operative counterpart in Spohn's system.[3]

All these formal differences flow from the central fact that the intended interpretation of grades in each account (Spohn's and ours) is fundamentally different. Spohn's account is a purely doxastic account where ranks can be (roughly) interpreted as the orders of magnitude of infinitesimal probabilities. As we explain above Spohn's main goal is to develop a non-probabilistic articulation of *degrees of disbelief.* In our account the grades are induced by a probability-based function measuring the *value* of information.

In spite of the aforementioned differences with ranking functions and Grove systems, there are also some formal connections between shell systems and both Grove and ranking systems that we will exploit below to present a representation result not tackled yet in the literature (including (Levi, 200+)). These connections make possible the use of well known techniques standardly used in the study of notions of contraction not motivated in decision-theoretical terms. The use of shell systems permits also the presentation of a considerably simpler soundness proof (in terms of the axioms presented in this coming section) than the one presented in (Hansson and Olsson, 1999) – which proceeds mostly syntactically, and only considers a subset of the axioms studied here.

4. Mild contractions

Here we will proceed axiomatically. The axioms used here are well known in the literature and their names are also more or less standard (see, for example, (Hansson, 1999)). A contraction operator relative to K and Π is a function $\div\colon K \times L \to \Phi$, obeying the following postulates:

[3] Such notion can, of course, be defined for Spohn's ranking functions as well. According to Spohn's official interpretation the 'upper' rank of a proposition would be determined by the degree of disbelief assigned to its most disbelieved points.

($\div$ 0) There are cells $C_1, ..., C_n$ and A-cells $c_1, ..., c_n$ in Δ such that $K \div A \cap \neg A = \cap_{1,n} C_i$ and $K \div A = \cap_{1,n} C_i \cap \cap_{1,n} c_i \cap K$.

($\div$ 1) $K \div A = Cn(K \div A)$ [closure]

($\div$ 2) $K \div A \subseteq K$ [inclusion]

($\div$ 3) If $A \notin K$ or $A \in Cn(LK)$, then $K \subseteq K \div A$ [vacuity]

($\div$ 4) If $A \notin Cn(LK)$, then $A \notin K \div A$ [success]

($\div$ 6) If $Cn(A) = Cn(B)$, then $K \div A = K \div B$ [extensionality]

($\div$ 7) If $A \notin Cn(LK)$, then $K \div A \subseteq K \div (A \wedge B)$ [antitony]

($\div$ 8) If $A \notin K \div (A \wedge B)$, then $K \div (A \wedge B) \subseteq K \div A$ [conjunctive inclusion]

All the conditions, except antitony and the first strictural condition, are AGM properties. On the other hand there is a notorious postulate, AGM's axiom of recovery, which is not in the previous list and that is not derivable from the list:

($\div$ 5) $K \subseteq Cn((K \div A) \cup \{A\})$ [recovery]

Antitony is perhaps the most controversial postulate from the list. For example Hansson reports in (Hansson, 1999) that antitony does not hold '[...] for any sensible notion of contraction'; while Rott and Pagnuco report in page 513 of (Rott and Pagnucco, 1999) that '[...] intuitively antitony makes quite a bit of sense'.

The axiomatic base given here is exactly the one proposed in (Rott and Pagnucco, 1999) to characterize *severe withdrawals*. Here we will show that this axiomatic base is indirectly supported by the intuitiveness of the postulates of Economy and Entrenchment. This gives, in turn, indirect support to Antitony.

5. A representation result for mild contractions

We will focus first on presenting some of the main lemmas that are needed in order to have a soundness result.

LEMMA 5.1. *Any standard operator of informational value satisfies all the postulates of mild contractions.*

(Hansson and Olsson, 1999) offers a proof that any *basic* operator of informational value obeys postulates ($\div$ 1)-($\div$ 4), ($\div$ 6) and ($\div$ 8) as long as the underlying partition is universal and constituted by maximal and consistent theories of L. The proof of ($\div$ 8) requires some work if only weak monotony is assumed (as a matter of fact this proof is one of the substantial arguments presented in (Hansson and Olsson, 1999)). Things are simpler when we have an operator of informational value which obeys the Weak Min. The use of shells of informational value permits a more direct proof in the general case (when partitions need not be either universal or opinionated) - we offer above a simpler proof of ($\div$ 8) to illustrate this.

LEMMA 5.2. *Any standard operator of informational value satisfies all the postulates of mild contractions.*

Aside from soundness we can also establish the following completeness result:

THEOREM 5.1. *If '$\div$' is a mild contraction function obeying the correspondent postulates, then '$\div$' can be represented as an operator of informational value.*

6. The Principle of Entrenchment and informational value

So far nothing has been said about the Principle of Entrenchment invoked in the introduction. Let's first introduce a relation of entrenchment formally. Let $\leq$ be an ordering of the sentences of L. $\leq$ is a relation of entrenchment for a theory K if and only if the following postulates are satisfied

(i) If $A \leq B$ and $B \leq C$, then $A \leq C$ (transitivity)

(ii) If $A \in Cn(LK)$, then $B \leq A$ (dominance)

(iii) $A \leq A \wedge B$ or $B \leq A \wedge B$ (conjunctiveness)

(iv) If $K \neq L$, then $A \leq B$ for every $B \in$ L if and only if $A \notin K$ (minimality)

(v) If $A \leq B$ for every $A \in$ L, then $B \in Cn(LK)$

Now, we remind the reader that our principle of entrenchment said that in giving up a sentence A from the current view one should preserve the sentences better entrenched than A. This translates formally into:

DEFINITION 6.1. *If $A \in K$ and $A \notin Cn(LK)$, then $K \div A = K \cap \{B : A < B\}$ and K otherwise.*

This simple and elegant manner of characterizing contraction in terms of entrenchment was first proposed by Rott in (Rott, 1991). AGM cannot be characterized in terms of entrenchment in this way. The non-equivalent bridge connecting entrenchment and AGM contractions stipulates that: $K \div A = K \cap \{B : A < A \vee B\}$

OBSERVATION 6.1. *(Rott and Pagnuco) If $\leq$ satisfies the postulates (i) to (v) then the function $\div$ obtained from $\leq$ by definition 5.1 is a mild contraction.*

Therefore, via the completeness result offered above, the informational value contractions can be retrieved by using the Principle of Entrenchment. Moreover, we can also appeal to a result recently proved by Rott and Pagnuco (and to our completeness result) in order to retrieve the relevant notion of entrenchment from an informational value contraction.

DEFINITION 6.2. *If $A \leq B$ if and only if $A \notin K \div B$, or $B \in Cn(LK)$.*

OBSERVATION 6.2. *(Rott and Pagnuco) If $\div$ is a mild contraction then the relation $\leq$ obtained via definition 5.2 satisfies the postulates (i) to (v) for the entrenchment relation.*

So, the Principle of Entrenchment and the Principle of Economy give exactly the same account of contraction. AGM contractions are also mirrored by a corresponding notion of entrenchment, but this notion does not obey the Principle of Entrenchment. One of the consequences of this divergence is the fact that AGM contractions satisfy the controversial principle of recovery, which is not satisfied by mild contractions (alias severe withdrawals according to Rott and Pagnuco's terminology).

7. Conclusions

The article focuses on determining the logical commitments entailed by our formulation of the principle of Economy (in terms of informational value) and the principle of Entrenchment. Together they give a holistic justification of the axioms of mild contractions. We pointed out above, nevertheless, that the principle of Economy is more accommodating than the principle of Entrenchment. In fact, minimizing loses of informational value under the constraints imposed by the three core postulates of (damped) informational value is coherent with endorsing our version of Economy. But this does not guarantee the satisfaction of:

($\div$ 7) If $A \notin Cn(\emptyset)$, then $K \div A \subseteq K \div (A \wedge B)$ [antitony]

In order to satisfy Antitony we need to assume in addition the full force of Min. One could justify Min (and therefore Antitony) by pointing out that this is exactly the requirement that is needed in order to have a theory of contraction where both Economy and Entrenchment are satisfied.

Apparent counterexamples against Antitony are easy to concoct, nevertheless, by considering scenarios where the sentences A and B (in the formulation of Antitony) express propositions that in some pre-systematic sense are irrelevant to each other. The serious study of these apparent counterexamples requires nevertheless a minimal articulation of the notion of relevance presupposed by the examples. Even when formalizing a notion of relevance is a complicated problem in itself, there are some proposals in the literature that articulate precisely the degree of relevance that two formulas might have to each other (Parikh, 1999). It should be pointed out here in passing, nevertheless, that an unmodified version of our model is not insensitive to the problem of relevance. The theory of contraction we are proposing has various contextual parameters that are useful for this problem. In particular the space of options $\mathcal{M}$ is determined by the adoption of a basic partition encoding a set of potential answers that are relevant to a problem. So our model, like some of the existing syntactic approaches to the problem of relevance focuses on certain relevant (syntactic) partitions of the set of all expansions of the basic theory LK. Unlike most of the existing theories of belief change we restrict the set of potential options and the set of potential contractions to a set relevant to the solution of a cognitive problem from scratch. So, counterexamples in terms of relevance are not crucially threatening to the theory presented here. In any case, it is important to realize that potential counterexamples against Antitony do not seem to threaten the core postulates (i.e. the notion of informational value) but the full force of Min.

There are some obvious extensions of our proposal and some ways of weakening it as well. An obvious extension is the consideration of the infinite case. In this case the density of the real interval $[0, 1]$ is important. Otherwise we can as well define a value function with range over the natural numbers. In addition, it would be interesting to study the shape of sequential change of view in this setting. It is also clear that more expressive languages (taking advantage of the numerical features of the semantics in terms of system of shells) can also be studied.

Concerning weakenings of the proposed theory, when Min fails there are a variety of possible contraction theories obeying the core postulates. Their parametric study can help to identify the constraints on value that completely characterize various *withdrawal operators*. Finally it would be interesting to study the more realistic case which permits value to be indeterminate (we conjecture that this is also a weakening of the theory that leads to the failure of Antitony).

We would like to close this section by pointing out that recent foundational work on the nature of contraction functions (Rott and Pagnucco, 1999) has converged in defending exactly the same axiom system that we presented here in section 4. Nevertheless, the foundational reasons offered by Rott and Pagnucco in (Rott and Pagnucco, 1999) in defense of the same syntactic principles are quite different form ours. Among other things they do not offer a decision-theoretic argument and therefore the technical aspects of their models differ substantially from ours. In spite of these differences it is indeed remarkable that the two theories can be characterized by the same axiom system – departing from the standard AGM characterization of contraction.

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Appendix

A. Proofs

OBSERVATION 2.1: Weak monotony, extended weak monotony and weak intersection equality imply Weak Min.

Proof.

Focus first on the set $S(K, A)$. List all the saturatable contractions in a finite family $T \subseteq S(K, A)$ with S_i and $1 \leq i \leq k$. Consider then:

(P) $V(\cap T) = min(V(S_i) : S_i \in T)$.

(P) holds trivially for $T_1 = S_1$ and we should show that if it holds for all non-empty subsets of $T_n = \{S_1, ..., S_n\}$ with $n < k$, then it holds for any non-empty subset of $T' = \{S_1, ..., S_n, S_{n+1}\}$. Then (P) holds for all non-empty subsets of $T_k = T \subseteq S(K, A)$.

So, assume (P) holds for all subsets of T_n. Consider then $\cap T_n \cap S_j$ with $S_j \notin T_n$. Let M be the set of partition cells of Δ entailing $\neg A$

used in order to construct staturatable contractions in T_n. We will consider the most general case where $S_j = K \cap M_A \cap X$, were M_A is an intersection of partition cells of Δ entailing A and X is a partition cell entailing $\neg A$ such that $X \notin M$. We will also assume that there is a non-empty subset of elements of M_A not used in order to construct saturatable contractions in T_n. Call this subset S_A. Notice that $S = K \cap S_A \cap X$ is a contraction in $S(K, A)$ which cannot be in T_n. Moreover $\cap T_n \cap S_j = \cap T_n \cap S$. Finally we can construct another potential contraction removing A which will be useful below: $Y' = \cap T_n \cap S_A$. Notice that X is now a theory incompatible with all saturatable contractions in T_n, a fact that will also be useful below. Weak Min holds for $\cap T_n$. Let Y be a member of T_n such that $V(\cap T_n) = V(Y)$. We need then to show that $V(\cap T_n \cap S_j) = min(S, Y)$.

Consider first the case $V(X) \leq V(\cap T_n) = V(Y)$, where Y is a member of T_n. Let Z be a partition cell entailing $\neg A$ that is distinct from X and where $V(Z) = V(X)$. Z should also be incompatible with the saturatable contractions in T_n. There need not be such a Z in $S(K, A)$. If that were the case we can always embed hypothetically Δ into Δ' containing a cell for Z and such that the original structure of values remains unaltered by the partition change. In order to do so consider the (logically finite) underlying language L and its expansion $L' = L \cup t$, where t is a fresh atom not occurring in L. For every theory S in L, where $V(S) = x$, and for every S' over L', such that $S' \cap L = S$, $V(S') = x$. So, we can construct an embedding partition Π'. For each original cell of Π which is the intersection of a set of maximal and consistent theories of L, consider now the theory determined by the t-counterpart of each one of these maximals. Now let Z be the $\neg t$-counterpart of X. One does so by adding to Π' a cell determined by intersecting the $\neg t$-counterparts of each maximal and consistent theory determining the cell containing X. It is obvious that Z exists and that $V(X) = V(Z)$.

If $T'_{L'}$ is a set of saturatable contractions expressible in L' over Π', it should be clear as well that if we manage to prove that $V(\cap T'_{L'}) = min(V(R) : R \in T'_{L'})$ then the result also shows that $V(\cap T') = min(V(S_i) : S_i \in T')$ – the reason being that $V(\cap T'_{L'} \cap L) = V(\cap T')$. It is important to realize for what follows that $Z_{L'}$ is incompatible both with $X_{L'}$ and with $Y_{L'}$. In order not to inflate terminology we will drop from now on the sub-index L'.

Notice first that weak intersection equality and weak monotony yield that $V(K \cap X) = V((K \cap X) \cap (K \cap Z))$ – it is clear that the two contractions in the RHS of the last equality are A-equivalent. By WM we have that $V((K \cap X) \cap (K \cap Z)) \leq V(Z \cap X)$. In addition WIE yields that $V(Z \cap X) \leq V(Y \cap Z) = V(Y \cap X)$. Therefore we have that

$V(K \cap X) \leq V(Y \cap K \cap X)$. Finally by one more application of WIE we have that $V(K \cap X \cap S_A) \leq V(Y \cap K \cap X \cap S_A)$. So, $V(S) \leq V(S \cap Y)$. The converse holds trivially by WM. This completes the proof of the first case.

For the second case consider $V(\cap T_n) = V(Y) < V(X)$. Replace Y in T_n with X. Since the result contains only n contractions, the min-rule applies. Moreover, since the result must carry informational value no less than the original, it carries informational value at least as great as Y. In effect, case 2 has been converted into the first case•

OBSERVATION 3.1: $|K \div \neg A| = S_A$

Proof.

Consider $w \in S_A \cap |A|$. Let $C(A) = S_A - (S_A \cap |A|)$. Notice that $M_w = \cap C(A) \cap w$ is a saturatable contraction removing $\neg A$ from K. M_w is one of the contractions of maximal value in $S(K, \neg A))$.

In general these contractions have the form $K \cap C_i \cap (\cap S)$ where $C_i \in S_A \cap |A|$ is a partition cell in Δ, and $S \subseteq C(A)$. In fact, it is easy to see that if either of these conditions fails the resulting saturatable contraction is not maximal. Say that we consider $(K \cap C_i \cap (\cap S)) = C$ where C_i is an A-partition cell in Δ, but $C_i \notin S_A \cap |A|$. Then the contraction can be represented as $(K \cap C_i) \cap (K \cap (\cap S))$ where $V(K \cap C_i) < V(K \cap (\cap S))$. The Min rule requires therefore that $V(C) < V(M_w)$. The reasoning is similar when $S \not\subseteq C(A)$ (in this case one has to focus on the intersection of S with the lowest rank intersecting S). It is then clear that the result of intersecting all the maximal contractions in $S(K, \neg A))$ (i.e. $K \div \neg A$) can be represented by just taking the intersection of all the saturatable contractions of the form M_w, with $w \in S_A \cap [A]$ – the value of this intersection is also maximal by WIE. But then it is obvious that $|K \div \neg A| = S_A$, as desired•

LEMMA 5.2: Any standard operator of informational value satisfies all the postulates of mild contractions.

Proof.

We will exhibit the proofs of two key postulates. All other proofs proceed in a similar manner.

Let's first focus on Conjunctive Inclusion. Assume that $A \notin K \div (A \wedge B)$. We need to show that $K \div (A \wedge B) \subseteq K \div A$. We have that $|K \div A| = S_{\neg A}$, i.e. we know that $|K \div A|$ is identical to the smallest m-shell of informational value intersecting $|\neg A|$. And by the same token we have that $|K \div (A \wedge B)| = S_{\neg(A \wedge B)}$. Given the assumption of the proof we have that $S_{(\neg A \vee \neg B)} \cap |\neg A| \neq \emptyset$. This is enough to guarantee that $S_{(\neg A)} \subseteq S_{(\neg A \vee \neg B)}$. Therefore $|K \div A| \subseteq |K \div (A \wedge B)|$, and $K \div (A \wedge B) \subseteq K \div A$, as desired.

Considering Antitony, we have to show that when $A \notin Cn(\emptyset)$, $K \div A \subseteq K \div (A \wedge B)$. So, assume that $A \notin Cn(LK)$. We need to show that $K \div A \subseteq K \div (A \wedge B)$. In other words, we need to show that $|K \div (A \wedge B)| \subseteq |K \div A|$. So, we need to show that $S_{\neg(A \wedge B)} \subseteq S_{\neg A}$, or, equivalently that $S_{(\neg A \vee \neg B)} \subseteq S_{\neg A}$. By the assumptions we know that $(A \wedge B) \notin Cn(LK)$. So, since both $\neg A$ and $\neg A \vee \neg B$ are in $\mathbf{L}$ and $|\neg A| \subseteq |\neg A \vee \neg B|$, we have that $S_{\neg(A \wedge B)} \subseteq S_{\neg A}$, as desired. Of course, as we showed in a previous observation, the appeal to the identity $|K \div \neg A| = S_A$ in the proof presupposes that our operator of informational value obeys not only all the core postulates but also Min, i.e. that it is a standard operator•

THEOREM 5.1: If '$\div$' is a mild contraction function obeying the correspondent postulates, then '$\div$' can be represented as an operator of informational value.

Proof.

We need to show that starting with an operator $\div$ obeying the postulates of mild contractions we can explicitly construct a system of shells of informational value. We have to show as well that the operator $\div'$ obtained from the defined system of shells of informational value by requiring $[K \div' A] = S_{\neg A}$, where $S_{\neg A}$ is the smallest m-shell of informational value intersecting $\neg A$, is identical to $\div$. The proof proceeds in three stages. First we follow what now is a standard procedure in order to construct a Grove system in terms of the operator '$\div$' (Grove, 1988). Then we show how to build a system of shells of informational value in terms of the constructed Grove system. In order to do so we show how to build a shell of informational value for the constructed Grove system. Finally we have to show that the operation $\div'$ obtained from the defined system of shells of informational value by requiring $[K \div' A] = S_{\neg A}$, is identical to $\div$.

Let's first focus on how to construct a Grove system in terms of the operation $\div$. As we explained above, the method for constructing a Grove system from a contraction operation is well-known. We will introduce some minor modifications in the standard proof, while trying to skip unnecessary details. The proof sketched here follows also a suitable modification of Grove's original proof as presented in (Rott and Pagnucco, 1999). The central idea is quite simple: a Grove system of spheres $\mathbf{S}$ centered on $|X|$, is determined by identifying a sphere in $\mathbf{S}$ with the collection $|X \div A|$ for some $A \in L$. More precisely:

(d) $X_A = |X \div A|$.

We can define now the system of spheres $\mathbf{S}$ as follows;

$\mathbf{S} = \{X_A \colon A \in L\} \cup \Pi$, when $K \neq L$, and $\mathbf{S} = \{X_A \colon A \in L\} \cup \Pi \cup \emptyset$ otherwise.

The gist of this first part of the proof consists on showing that the system $\mathbf{S}$ is indeed a Grove system of spheres centered on $|X|$. Since this is important for the rest of the result we are showing, we will remind the reader immediately of the definition of a Grove system of spheres centered on $|X|$.

Let $\mathbf{S}$ be a collection of subsets of Δ. $\mathbf{S}$ is a system of spheres, centered on $X = |K| \subseteq \Pi$ and satisfying:

(1) $\mathbf{S}$ is totally ordered by $\subseteq$.

(2) X is the $\subseteq$-minimum of $\mathbf{S}$.

(3) Π is the $\subseteq$-maximum of $\mathbf{S}$.

(4) If $A \in L$ and $\emptyset \neq [A] \in 2^{\Delta}$, then there is a smallest sphere S_A in $\mathbf{S}$ intersecting the set $|A|$.

Condition (1) is directly satisfied in virtue of (d) above and the fact that the following property can be deduced from the axioms of mild contractions:

(d1) Either $K \div A \subseteq K \div B$ or $K \div B \subseteq K \div A$.

Conditions ($\div$ 2) and ($\div$ 3) guarantee that $K \div \mathbf{true} = K$. This and (d) are enough to show that $X = |K|$ is a sphere. That this is the innermost sphere follows immediately from ($\div$ 2) and (d). This takes care of condition (2). Condition (3) is automatically satisfied by the given definition of $\mathbf{S}$.

Condition (4) is slightly harder. Let A be such that $\emptyset \neq [A] \in 2^{\Delta}$. Now we need to show that there is a sphere $U \in \mathbf{S}$, such that $U \cap |A| \neq \emptyset$ and for every other $V \in \mathbf{S}$, such that $V \cap |A| \neq \emptyset$, we have $U \subseteq V$. The basic idea of the proof, which we skip here, is to show that $|K \div \neg A|$ $= X_{\neg A}$ satisfies this constraint.

The proof of condition (4) establishes that (when A is not a tautology) $S_{\neg A} = |K \div A| = X_A$, where $S_{\neg A}$ denotes the smallest sphere intersecting $|\neg A|$ - we used basically the same notation $S_{.}$ for shells above, indicating the smallest m-shell intersecting $|.|$. This fact will be useful below.

Now we should focus on the second step of the proof, namely the construction of a system of shells of informational value for the constructed system of spheres. Here is the recipe in order to do so.

First construct a system of ranks out of the given system of spheres as follows: index spheres with natural numbers starting with 0 assigned to the innermost core in such a way that S_i denotes the set of partition cells in the sphere indexed by i. This is done via an indexing function mapping propositions to natural numbers. Then define a function δ from the range of the indexing function to propositions, such that $\delta(k)$ $= S_{k+1}$ - S_k.

As a second step assign an arbitrary uniform V-value x to the partition cells in the innermost sphere of $\mathbf{S}$, $|K|$ as long as x is greater than k, where k is the index of the outermost sphere S_k.

As a third step we need to give a value to each partition cell in Δ. In order to do so assign a uniform value $y > x$ to each partition cell in $\delta(0)$ and, in general, for every $\delta(i+1)$, for $i \geq 0$, assign a uniform value $z > z'$ to the maximals in $\delta(i+1)$, where z' is the uniform value of maximals in $\kappa(i)$.

As a fourth step we need to define ranks and m-shells of informational value. In order to do so we need to re-index the ranks we just defined from the Grove system. Assign to $\delta(0)$ a positive index $x' < x$ and, in general, for every $\delta(i+1)$, with $i \geq 0$, assign to $\delta(i+1)$ a positive index z, where $z < z'$ and z' is the value of $\delta(i)$. So, for an arbitrary $\delta(i)$, we have a positive number m assigned to it such that $m < x$. Create then the ranks of informational value R_i^m and the corresponding shells S_i^m as follows: $R_i^m = \delta(i)$ and $S_i^m = S_i$

The last definition allows us to complete the definition of the V-measure, by requiring that (i) for every $Y \subseteq R_i^x$, $V((\cap Y) \cap K) = x$; (ii) that for every $Y \subseteq S_i^x$, such that R_i^x is the outermost rank such that $R_i^x \cap Y \neq \emptyset$, $V((\cap Y) \cap K) = x$.

It is obvious that V, as defined, is a function from Φ to the natural numbers (it could be easily normalized to [0, 1]). In fact, for every member T of Φ there is, by construction, an outermost rank overlaping $|T|$. It is also clear that the function satisfies WM. In order to verify it we need to check that for any two contractions of K, X, Y, such that $X \subset Y$, then $V(X) \leq V(Y)$. Let $R^X = R_i^m$ be the outermost rank intersecting $|X|$. Then it is clear that the outermost rank intersecting $|Y|$, $R^Y = R_{i'}^{m'}$ is such that $i' \leq i$ and $m' \geq m$. Since $|Y|$ is a subset of $S_{i'}^{m'}$ Y receives value m', and since $|X|K$ is a subset of R_i^m X receives value m. And since $m' \geq m$, weak monotony is satisfied.

Consider now $Y = \cap X$, where X is a family of maxichoice contractions of K, and its associated set $|Y|$. Consider again the outermost rank intersecting $|Y|$, $R^Y = R_i^m$. Then according to the proposed explicit definition of V, $V(Y) = m$. Take now an arbitrary $Z \in X$ such that $|Z|$ does not intersect R^Y. Then, by construction, $V(Y) < V(Z)$. And if $|Z|$ intersects R^Y, $V(Z) = m$. So, clearly we do have that $V(Y) = min\{V(Z) : Z \in X\}$.

Finally we need to check that an operation of contraction $\div'$ defined from the explicitly constructed system of shells of informational value, coincides with the operator $\div$ characterized by the postulates of mild contractions. We will define:

$$|K \div' \neg A| = S_A$$

where S_A is the smallest m-shell of informational value intersecting $|A|$ union $|K|$. The non-trivial case to consider is when A is not in LK. We will work under this assumption. Assume first that $B \in K \div \neg A$, $B \in L$ This entails that $|K \div \neg A| \subseteq |B|$. The proof of condition (4) for Grove systems above tell us that $[K \div \neg A]$ is a sphere S_i, and since in our construction an m-shell is obtained from it by taking S_i^m; then we have that $|K \div \neg A|$ is also a m-shell of informational value. Moreover, the proof of (4) also tells us that the smallest sphere overlapping $[A]$ is identical to $[K \div \neg A]$. Therefore the smallest m-shell of informational value overlapping $|A|$ is identical with $|K \div \neg A|$. Therefore we have that $|K \div' \neg A| \subseteq |B|$, and $B \in K \div' \neg A$, as desired. Proving that $K \div' \neg A \subseteq K \div \neg A$ only requires reversing the strategy used for the RTL inclusion•

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