Algebraic Models of Intuitionistic Theories of Sets and Classes

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Abstract

This paper constructs models of intuitionistic set theory in suitable categories. First, a Basic Intuitionistic Set Theory (BIST) is stated, and the categorical semantics are given. Second, we give a notion of an *ideal* over a category, using which one can build a model of BIST in which a given topos occurs as the sets. And third, a sheaf model is given of a Basic Intuitionistic Class Theory conservatively extending BIST. The paper extends the results in [2] by introducing a new and perhaps more natural notion of ideal, and in the class theory of part three.

1 Introduction

We begin with a very brief and informal sketch (elaborated in section 2 below) of the leading ideas of algebraic set theory, as it was recently presented in [2], and first proposed in [7] (see also [3, 9, 8]). The basic tool of algebraic set theory is the notion of a *category with class structure*, which provides an axiomatic framework in which models of set theory are constructed. Such a category $\mathcal{C}$ is equipped with three interrelated structures: a subcategory $\mathcal{S} \hookrightarrow \mathcal{C}$ of *small maps*, a *powerclass* functor $\mathcal{P} : \mathcal{C} \longrightarrow \mathcal{C}$, and a *universal class* $\mathcal{U}$ in $\mathcal{C}$. The small maps determine which classes are sets, the powerclass $\mathcal{P}(C)$ is the class of all sub*sets* of a class C, and the universe $\mathcal{U}$ is a fixed point of $\mathcal{P}$, in the sense that $\mathcal{P}(\mathcal{U}) \cong \mathcal{U}$ (related conditions like $\mathcal{P}(\mathcal{U}) \subseteq \mathcal{U}$ are also considered).

The language of elementary set theory (first–order logic with a binary 'membership' relation ϵ) can be interpreted in any such universe $\mathcal{U}$, and the elementary theory of all such universes can be completely axiomatized by a very natural system of set theory, called Basic Intuitionistic Set Theory (BIST), first formulated in [2]. It is noteworthy for including the unrestricted axiom scheme of Replacement in the absence of the full axiom scheme of Separation (a combination that can not occur in classical logic, where Replacement implies Separation).

The objects of a category with class structure that have a small morphism into the terminal object are called *small* objects or *sets*. These are easily shown to be a topos. In [2] it is shown that any topos whatsoever occurs as the subcategory of small objects in some category with class structure. This is achieved by defining a notion of an *ideal on a topos*. The main part of this paper consists in a modification of this notion (elaborated in section 3). It is shown that a useful notion of ideal on a topos can be obtained by considering certain sheaves on the topos under the coherent (or finite epimorphic families) covering. Namely, these are those sheaves that occur as colimits of filtered diagrams of representables, in which every morphism is a monomorphism. Following a suggestion by André Joyal, these sheaves are characterized as satisfying a "small diagonal" condition with respect to maps with representable fibers. The subcategory of such ideals then forms a category with class structure in which one can solve for fixed points of the powerobject functor.

If the powerclass of an ideal C is thought of as the class of all sub*sets* of C, then the powerobject of C in the category of sheaves can be thought of as the "hyper class" of all sub*classes* of C, since ideals are closed under sub(pre)sheaves. The first step in a comparison between these two kinds of powerobjects is carried out in section 4, where it is shown that there is a model in the category of sheaves of a Morse-Kelley style theory of sets and classes which is a conservative extension of BIST.

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2 BIST and categories with class structure

2.1 Basic Intuitionistic Set Theory

We recall the following *Basic Intuitionistic Set Theory*—BIST— from [2]: The language has, in addition to the membership relation ϵ, a predicate S for "Sethood". We make use of the shorthand notation of $\text{Ƨ}x.\ \phi$ for $\exists y.\ \mathsf{S}(y) \wedge \forall x.\ (x \epsilon y \leftrightarrow \phi)$, where y is not free in ϕ. The expression $x \subseteq y$ stands for $\mathsf{S}(x) \wedge \mathsf{S}(y) \wedge \forall z \epsilon x.\ z \epsilon y$.

BIST 1. (Membership) $y \epsilon x \to \mathsf{S}(x)$

BIST 2. (Extensionality) $x \subseteq y \wedge y \subseteq x \to x = y$

BIST 3. (Empty Set) $\text{Ƨ}z.\bot$

BIST 4. (Pairing) $\text{Ƨ}z.\ z = x \vee z = y$

BIST 5. (Union) $\mathsf{S}(x) \wedge (\forall y \epsilon x.\ \mathsf{S}(y)) \to \text{Ƨ}z.\ \exists y \epsilon x.\ z \epsilon y$

BIST 6. (Replacement) $\mathsf{S}(x) \wedge (\forall y \epsilon x.\ \exists! z.\ \phi) \to \text{Ƨ}z.\ \exists y \epsilon x.\ \phi$

BIST 7. (Power Set) $\mathsf{S}(x) \to \text{Ƨ}y.\ y \subseteq x$

BIST 8. (Intersection) $\mathsf{S}(x) \wedge (\forall y \epsilon x. y \subseteq w) \to \text{Ƨ}z.\ z \epsilon w \wedge \forall y \epsilon x. z \epsilon y$

These axioms allow us to define the usual notions of "ordered pair" and "function", which allow us to state the last axiom of BIST, which we state informally as:

BIST 9. (Infinity) There exists a set I with an injection $s : I + 1 \to I$.

For the purposes of this paper, the axiom of Infinity is of no special interest.

We introduce the notation $!\phi$ (read 'ϕ is simple') as a shorthand for:

$$\text{Ƨ}z.\ z = \emptyset \wedge \phi$$

where z is not free in ϕ. Separation for simple formulas is provable in BIST [2]:

Proposition 2.1.1 (!-Sep) BIST $\vdash (\forall y \epsilon x.\ !\phi) \rightarrow (\mathsf{S}(x) \rightarrow \text{Ƨ}y.\ y \epsilon x \wedge \phi)$

Certain closure conditions also hold for provably simple formulas [2]:

Lemma 2.1.2 *In* BIST, *the following are provable:*

1. $!(x = y)$
2. $\mathsf{S}(x) \rightarrow !(y \epsilon x)$
3. $!\phi \wedge !\psi \rightarrow !(\phi \wedge \psi) \wedge !(\phi \vee \psi) \wedge !(\phi \rightarrow \psi)$
4. $\mathsf{S}(x) \wedge \forall y \epsilon x.\ !\phi \rightarrow !(\exists y \epsilon x.\ \phi) \wedge !(\forall y \epsilon x.\ \phi)$
5. $(\phi \vee \neg\phi) \rightarrow !\phi$

The following form of Δ_0 separation therefore holds.

Proposition 2.1.3 (Δ_0-Sep) *In* BIST, *separation holds for* S*–predicate free* Δ_0 *formulas in the context of a "well–typing", in the following sense: For a* Δ_0 *formula* ϕ *in which the* S*-predicate does not occur, let* $x_1, \ldots, x_n$*, be a list of all the variables occurring on the right hand side of an* ϵ *in* ϕ*. Construct a formula* ψ_n *by induction on* n *as follows:* $\psi_0 = \top$*. If* x_i *is free in* ϕ*, then* $\psi_i = \psi_{i-1} \wedge \mathsf{S}(x_i)$*. If* x_i *is bound by a quantifier* $\forall x_i \epsilon t$ *or* $\exists x_i \epsilon t$ *and* t *is free in* ϕ*, then* $\psi_i = \psi_{i-1} \wedge \mathsf{S}(t) \wedge \forall s \epsilon t.\, \mathsf{S}(s)$*. If* t *itself is bound by a formula* $\forall t \epsilon u$ *or* $\exists t \epsilon u$ *and* u *is free in* ϕ*, then* $\psi_i = \psi_{i-1} \wedge \mathsf{S}(u) \wedge \forall s \epsilon u.\, \mathsf{S}(s) \wedge \forall p \epsilon s.\, \mathsf{S}(p)$*. If* u *is bound as well, then continue in the same way. We have then that:*

$$\text{BIST} \vdash \mathsf{S}(x) \wedge \psi_n \rightarrow \text{Ƨ}y \epsilon x.\, \phi$$

and if x_n *is free in* ϕ *that:*

$$\text{BIST} \vdash \mathsf{S}(x) \wedge (\forall y \epsilon x.\, \mathsf{S}(y)) \wedge \psi_{n-1} \rightarrow \text{Ƨ}x_n \epsilon x.\, \phi$$

We remark that in order to have unrestricted Δ_0 separation, it is sufficient to add to BIST an axiom stating that the S–predicate is simple.

2.2 Categories with class structure

Let $\mathcal{C}$ be a positive Heyting category, i.e. a Heyting category with finite disjoint coproducts that are stable under pullback (see [5, A1.4.4]). A *system of small maps* on $\mathcal{C}$ is a collection of morphisms of $\mathcal{C}$ satisfying the following conditions:

(S1) Every identity map $Id_A : A \longrightarrow A$ is small, and the composite $g \circ f : A \longrightarrow C$ of any two small maps $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is again small.

(S2) The pullback of a small map along any map is small. Thus in an arbitrary pullback diagram,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ C \arrow[r] & D \end{tikzcd}$$

f' is small if f is small.

(S3) Every diagonal $\Delta : A \longrightarrow A \times A$ is small.

(S4) If $f \circ e$ is small and e is regular epic, then f is small, as indicated in the diagram:

$$\begin{tikzcd} A \arrow[rr, "f \circ e"] \arrow[dr, "e"'] & & B \\ & C \arrow[ur, "f"'] & \end{tikzcd}$$

(S5) Copairs of small maps are small. Thus if $f : A \longrightarrow C$ and $g : B \longrightarrow C$ are small, then so is the copairing $[f, g] : A + B \longrightarrow C$.

A relation $r : R \rightarrowtail A \times B$ is called a *small relation* if the second projection $\pi_2 \circ r : R \rightarrowtail A \times B \longrightarrow B$ is a small map. We make the small relations representable in requiring that $\mathcal{C}$ has *(small) powerobjects* consisting of, for every object A in $\mathcal{C}$, an object PA and a small relation $\epsilon_A \rightarrowtail A \times PA$, such that the following two axioms are satisfied:

(P1) For any small relation $R \overset{m}{\rightarrowtail} A \times B$, there exists a unique *classifying map* $\rho : B \longrightarrow PA$ such that the following is a pullback:

$$\begin{tikzcd} R \arrow[r] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \epsilon_A \arrow[d] \\ A \times B \arrow[r, "Id \times \rho"'] & A \times PA \end{tikzcd}$$

(P2) The internal *subset* relation $\subseteq_A \rightarrowtail PA \times PA$ is a small relation (the definition of $\subseteq_A$ is as expected, see [2]).

For any morphism $f : A \longrightarrow B$ in $\mathcal{C}$, the image of ϵ_A along $f \times Id_{PA}$ is a small relation by S4. Its classifying map $Pf : PA \longrightarrow PB$, also known as the *internal direct image map*, is the morphism part of the *powerobject functor* $\mathcal{P} : \mathcal{C} \longrightarrow \mathcal{C}$.

A *universal object* $\mathcal{U}$ in $\mathcal{C}$ is an object such that for every object A in $\mathcal{C}$, there exists a monomorphism $A \rightarrowtail \mathcal{U}$. A universal object is in particular a *universe*, that is, an object U such that there exists a monomorphism $PU \rightarrowtail U$. We require the existence of a universal object:

(U) $\mathcal{C}$ has a universal object $\mathcal{U}$.

A positive Heyting category $\mathcal{C}$ having a system of small maps satisfying S1-S5, powerobjects satisfying P1-P2, and a universal object we call a *category with class structure*, or briefly *class category*. We denote a category with this structure $(\mathcal{C},\mathcal{S},\mathcal{P},\mathcal{U})$ or briefly $\mathcal{C}$.

For a class category $\mathcal{C}$, the universal object $\mathcal{U}$ together with a choice of inclusion $\iota : P\mathcal{U} \rightarrowtail \mathcal{U}$ gives us a structure for a first-order set theory (ϵ, S), by defining the interpretation $[\![x \mid \mathsf{S}(x)]\!]$ to be the mono:

$$\iota : P\mathcal{U} \rightarrowtail \mathcal{U}$$

and $[\![x, y \mid x\epsilon y]\!]$ as the composite:

$$\epsilon_{\mathcal{U}} \rightarrowtail \mathcal{U} \times P\mathcal{U} \rightarrowtail \mathcal{U} \times \mathcal{U}$$

The following is proved in [2, Section 3]:

Theorem 2.2.1 *The set theory* BIST *is sound and complete with respect to such models* $(\mathcal{U}, \iota)$ *in class categories* $\mathcal{C}$*:*

$$\text{BIST} \vdash \phi \textit{ iff, for all } \mathcal{C}, \textit{ one has } \mathcal{U} \models_{\mathcal{C}} \phi$$

The completeness result is proved by defining a class structure on the first-order syntactic category of BIST. Briefly, a morphism $[x, y \mid \phi] : [x \mid \psi] \longrightarrow [y \mid \sigma]$ is small if $\text{BIST} \vdash_y \text{ج}x.\ \phi$; the powerobject of $[x \mid \psi]$ is $[u \mid \mathsf{S}(u) \wedge \forall x\epsilon u.\ \psi]$; and $\mathcal{U} = [x \mid x = x]$.

3 Ideals over a topos

3.1 Small maps in sheaves

In a class category $\mathcal{C}$, a *small object* is an object A such that the unique map $A \longrightarrow 1$ is small. By [2], the small objects in $\mathcal{C}$ form a topos, and every topos occurs as the category of small objects in a category with class structure. The purpose of this section is to provide a new proof of the latter fact, using a more canonical construction that avoids some of the difficulties in the original proof.

Let a small topos (or, for this subsection, just a pretopos) $\mathcal{E}$ be given. Consider the category $\mathrm{Sh}(\mathcal{E})$ of sheaves on $\mathcal{E}$, for the coherent covering [5, A2.1.11(b)]. Recall that the Yoneda embedding $y : \mathcal{E} \hookrightarrow \mathrm{Sh}(\mathcal{E})$ is a full and faithful Heyting functor [6, D3.1.17].

We intend to build a class category in $\mathrm{Sh}(\mathcal{E})$ where the representables are the small objects. First, we define a system $\mathcal{S}$ of small maps on $\mathrm{Sh}(\mathcal{E})$ by including in $\mathcal{S}$ the morphisms of $\mathrm{Sh}(\mathcal{E})$ with "representable fibers" in the following sense:

Definition 3.1.1 (Small Map) A morphism $f : A \longrightarrow B$ in $\mathrm{Sh}(\mathcal{E})$ is *small* if for any morphism with representable domain $g : yD \longrightarrow B$, there exists an object C in $\mathcal{E}$, and morphisms f', g' in $\mathrm{Sh}(\mathcal{E})$ such that the following is a pullback:

$$\begin{tikzcd} yC \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & yD \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Thus, in this sense, *small maps pull representables back to representables.*

Proposition 3.1.2 $\mathcal{S}$ *satisfies axioms S1, S2, and S5.*

PROOF S1 and S2 follow easily from the Two Pullback lemma.

For S5, the pullback of, say, $yD \xrightarrow{h} C$ along $[f,g] : A + B \longrightarrow C$ is the coproduct of the pullback of h along f and of h along g. But this is representable, since representables are closed under finite coproducts in $\mathrm{Sh}(\mathcal{E})$. $\dashv$

We move to consider S3. A *directed diagram* (in a category $\mathcal{C}$, say) is a functor $I \longrightarrow \mathcal{C}$ where I is a directed preorder. A small directed diagram in $\mathcal{C}$ in which (the image of) every morphism is a monomorphism in $\mathcal{C}$we shall call an *ideal diagram*. An ideal diagram has no non-trivial parallel pairs, and is therefore also a filtered diagram.

Definition 3.1.3 (Ideal over $\mathcal{E}$) An object A in $\mathbf{Sets}^{\mathcal{E}^{\mathrm{op}}}$ is an *ideal over $\mathcal{E}$* if it can be written as a colimit of an ideal diagram $I \longrightarrow \mathcal{E}$ of representables,

$$A \cong \varinjlim_I (yC_i) \qquad \lhd$$

Lemma 3.1.4 *Every ideal is a sheaf.*

PROOF Since an ideal diagram is a filtered diagram, filtered colimits commute with finite limits, being a sheaf is a finite limit condition, and all representables are sheaves, all such presheaves are also sheaves. $\dashv$

In accordance with a conjecture by André Joyal, it now turns out that the ideals over $\mathcal{E}$ are exactly the sheaves for which S3 holds, i.e. for which the diagonal $A \rightarrowtail A \times A$ is small:

Lemma 3.1.5 *Any sheaf F can be written as a colimit (in $\mathbf{Sets}^{\mathcal{E}^{\mathrm{op}}}$) of representables $\varinjlim_I (yC_i)$ where I has the property that for any two objects i, j in I, there is an object k in I and morphisms $i \longrightarrow k$ and $j \longrightarrow k$ (i.e. I is directed).*

PROOF We may write a sheaf F as the colimit of the composite functor $\int F \xrightarrow{\pi} \mathcal{E} \xrightarrow{y} \mathbf{Sets}^{\mathcal{E}^{\mathrm{op}}}$, where $\int F$ is the category of elements of F, and π is the forgetful functor. The objects in $\mathrm{Sh}(\mathcal{E})$ can be characterized as the functors $\mathcal{E}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$ which preserve monomorphisms and finite products. It follows that $\int F$ has the required property, since for any two objects (A, a), (B, b) in $\int F$ (with $a \in FA$, $b \in FB$),

$$(A, a) \longrightarrow (A + B, \langle a, b \rangle) \longleftarrow (B, b)$$

(By the coproduct $A + B$, we mean the coproduct in $\mathcal{E}$, hence the product $A \times B$ in $\mathcal{E}^{\mathrm{op}}$, which is sent to the product $FA \times FB$ in $\mathbf{Sets}$.) $\dashv$

Theorem 3.1.6 *For any sheaf F, the following are equivalent:*

1. *F is an ideal.*
2. *The diagonal $F \rightarrowtail F \times F$ is a small map.*
3. *For all arrows with representable domain $yC \xrightarrow{f} F$, the image of f in sheaves is representable, $f : yC \twoheadrightarrow yD \rightarrowtail F$, for some D in $\mathcal{E}$.*

PROOF (1)⇒(2):

We write F as an ideal diagram of representables, $F = \underrightarrow{Lim}_I(yC_i)$. Note that the pullback of any arrow $A \xrightarrow{f} F \times F$ along $F \xrightarrow{\Delta} F \times F$ is the equalizer of the pair $A \overset{\pi_1 f}{\underset{\pi_2 f}{\rightrightarrows}} F$. Thus let $g, h : yD \rightrightarrows F$ be given, and we must verify that their equalizer $e : E \rightarrowtail yD$ is representable. Recall that, in $\mathbf{Sets}^{E^{\mathrm{op}}}$, if we are given a colimit $\underrightarrow{Lim}_I(yC_i)$ and an arrow $yX \xrightarrow{f} \underrightarrow{Lim}_I(yC_i)$, f factors through the base of the colimiting cocone, i.e.

$$\begin{tikzcd}
yX \arrow[rr, "e"] \arrow[dr, "f"'] & & yC_i \arrow[dl, "f_i"] \\
& \underrightarrow{Lim}_I(yC_i) &
\end{tikzcd}$$

for some i (where f_i is an arrow of the colimiting cocone). Hence we may factor h as $yX \xrightarrow{e_h} yC_i \xrightarrow{f_i} \underrightarrow{Lim}_I(yC_i)$ and g as $yX \xrightarrow{e_g} C_j \xrightarrow{f_j} \underrightarrow{Lim}_I(yC_i)$. Since the diagram is directed, there is a C_k and arrows u, v such that the two triangles in the following commute:

$$\begin{tikzcd}
yD \arrow[r, "e_h"] \arrow[d, "e_g"'] & yC_i \arrow[d, "u"] \arrow[ddr, "f_i"] & \\
yC_j \arrow[r, "v"'] \arrow[drr, "f_j"'] & yC_k \arrow[dr, "f_k"] & \\
& & F
\end{tikzcd}$$

Since f_k is monic, the equalizer $e : E \rightarrowtail yD$ of $h = f_k u e_h$ and $g = f_k v e_g$ is precisely the equalizer of $u e_h$ and $v e_g$. But Yoneda preserves and reflects equalizers, so we may conclude that the equalizer of h and g is representable, $E \cong yC$.

(2)⇒(3):

Let $yD \xrightarrow{f} F$ be given. The kernel pair k_1, k_2 of f can be described as the pullback:

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "(k_1,k_2)"'] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, "\Delta"] \\
yD \times yD \arrow[r, "f \times f"'] & F \times F
\end{tikzcd}$$

Since $yD \times yD \cong y(D \times D)$ is representable and the diagonal of F is small, K is representable ($K \cong yK$, with some abuse of notation). Hence we may rewrite the kernel pair as

$$yK \overset{yk_1}{\underset{yk_2}{\rightrightarrows}} yD \xrightarrow{f} F$$

The kernel pair is an equivalence relation in $\widehat{\mathcal{E}}$. Since Yoneda is full and faithful and cartesian, $K \underset{k_2}{\overset{k_1}{\rightrightarrows}} D$ is an equivalence relation in $\mathcal{E}$. Since $\mathcal{E}$ is effective, there is a coequalizer

$$\begin{tikzcd} K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & D \arrow[r, two heads, "e"] & E \end{tikzcd}$$

such that k_1 and k_2 is the kernel pair of e. Since Yoneda preserves pullbacks and regular epis into $\mathrm{Sh}(\mathcal{E})$,

$$\begin{tikzcd} yK \arrow[r, shift left, "yk_1"] \arrow[r, shift right, "yk_2"'] & yD \arrow[r, two heads, "ye"] & yE \end{tikzcd}$$

is a coequalizer diagram in $\mathrm{Sh}(\mathcal{C})$. This gives us, then, the required epi–mono factorization:

$$\begin{tikzcd} yK \arrow[r, shift left, "yk_1"] \arrow[r, shift right, "yk_2"'] & yD \arrow[rr, "f"] \arrow[dr, two heads, "ye"'] & & F \\ & & yE \arrow[ur, tail] & \end{tikzcd}$$

(3)$\Rightarrow$(1):

Step 1: To construct an ideal diagram of representables.

We write F as a colimit $F = \varinjlim_I (yD_i)$, in accordance with Lemma 3.1.5 (so that I is the category of elements of F). Now, for each $i \in I$, factor in sheaves the cocone arrow $yD_i \xrightarrow{f_i} F$:

$$\begin{tikzcd} yD_i \arrow[rr, "f"] \arrow[dr, two heads, "ye_i"'] & & F \\ & yE_i \arrow[ur, tail, "m_i"'] & \end{tikzcd}$$

For $yD_i \xrightarrow{u} yD_j$ in the diagram I, consider the diagram:

$$\begin{tikzcd} yD_i \arrow[r, "ye_i"] \arrow[d, "u"'] & yE_i \arrow[d, tail, dashed, "v"] \arrow[dr, "m_i"] & \\ yD_j \arrow[r, "ye_j"'] & yE_j \arrow[r, tail, "m_j"'] & F \end{tikzcd}$$

Since $f_i = f_j u$, it follows that f_i factors through yE_j, which gives us the mono v, making the triangle in the diagram commute (to see this, the diagram must be considered in $\mathrm{Sh}(\mathcal{E})$, where e_i is a cover). Since m_j is monic, the square commutes.

The new diagram I' of the yE_i and v thus obtained is directed, since I has the property described in Lemma 3.1.5 and any parallel pair of arrows collaps by the construction.

Step 2: To show $F \cong \varinjlim_{I'} (yE_i)$

Observe that the ye_i's in the diagram above give us a morphism $e : \underrightarrow{Lim}_I yD_i \longrightarrow \underrightarrow{Lim}_{I'} yE_i$, while the m_i's give us a monomorphism $\underrightarrow{Lim}_{I'} yE_i \rightarrowtail F$, such that the following commutes:

$$\begin{tikzcd}
\underrightarrow{Lim}_I yD_i \arrow[rr, "e"] \arrow[dr, "\cong"'] & & \underrightarrow{Lim}_{I'} yE_i \arrow[dl, "m"] \\
& F &
\end{tikzcd}$$

Thus m is also an isomorphism. ⊣

In order to ensure that S3 is satisfied, we therefore narrow our attention from $\mathrm{Sh}(\mathcal{E})$ to the full subcategory of ideals, denoted $\mathrm{Idl}(\mathcal{E})$. We shall see that no further restriction is needed. First, we verify that $\mathrm{Idl}(\mathcal{E})$ is a positive Heyting category:

Lemma 3.1.7 $\mathrm{Idl}(\mathcal{E})$ *is closed under (presheaf) subobjects and finite limits.*

PROOF We use the description of ideals as sheaves with small diagonal. That $\mathrm{Idl}(\mathcal{E})$ is closed under subobjects follows from S2.

$1 \xrightarrow{\Delta} 1 \times 1$ is iso, hence small.

If A, B are ideals and C is any sheaf, we consider the pullback:

$$\begin{tikzcd}
D \arrow[r, "k_2"] \arrow[d, "k_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

Now, if we pull the diagonals back:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\Delta"] \\
D \times D \arrow[r, "k_1 \times k_1"'] & A \times A
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r] \arrow[d, "\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\Delta"] \\
D \times D \arrow[r, "k_2 \times k_2"'] & B \times B
\end{tikzcd}$$

By a diagram chase, the diagonal of D is $A_1 \cap B_1$, which is small since smallness is preserved by pullback and composition.

Lemma 3.1.8 $\mathrm{Idl}(\mathcal{E})$ *is closed under finite coproducts (of sheaves), and inclusion maps are small.*

PROOF $0 \longrightarrow 0 \times 0$ is iso, so small.

Now, the terminal object 1 in $\mathrm{Sh}(\mathcal{E})$ is representable, and so is $1+1$, since Yoneda preserves finite coproducts. The inclusion $i_1 : 1 \longrightarrow 1+1$ is therefore small. But coproducts in $\mathrm{Sh}(\mathcal{E})$ being disjoint, the following is a pullback:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i_A"'] & 1 \arrow[d, "i_1"] \\
A + B \arrow[r, "!_A + !_B"'] & 1 + 1
\end{tikzcd}$$

So by S2, the inclusion map i_A is small.

The diagonal of $A+B$ can be regarded as the disjoint union of the diagonal of A and of B:

$$\begin{tikzcd}
A \arrow[rr, "p_A"] \arrow[d, "\Delta"'] & & A+B \arrow[d, "\Delta"] & & B \arrow[ll, "p_B"'] \arrow[d, "\Delta"] \\
A \times A \arrow[rr, dashed] \arrow[dr, "p_{A\times A}"'] & & (A+B)\times(A+B) & & B \times B \arrow[ll, dashed] \arrow[dl, "p_{B\times B}"] \\
& (A\times A)+(A\times B)+(B\times A)+(B\times B) \arrow[ur, "\cong"] & & &
\end{tikzcd}$$

By smallness of coproduct inclusions and isos, and applying S5, if A, B are ideals then so is $A+B$. $\dashv$

Proposition 3.1.9 $\mathrm{Idl}(\mathcal{E})$ *is positive Heyting, and this structure can be calculated in* $\mathrm{Sh}(\mathcal{E})$.

PROOF We have done finite limits and finite coproducts. For a morphism $f : A \longrightarrow B$ of ideals, $Im(f)$ is an ideal, since there is a monomorphism $Im(f) \rightarrowtail B$. The cover $e : A \twoheadrightarrow Im(f)$ is the coequalizer of its kernel pair in $\mathrm{Sh}(\mathcal{E})$, the kernel pair is the same in $\mathrm{Idl}(\mathcal{E})$, so e is also a regular epimorphism in $\mathrm{Idl}(\mathcal{E})$.

For dual images, since $\mathrm{Idl}(\mathcal{E})$ is closed under subobjects and finite limits can be taken in sheaves, dual images can also be taken in sheaves.

Lemma 3.1.10 *S4 is satisfied in* $\mathrm{Idl}(\mathcal{E})$.

PROOF Let $A \overset{a}{\twoheadrightarrow} B \overset{b}{\longrightarrow} C$ be given, and assume $b \circ a$ is small. Let $yG \longrightarrow C$ be given, and consider the following two pullback diagram:

$$\begin{tikzcd}
yD \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & yG \arrow[d] \\
A \arrow[r, two heads, "a"'] & B \arrow[r, "b"'] & C
\end{tikzcd}$$

By Theorem 3.1.6, the image of a representable is a representable in $\mathrm{Idl}(\mathcal{E})$. Hence E in the diagram above is (isomorphic to) a representable. $\dashv$

We summarize the results of this subsection:

Theorem 3.1.11 *For any pretopos $\mathcal{E}$, the full subcategory $\mathrm{Idl}(\mathcal{E}) \hookrightarrow \mathrm{Sh}(\mathcal{E})$ of ideals is a positive Heyting category with a system of small maps satisfying axioms S1–S5.*

3.2 Powerobjects and universes in $\mathrm{Idl}(\mathcal{E})$

We end the section with a brief discussion of the remaining part of the class structure in $\mathrm{Idl}(\mathcal{E})$, powerobjects and universes. In this subsection, we require $\mathcal{E}$ to be a topos, for we shall use the powerobjects in $\mathcal{E}$ to build powerobjects for ideals. Here we rely heavily on the characterization of $\mathrm{Idl}(\mathcal{E})$ as the colimits of ideal diagrams of representables. As such, $\mathrm{Idl}(\mathcal{E})$ is a subcategory of $\mathrm{Ind}(\mathcal{E})$, the category of filtered colimits of representables. We refer to [6, section C2] for the properties of $\mathrm{Ind}(\mathcal{E})$. Much of what is said about $\mathrm{Idl}(\mathcal{E})$ here are just special cases of that.

Lemma 3.2.1 $\mathrm{Idl}(\mathcal{E})$ *has colimits of ideal diagrams ("*ideal colimits*").*

PROOF Any such diagram is an ideal diagram of representables, see [6, section C2]. $\dashv$

Proposition 3.2.2 *If $\mathcal{C}$ is a category with ideal colimits, and $F : \mathcal{E} \longrightarrow \mathcal{C}$ is a functor which preserves monomorphisms, then there is a unique (up to natural isomorphism) extension $\tilde{F} : \mathrm{Idl}(\mathcal{E}) \longrightarrow \mathcal{C}$ of F such that $\tilde{F}$ is* continuous, *in the sense of preserving ideal colimits, and such that the following commutes:*

$$\begin{tikzcd} \mathrm{Idl}(\mathcal{E}) \arrow[r, "\tilde{F}"] & \mathcal{C} \\ \mathcal{E} \arrow[u, "y"] \arrow[ur, "F"'] & \end{tikzcd}$$

PROOF Write $E = \varinjlim_I(yC_i)$ and set $\tilde{F}(E) = \varinjlim_I(FC_i)$. $\dashv$

The powerobject functor $P : \mathcal{E} \longrightarrow \mathcal{E}$ preserves monomorphisms, as does $y : \mathcal{E} \longrightarrow \mathrm{Idl}(\mathcal{E})$, and so $y \circ P$ extends to a continuous functor $\mathcal{P} : \mathrm{Idl}(\mathcal{E}) \longrightarrow \mathrm{Idl}(\mathcal{E})$. If $\varinjlim_I(yA_i)$ is an ideal then $\varinjlim_I(yPA_i)$ is its powerobject. The epsilon subobject is similarly constructed.

Proposition 3.2.3 *With these powerobjects,* $\mathrm{Idl}(\mathcal{E})$ *satisfies P1 and P2.*

PROOF We do one example: Let a yC be a small subobject of the ideal $\varinjlim_I(yA_i)$. The inclusion arrow $yC \rightarrowtail \varinjlim_I(yA_i)$ factors through some yA_i, and we get the following diagram:

$$\begin{tikzcd} yC \arrow[rr] \arrow[d, tail] & & y\epsilon_{A_i} \arrow[d, tail] \\ yA_i \times 1 \arrow[rr, "Id\times\alpha"] \arrow[d, tail] & & yA_i \times yPA_i \arrow[d, tail] \\ \varinjlim_I(yA_i) \times 1 \arrow[r, tail] & \varinjlim_I(yA_i) \times \varinjlim_I(yPA_i) & \end{tikzcd}$$

whence we get the global point $1 \longrightarrow \varinjlim_I(yPA_i)$ classifying yC. $\dashv$

The main point is that every small subobject $B \rightarrowtail \varinjlim_I(yA_i)$ of an ideal is already a (small) subobject $B \rightarrowtail yA_i$ of some yA_i in the diagram.

Since the powerobject functor $\mathcal{P}$ is continuous in the above sense, we can find fixed points for it. For one example, we compose $\mathcal{P}$ on $\mathrm{Idl}(\mathcal{E})$ with the continuous functor $C \mapsto A + C$ for a fixed A in $\mathrm{Idl}(\mathcal{E})$. To construct a universal object, we wish for every representable to have a monomorphism into our universe, so take as our starting point $A := \coprod_{C \in \mathcal{E}} yC$ (where the coproduct is taken in sheaves). This is an ideal, for it is the colimit of the ideal diagram of finite coproducts of representables, which themselves are representable, with arrows the coproduct inclusions.

Now consider the diagram

$$A \overset{i_A}{\rightarrowtail} A + PA \overset{Id_A + Pi_A}{\rightarrowtail} A + P(A + PA) \rightarrowtail \ldots$$

Call the colimit U. Then $A + PU \overset{\cong}{\rightarrowtail} U$, so we have a universe consisting of the class A of atoms and the class PU of sets. (U is the free algebra on A for the endofunctor $\mathcal{P}$.)

Although it is a universe, U is not yet a universal object. We obtain, finally, our category with class structure containing $\mathcal{E}$ as the small objects by cutting out the part of $\mathrm{Idl}(\mathcal{E})$ we need (as in [9]):

Proposition 3.2.4 *If $(\mathcal{C}, \mathcal{S}, \mathcal{P})$ is a category with class structure (with or without a universal object) and U is a universe in $\mathcal{C}$, then the full subcategory $\downarrow(U)$ of objects A in $\mathcal{C}$ such that there exists a monomorphism $A \rightarrowtail U$ is a category with class structure, with the structure it inherits from $\mathcal{C}$, and with U as its universal object.*

PROOF We can demonstrate the existence of a *encoded ordered pair* map $U \times U \longrightarrow PPU$ which, when composed with the inclusions $PPU \rightarrowtail PU \rightarrowtail U$ gives a monomorphism $U \times U \rightarrowtail U$. The rest is straightforward (see also [9]). $\dashv$

There are of course a number of universes in $\mathrm{Idl}(\mathcal{E})$ that contain the representables, in the sense above. From [2], we know that:

Theorem 3.2.5 BIST+*Coll is sound and complete with respect to class categories of the form $\downarrow(U) \rightarrowtail \mathrm{Idl}(\mathcal{E})$, for toposes $\mathcal{E}$ and universes U containing the representables.*

Here, we either do not consider BIST to include an axiom of infinity, or we restrict attention to topoi containing a natural numbers object, see [2] and [1] for details. *Coll* is the axiom scheme of Collection which says that for any total relation R on a set A, there is a set B contained in the range of R:

(Coll) $\mathsf{S}(z) \wedge (\forall x \epsilon z.\ \exists y.\ \phi) \rightarrow \exists w.\ (\mathsf{S}(w) \wedge (\forall x \epsilon z.\ \exists y \epsilon w.\ \phi) \wedge (\forall y \epsilon w.\ \exists x \epsilon z.\ \phi))$

4 BICT and full powerobjects

4.1 BICT

We introduce the following *Basic Intuitionistic Class Theory*:[1] Its language is a two-typed first-order language, where we use lower case variables for the "type of elements" and upper case variables for the "type of classes". There is a "Sethood" predicate S and a binary "element" relation ϵ, both of which are "element"-typed. In addition, there is a binary predicate η which takes elements on the left and classes on the right:

BICT**1.** (BIST axioms)

All axioms of BIST, i.e. BIST1–BIST**9**, except, if one prefers, Replacement, which gets covered below.

[1] It is intended to stand to BIST as Morse–Kelley stands to ZF, and is patterned on the system B in [4].

BICT**2.** (Class extensionality)

$$(\forall z.\ z\eta X \leftrightarrow z\eta Y) \rightarrow X = Y$$

BICT**3.** (Replacement) For any formula ϕ:

$$\mathsf{S}(x) \wedge (\forall y\epsilon x.\ \exists! z.\ \phi) \rightarrow \reflectbox{S} z.\ \exists y\epsilon x.\ \phi$$

BICT**4.** (Comprehension) For any formula ϕ (X not free in ϕ):

$$\exists X.\ \forall z.\ z\eta X \leftrightarrow \phi$$

We say that a class is *represented* by a set if they have the same elements. Replacement now holds also for formulas ϕ with class quantifiers. Actually, it need not be a schema, since one could reformulate it as: If the domain of a functional class of ordered pairs is represented by a set, then so is the image.

4.2 A model of BICT in sheaves

Let two categories and a functor $z : \mathcal{C} \hookrightarrow \mathcal{G}$ be given, and assume that

- $\mathcal{C}$ is a category of classes.
- $\mathcal{G}$ is a topos.
- z is full and faithful and Heyting.
- $z\mathcal{C}$ generates $\mathcal{G}$ (in the sense that if $f \neq g : G \rightrightarrows G'$ in $\mathcal{G}$ then there is some $h : zC \longrightarrow G$ such that $fh \neq gh$.)

Then we shall conclude that $\mathcal{G}$ contains a model of BICT.

Lemma 4.2.1 *If $C \in \mathcal{C}$, then $\mathcal{C}/C$ and $\mathcal{G}/zC$ and $z/C : \mathcal{C}/C \longrightarrow \mathcal{G}/zC$ inherit the above-listed properties.*

Proof The class structure is preserved by slicing by [2]. $\dashv$

Denote by U the universal object of $\mathcal{C}$, and consider the relation $z\epsilon_U \rightarrowtail zU \times zPU$. Then there exists a unique classifying arrow $\kappa : zPU \longrightarrow PzU$ making the following a pullback in $\mathcal{G}$:

$$\begin{tikzcd}
z\epsilon_U \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \epsilon_{zU} \arrow[d, tail] \\
zU \times zPU \arrow[r, "Id\times\kappa"'] & zU \times PzU
\end{tikzcd}$$

where (PzU, ϵ_{zU}) is the (full) powerobject in the topos $\mathcal{G}$. We claim that the objects zU and PzU model BICT, with $[\![x \mid \mathsf{S}(x)]\!]$ interpreted as $zPU \xrightarrow{z\iota} zU$, $[\![x, y \mid x\epsilon y]\!]$ as the composite $z\epsilon_U \rightarrowtail zU \times zPU \xrightarrow{Id\times z\iota} zU \times zU$, and $[\![x, Y \mid x\eta Y]\!]$ as $\epsilon_{zU} \rightarrowtail zU \times PzU$. Let us call this structure $\mathcal{M}$.

Because the standard notation for class categories is so similar to the notation usually employed for topoi, we introduce some modifications. We shall often not bother to write out the z denoting the embedding. Instead, we write $P_s A$ for the powerobject of A in $\mathcal{C}$, and call it the *small* powerobject of A. ϵ_A

remains the notation for the (small) elementhood relation in $\mathcal{C}$. We continue to use PA for the powerobject in $\mathcal{G}$ (or PzA, if it is not clear from context that A is in the image of z), but we denote the full elementhood relation by η_A. Our pullback diagram above will then look like this:

$$
\begin{tikzcd}
\epsilon_U \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \eta_U \arrow[d, tail] \\
U \times P_s U \arrow[r, "Id \times \kappa"'] & U \times PU
\end{tikzcd}
$$

Proposition 4.2.2 $\mathcal{M}$ *models* BICT.

PROOF BICT1(BIST axioms): Since z is Heyting.

BICT2(Class extensionality): By topos extensionality.

BICT3(Replacement): The proof in [2] that replacement (BIST6) holds in a category with class structure carries over to our case, using Lemma 4.2.1 and the fullness of z and the fact that $z\mathcal{C}$ generates $\mathcal{G}$. (Briefly, the idea is that replacement holds in $\mathcal{C}$ and there are no "new" arrows in $\mathcal{G}$ between objects from $\mathcal{C}$.)

BICT4(Comprehension): By topos comprehension.

Corollary 4.2.3 BICT *is a conservative extension of* BIST.

PROOF Let $\mathcal{C}$ be the syntactic category of BIST, $\mathcal{G}$ the category of sheaves (coherent covering) on $\mathcal{C}$ and z the Yoneda embedding. $\dashv$

Another instance worth considering is when $\mathcal{C}$ is $\downarrow(U)$ in $\mathrm{Idl}(\mathcal{E})$ for some topos $\mathcal{E}$, and $\mathcal{G}$ is the category of sheaves on $\mathcal{E}$ (coherent covering), and z is the embedding of $\downarrow(U)$ into $\mathrm{Sh}(\mathcal{E})$.

We end this paper with some further observations concerning the case where $\mathcal{C}$ is a syntactic category. Observe that BIST may be extended to many a familiar set theory by adding appropriate axioms. The syntactic category of this set theory is still a class category, and the model $\mathcal{M}$ in sheaves is then a model of BICT extended by the axioms originally added to BIST.

For instance, BIST may be extended to ZF by adding an axiom of universal sethood, an axiom of foundation, and the law of excluded middle (LEM) for every formula of the language: Adding these same axioms to BICT, then, preserves conservativity. Call the theory consisting of BICT and these new axioms BIMK (Basic Intuitionistic Morse–Kelley). Note that we have added LEM only for formulas of BIST, i.e. those without class variables.

Corollary 4.2.4 BIMK *is a conservative extension of ZF.*

Separation—the assertion that the intersection of a class with a represented class yields a represented class—fails in general in BICT and in BIMK (and as a result, these extensions may at first strike the reader as rather pointless). However, we may repeat the analysis of simplicity from BIST to yield the following:

Lemma 4.2.5

$$\mathrm{BICT} \vdash \ (\mathsf{S}(x) \wedge \forall y \epsilon x.\ \text{Ƨ} z.\ z = \varnothing \wedge y \eta X) \rightarrow \text{Ƨ} y.\ y \epsilon x \wedge y \eta X$$

PROOF As in BIST. ⊣

Finally, we introduce a predicate $!X$ for $\forall x.!(x\eta X)$, to be read "X is *simple*". Observe that in BIMK, $!X$ iff $\forall x.(x\eta X \vee \neg(x\eta X))$, and the simple classes are just the complemented ones. The subobject $[\![X \mid !X]\!] \rightarrowtail PU$ is of independent interest, as it is the exponent $(P_s 1)^U$. We will not go into the analysis of these objects here, but simply point out that if we use these objects, instead of the sheaf powerobjects, as our "types of classes", we will get a class theory with full separation but restricted comprehension, instead of full comprehension and restricted separation.

In the case where $\mathcal{C}$ is the syntactic category of ZF, it can be shown that we have comprehension for any formula ϕ in which every class variable is free. We state the implication of this for BIMK, but leave the proof for a proper presentation of simplicity, in BIST and BICT, and the *simple powerobjects* $(P_s 1)^A$ in locally cartesian closed categories with class structure.

Proposition 4.2.6 *If a formula ϕ in* BIMK *is such that all class variables $X, Y, \ldots, Z$ occurring in ϕ are free, then*

$$\mathcal{M} \models !X \wedge !Y \wedge \ldots \wedge !Z \rightarrow (\forall x.\ \exists y.\ \forall z.\ z\epsilon y \leftrightarrow z\epsilon x \wedge \phi)$$

(where x and y are not free in ϕ).

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