Predicative Algebraic Set Theory

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The purpose of these notes is to generalize the machinery and results developed in [4] to the predicative case. Specifically, in *ibid.* it was shown that:

1. every category of classes contains a model of the intuitionistic, elementary set theory **BIST**,
2. **BIST** is logically complete with respect to such class category models,
3. the category of sets in such a model is an elementary topos,
4. every topos occurs as the sets in such a category of classes.

It follows, in particular, that **BIST** is sound and complete with respect to topoi as they can occur in categories of classes.[1] Thus, in a very precise sense, **BIST** represents exactly the elementary set theory whose models are the elementary topoi.

In the current notes, we show that the same situation obtains with respect to a weaker, "predicative" set theory without the powerset axiom (called "**CST**"), and the new notion of a "predicative topos" (called a "Π–pretopos", and defined as a locally cartesian closed pretopos).[2] As in the impredicative case, the correspondence between the set theory and the category is mediated by a suitable category of classes, now weakened by the omission of the small powerset condition **(P2)**. This condition essentially asserted that the powerobject $\mathcal{P}_s(A)$ of a small object A is again small; in its place, we essentially have the requirement that the exponential B^A of small objects A, B is again small. We also consider an even weaker, basic set theory "**BCST**" without the function set axiom, for which the corresponding categories of sets are exactly the Heyting pretopoi.

The categories of sets at issue are briefly introduced in section 1 below, and the elementary set theories in section 2. Section 3 then develops the predicative categories of classes and shows that the set theories are indeed sound and complete for such class category models. This development follows that of [4] quite closely, and it displays just how flexible and powerful the method developed

[1] It was also shown that every category of classes embeds into one of a special kind (the "ideal completion" of a topos), strengthening the completeness statement to topoi occurring in this special way. The predicative analogue of that result will not be considered here.

[2] A better notion of "predicative topos" is a Π–pretopos with W–types (cf. [12] and [13]). However, such categories will not be considered in this paper.

there proves to be. To establish point (4) above in *ibid.*, the notion of an *ideal in a topos* was invented and exploited. This concept has turned out to be quite robust and important. We here follow a suggestion of Joyal's to reformulate it as a certain 'diagonal condition' on sheaves.[3] As such, it also becomes a very flexible tool for the construction of class categories of various kinds.

The main technical result in these notes is Proposition 4.4.4, stating that for any Heyting pretopos $\mathcal{E}$, the small powerobject $\mathcal{P}_s(A)$ of an ideal A on $\mathcal{E}$ is again an ideal; this is key for the possibility of constructing a predicative category of classes with $\mathcal{E}$ as its category of sets. The construction makes use of the fact that the category of ideals over any pretopos $\mathcal{E}$ already satisfies the axioms for small maps as was shown in [5]. These topics are presented in section 4.

Taken together, these results show that **CST** is exactly the elementary set theory of Π-pretopoi, while **BCST** is the set theory of heyting pretopoi. Indeed, syntactic versions of these facts, involving translations of theories, can even be given, although we do not pursue that here.

Finally, we wish to acknowledge many helpful discussions with Carsten Butz, Henrik Forssell, Nicola Gambino, André Joyal, Ivar Rummelhoff, Dana Scott, Thomas Streicher, and especially Alex Simpson. A more thorough accounting of our debts and gratitudes will have to wait for a more polished presentation of these results.

1 Π–Pretopoi

In the following we attempt to follow the nomenclature of Johnstone [8] and [9] as much as possible. In particular, a *cover* is (in all of the categories with which we are concerned) a regular epimorphism. As regards notation, for $f : C \longrightarrow D$, we write $\Delta_f : \mathcal{C}/D \longrightarrow \mathcal{C}/C$ for the pullback functor, Σ_f for its left adjoint and Π_f for its right adjoint.

As mentioned in the introduction we adopt the following definition:

Definition 1.0.1 A Π–pretopos is a locally cartesian closed pretopos. $\lhd$

As an easy consequence of the definition we have the following:

Proposition 1.0.2 *If $\mathcal{R}$ is a Π-pretopos, then $\mathcal{R}$ is Heyting.*

The reader should recall the following theorem which affirms a tight connection between locally cartesian closed categories and dependent type theory (cf. [9] or [7]):

Theorem 1.0.3 (LCCC Soundness and Completeness) *For any judgement in context $\Gamma|\varphi$ of dependent type theory (DTT),*

$$\mathrm{DTT} \vdash \Gamma|\varphi \text{ iff, for every lccc } \mathcal{C}, \mathcal{C} \models \Gamma|\varphi.$$

Since every Π–pretopos is locally cartesian closed we obtain the following:

[3] See [5] for a fuller treatment.

Corollary 1.0.4 (Π–Pretopos Soundness and Completeness) *For any judgement in context $\Gamma|\varphi$ of dependent type theory,*

$$\mathrm{DTT} \vdash \Gamma|\varphi \text{ iff, for every } \Pi\text{–pretopos } \mathcal{R},\ \mathcal{R} \models \Gamma|\varphi.$$

PROOF Soundess is trivial since every Π–pretopos is locally cartesian closed. For completeness notice that if $\mathcal{C}$ is locally cartesian closed, then the Yoneda embedding $y : \mathcal{C} \longrightarrow \widehat{\mathcal{C}}$ preserves all of the locally cartesian closed structure and $\widehat{\mathcal{C}}$ is a Π–pretopos. Suppose that, for all Π–pretopoi $\mathcal{R}$, $\mathcal{R} \models \Gamma|\varphi$. Then, in particular, $\widehat{\mathcal{C}} \models \Gamma|\varphi$ for every LCCC $\mathcal{C}$. But since y is conservative (i.e., reflects isomorphisms) and has the aforementioned properties it follows that $\mathcal{C} \models \Gamma|\varphi$. By the foregoing theorem $\vdash \Gamma|\varphi$. $\dashv$

2 Constructive Set Theories

All of the set theories under consideration are first–order intuitionistic theories in the language $\mathcal{L} := \{\mathsf{S}, \in\}$ where S ('sethood') and $\in$ ('membership') are, respectively, unary and binary predicates. We include S in the language because we intend to allow urelements or non–sets. The majority of the results of this section are to be found, either explicitly or implicitly, in [1] or [4].

Where φ is a formula, $\mathrm{FV}(\varphi)$ denotes the set of free variables of φ. We will freely employ the class notation $\{x|\varphi\}$ which is common set theoretical practice. Frequently it will be efficacious to employ bounded quantification which is defined as usual:

$$\forall x \in y.\varphi(x) := \forall x.x \in y \Rightarrow \varphi(x) \quad \text{and} \quad \exists x \in y.\varphi(x) := \exists x.x \in y \wedge \varphi(x).$$

A formula φ is called Δ_0 if all of its quantifiers are bounded.

Another notational convenience is the introduction of the 'set–many' quantifier Ƨ defined as:

$$\text{Ƨ}x.\varphi \quad := \quad \exists y.(\mathsf{S}(y) \wedge \forall x.(x \in y \Leftrightarrow \varphi)),$$

where $y \notin \mathrm{FV}(\varphi)$. We also write:

$$x \subseteq y \quad := \quad \mathsf{S}(x) \wedge \mathsf{S}(y) \wedge \forall z \in x.z \in y.$$

We write $\mathrm{func}(f, a, b)$ to indicate that f is a functional relation on $a \times b$ (which will exist in any of the set theories we consider):

$$\mathrm{func}(f, a, b) \quad := \quad f \subseteq a \times b \wedge \forall x \in a.\exists! y \in b.(x, y) \in f$$

Finally, for any formula φ, we define:

$$\mathrm{coll}(x \in a, y \in b, \varphi) \quad := \quad (\forall x \in a.\exists y \in b.\varphi) \wedge (\forall y \in b \exists x \in a.\varphi).$$

For the sake of brevity we omit the obvious universal quantifiers in the following axioms and schemata for set theories:

Membership: $x \in a \Rightarrow \mathsf{S}(a)$.

Universal Sethood: $\mathsf{S}(x)$.

Extensionality: $\mathsf{S}(a) \wedge \mathsf{S}(b) \ \wedge (\forall z. z \in a \Leftrightarrow z \in b) \Rightarrow a = b$.

Emptyset: $\text{Ƨ}z.\bot$.

Pairing: $\text{Ƨ}z.z = x \vee z = y$.

Binary Intersection: $\mathsf{S}(a) \wedge \mathsf{S}(b) \Rightarrow \text{Ƨ}z.z \in a \wedge z \in b$.

Union: $\mathsf{S}(a) \wedge (\forall x \in a.\mathsf{S}(x)) \Rightarrow \text{Ƨ}z.\exists x \in a.z \in x$.

Infinity: $\exists a.\mathsf{S}(a) \wedge (\exists x.x \in a \wedge (\forall x \in a)(\mathsf{S}(x) \wedge \exists y \in a.\mathsf{S}(y) \wedge x \in y))$.

$\in$–Induction: $[\forall a.(\mathsf{S}(a) \wedge \forall x \in a.\varphi(x) \Rightarrow \varphi(a))] \Rightarrow \forall a.(\mathsf{S}(a) \Rightarrow \varphi(a))$.

Replacement: $\mathsf{S}(a) \wedge \forall x \in a.\exists! y.\varphi \Rightarrow \text{Ƨ}y.\exists x \in a.\varphi$

Strong Collection: $\mathsf{S}(a) \wedge (\forall x \in a.\exists y.\varphi) \Rightarrow \exists b.(\mathsf{S}(b) \wedge \mathrm{coll}(x \in a, y \in b, \varphi)$.

Exponentiation: $\mathsf{S}(a) \wedge \mathsf{S}(b) \Rightarrow \text{Ƨ}z.\,\mathrm{func}(z, a, b)$.

Subset Collection: $\mathsf{S}(a) \wedge \mathsf{S}(b) \Rightarrow$

$$\exists c.\mathsf{S}(c) \wedge [\forall v.\forall x \in a.\exists y \in b.\varphi \Rightarrow \exists d \in c.\mathsf{S}(d) \wedge \mathrm{coll}(x \in a, y \in d)\varphi].$$

Δ_0–Separation: $\mathsf{S}(a) \Rightarrow \text{Ƨ}z.z \in a \wedge \varphi$, if φ is a Δ_0 formula.

The particular set theories with which we will be primarily concerned are given in Table 1. In Table 1 we employ a solid bullet $\bullet$ to indicate that the axiom in question is one of the axioms of the theory and a hollow bullet $\circ$ to indicate a consequence of the axioms. There are several points worth mentioning

Axioms	**BCST**	**CST**	**CZF**
Membership	$\bullet$	$\bullet$	$\circ$
Extensionality, Pairing, Union	$\bullet$	$\bullet$	$\bullet$
Emptyset	$\bullet$	$\bullet$	$\circ$
Binary Intersection	$\bullet$	$\bullet$	$\circ$
Replacement	$\bullet$	$\bullet$	$\circ$
Δ_0–Separation	$\circ$	$\circ$	$\bullet$
Exponentiation		$\bullet$	$\circ$
Infinity			$\bullet$
$\in$–Induction			$\bullet$
Strong Collection			$\bullet$
Subset Collection			$\bullet$
Universal Sethood			$\bullet$

Table 1: Several Constructive Set Theories

in connection with Table 1. First, **CZF** is conventionally formulated in the language $\{\in\}$ with all of the axioms suitably reformulated. In the present setting this amounts to the addition of Universal Sethood. Secondly, the form of Δ_0–Separation which holds in **BCST** and **CST** is subject to the stipulation that φ is also well–typed in a sense which will be made precise shortly. Finally, the reader should note that although the theories we consider do not include an axiom of infinity the results of this paper are easily extended to theories augmented with (an appropriate version of) Infinity (cf. [15].[4]

We begin by showing that a particularly useful axiom schema holds in **BCST**; namely, *Indexed Union*:

$$\mathsf{S}(a) \wedge (\forall x \in a.\text{Ɂ}y.\varphi) \Rightarrow \text{Ɂ}y.\exists x \in a.\varphi.$$

Lemma 2.0.5 BCST $\vdash$ Indexed Union.

Proof Suppose $\mathsf{S}(a)$ and $\forall x \in a.\text{Ɂ}y.\varphi(x,y)$, then for any $x \in a$ there is a unique b such that $\mathsf{S}(b)$ and $(\forall y)(y \in b \Leftrightarrow \varphi(x,y))$. By Replacement there exists a c such that $\mathsf{S}(c)$ and:

$$c \quad = \quad \{z | \exists x \in a.\mathsf{S}(z) \wedge (\forall y)(y \in z \Leftrightarrow \varphi(x,y))\}.$$

Clearly $\mathsf{S}(y')$ for any $y' \in c$. By Union $\mathsf{S}(\bigcup c)$. Intuitively, we want to show that the class $w := \{z | \exists x \in a.\varphi(x,z)\}$ is a set. The claim then is that $w = \bigcup c$.

To see that this is so suppose that $y \in \bigcup c$. Then there exists a $d \in c$ such that $y \in d$. By the definition of c there exists an $e \in a$ with $(\forall z)(z \in d \Leftrightarrow \varphi(e,z))$. So, since $y \in d$ it follows that $\varphi(e,y)$ and $y \in w$.

Next, suppose that $y \in w$. Then there exists an $e \in a$ such that $\varphi(e,y)$. By the original assumption there exists a set d such that:

$$d \quad := \quad \{z | \varphi(e,z)\}.$$

Also $d \in c$ and since $\varphi(e,y)$ it follows that $y \in d$ and $y \in \bigcup c$. Thus, $\text{Ɂ}y.\exists x \in a.\varphi$, as required. $\dashv$

We now show that, although **BCST** lacks a separation axiom, it is possible to recover some degree of separation. To this end we define:

$$\varphi[a,x]\text{–Sep} := \mathsf{S}(a) \Rightarrow \text{Ɂ}x.(x \in a \wedge \varphi).$$

Here the free variables a and x need not occur in φ. Additionally we say that a formula φ is *simple* when the following, written $!\varphi$, is provable:

$$\text{Ɂ}z.(z = \emptyset \wedge \varphi)$$

and $z \notin \mathrm{FV}(\varphi)$. The intuition behind simplicity is that certain formulas are sufficiently lacking in logical complexity that their truth values are indeed sets. In particular, if $!\varphi$ then we will write t_φ for the set $\{z | z = \emptyset \wedge \varphi\}$ which we call the *truth value* of φ (and, if necessary, we will exhibit the free variable of φ: $t_{\varphi(x)}$). Separation holds for such simple formulae:

[4] These considerations will be addressed in a more complete version of this paper. In any event, the form of infinity will certainly be stronger than the form mentioned above.

Lemma 2.0.6 (Simple Separation) **BCST** $\vdash (\forall x \in a.!\varphi(x)) \Rightarrow \varphi[a,x]$–Sep.

PROOF We will show that, given the assumptions, $\{z|z = x \wedge \varphi(x)\}$ is a set for each $x \in a$. The conclusion then is an easy consequence of Union–Rep. By assumption $\mathsf{S}(a)$ and that for every $x \in a$ the truth value:

$$t_{\varphi(x)} := \{z|z = \emptyset \wedge \varphi(x)\}$$

of $\varphi(x)$ is a set. Suppose $y \in t_{\varphi(x)}$, then $y = \emptyset \wedge \varphi(x)$. But then $\exists! z.z = x \wedge y = \emptyset \wedge \varphi(x)$. By replacement:

$$q := \{z|\exists y \in t_{\varphi(x)}.z = x \wedge y = \emptyset \wedge \varphi(x)\}$$

is a set. But $\exists y \in t_{\varphi(x)}.z = x \wedge y = \emptyset \wedge \varphi(x)$ is equivalent to $z = x \wedge \varphi(x)$ so that $\{z|z = x \wedge \varphi(x)\}$ is a set, as required. ⊣

Lemma 2.0.7 *In* **BCST***:*

1. *If* $\mathsf{S}(a)$ *and* $\forall x \in a.!\varphi(x)$*, then* $!(\exists x \in a.\varphi(x))$ *and* $!(\forall x \in a.\varphi(x))$.
2. $!(a = b)$.
3. $!(x \in a)$*, when* $\mathsf{S}(a)$.
4. *If* $!\varphi$ *and* $!\psi$*, then* $!(\varphi \wedge \psi)$, $!(\varphi \vee \psi)$, $!(\varphi \Rightarrow \psi)$*, and* $!(\neg\varphi)$.
5. *If* $\varphi \vee \neg\varphi$*, then* $!\varphi$.

PROOF As in [4]. ⊣

Corollary 2.0.8 (Δ_0–Separation) *If* φ *is a* Δ_0 *formula in which there are no occurrences of* S *and* $x_1, \ldots, x_n$ *are all of those free variables of* φ *that occur on the right hand side of occurrences of* $\in$*, then:*

$$\mathbf{BCST} \vdash \mathsf{S}(x_1) \wedge \ldots \wedge \mathsf{S}(x_n) \wedge \mathsf{S}(y) \Rightarrow \text{Ƨ} z \in y.\varphi.$$

We now consider quotients of equivalence relations.

Lemma 2.0.9 *If* $\mathsf{S}(a)$ *and* $r \subseteq a \times a$ *is an equivalence relation, then for each* $x \in a$ *the equivalence class:*

$$[x]_r := \{z|z \in a \wedge (x,z) \in r\}$$

is a set.

PROOF Let $x \in a$ be given to show that Ƨ$z.z \in a \wedge (x,z) \in r$. In order to apply Simple Separation let an arbitrary $y \in a$ be given. It is an obvious consequence of part (2) of Lemma 2.0.7 that $\forall z \in r.!(z = (x,y))$. By part (1) of the lemma $!(\exists z \in r.z = (x,y))$. Since we shown that, for all $y \in a$, $!(\exists z \in r.z = (x,y))$ it follows from Simple Separation that Ƨ$y.(y \in a \wedge (\exists z \in r.z = (x,y))$. I.e., $[x]_r$ is a set, as required. ⊣

Lemma 2.0.10 *If* $\mathsf{S}(a)$ *and* $r \subseteq a \times a$ *is an equivalence relation, then the quotient*

$$a/r := \{[x]_r | x \in a\}$$

of a *modulo* r *is a set.*

PROOF This is an easy application of Replacement. $\dashv$

Let "**Sets**" be the category consisting of sets and functions between them in **BCST**, then, by the foregoing lemmas and some obvious facts that we omit, we have the following:

Theorem 2.0.11 BCST *proves that* "**Sets**" *is a Heyting pretopos.*

Now we regard "**Sets**" as the category of sets in **CST**:

Lemma 2.0.12 *For any object* I *of* "**Sets**"*, the category* "**Sets**"$/I$ *is equivalent to* "**Sets**"^{I} *where* I *is regarded as a discrete category.*

PROOF Define F : "**Sets**"$/I \longrightarrow$ "**Sets**"^{I} by:

$$\begin{array}{rcl} X \xrightarrow{f} I & \longmapsto & (X_i)_{i \in I}, \text{ and} \\ h : f \longrightarrow g & \longmapsto & (h_i)_{i \in I}, \end{array}$$

where X_i is the fiber $f^{-1}(i)$ of f over i and:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & I & \end{tikzcd}$$

commutes in "**Sets**". Notice that each X_i is a set by Simple Separation.

Let G : "**Sets**"$^I \longrightarrow$ "**Sets**"$/I$ by:

$$(X_i)_{i \in I} \longmapsto f : X \longrightarrow I,$$

where $X := \coprod X_i$ and, for any $x \in X$, $f(x)$ is the $i \in I$ such that $x \in X_i$. Here $\coprod X_i := \{(x, i) | x \in X_i\}$ is a set by Simple Separation.

It is easily verified that F and G constitute an equivalence of categories. $\dashv$

Given $f : X \longrightarrow Y$ the pullback functor Δ_f : "**Sets**"$/Y \longrightarrow$ "**Sets**"$/X$ serves to reindex a family of sets $(C_y)_{y \in Y}$ as $(C_{f(x)})_{x \in X}$. Note also that given a set I and a family of sets X_i for each $i \in I$, the class $\{X_i | i \in I\}$ is a set by Replacement.

Lemma 2.0.13 *For any map* $f : X \longrightarrow Y$ *in* "**Sets**"*, the pullback functor* Δ_f : "**Sets**"$/Y \longrightarrow$ "**Sets**"$/X$ *has both a left adjoint* Σ_f *and a right adjoint* Π_f.

PROOF We may employ the usual definitions of the adjoints:

$$\begin{array}{ccc} \text{“}\mathbf{Sets}\text{”}^X & \xrightarrow{\Sigma_f} & \text{“}\mathbf{Sets}\text{”}^Y \\ (C_x)_{x\in X} & \longmapsto & (S_y)_{y\in Y}, \end{array}$$

where $S_y := \coprod_{f(x)=y} C_x$, and Π_f:

$$(C_x)_{x\in X} \longmapsto (P_y)_{y\in Y},$$

where $P_y := \prod_{f(x)=y} C_x$. Here the arbitrary product:

$$\prod_{i\in I} X_i := \{f : I \longrightarrow \bigcup_{i\in I} X_i | \forall i \in I.f(i) \in X_i\}$$

is a set. In particular, $\bigcup X_i$ is a set by Union and $(\bigcup X_i)^I$ is a set by Exponentiation. The result follows directly from Lemma 2.0.7 and Simple Separation.⊣

By the foregoing lemmas we have proved:

Theorem 2.0.14 CST *proves that* “**Sets**” *is a* Π*-pretopos.*

3 Predicative Categories of Classes

In this section we introduce the axiomatic theory of categories of classes (as well as several variants of this notion) and derive soundness and completeness results for **BCST** and **CST**. Our approach is related to those developed in [10], [16], [6], [4], and [14].

3.1 AXIOMS FOR CATEGORIES WITH BASIC CLASS STRUCTURE

A *system of small maps* in a positive Heyting category $\mathcal{C}$ is a collection $\mathcal{S}$ of maps of $\mathcal{C}$ satisfying the following axioms:

(S1) $\mathcal{S}$ is closed under composition and all identity arrows are in $\mathcal{S}$.

(S2) If the following is a pullback diagram:

$$\begin{tikzcd} C' \arrow[r, "g'"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\ D' \arrow[r, "g"'] & D \end{tikzcd}$$

and f is in $\mathcal{S}$, then f' is in $\mathcal{S}$.

(S3) All diagonals $\Delta : C \longrightarrow C \times C$ are contained in $\mathcal{S}$.

(S4) If e is a cover, g is in $\mathcal{S}$ and the diagram:

$$\begin{tikzcd} C \arrow[rr, "e"] \arrow[dr, "g"'] & & D \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

commutes, then f is in $\mathcal{S}$.

(S5) If $f : C \longrightarrow A$ and $g : D \longrightarrow A$ are in $\mathcal{S}$, then so is the copair $[f, g] : C+D \longrightarrow A$.

A map f is *small* if it is a member of $\mathcal{S}$ and an object C is small if the canonical map $!_C : C \longrightarrow 1$ is small. Similarly, a relation $R \rightarrowtail C \times D$ is a *small relation* if the composite:

$$R \rightarrowtail C \times D \longrightarrow D$$

with the projection is a small map. Finally, a subobject $A \rightarrowtail C$ is a *small subobject* if $A \rightarrowtail C \times 1$ is a small relation; i.e., provided that A is a small object.

Definition 3.1.1 A *category with basic (predicative) class structure* is a positive Heyting category $\mathcal{C}$ with a system of small maps satisfying:

(P1) For each object C of $\mathcal{C}$ there exists a *(predicative) power object* $\mathcal{P}_s(C)$ and a small *membership relation* $\epsilon_C \rightarrowtail C \times \mathcal{P}_s(C)$ such that, for any D and small relation $R \rightarrowtail C \times D$, there exists a unique map $\rho : D \longrightarrow \mathcal{P}_s C$ such that the square:

$$\begin{tikzcd} R \arrow[r] \arrow[d, tail] & \epsilon_C \arrow[d, tail] \\ C \times D \arrow[r, "1_C \times \rho"'] & C \times \mathcal{P}_s C \end{tikzcd}$$

is a pullback. $\lhd$

As in topos theory we call the unique map ρ in **(P1)** the *classifying map* of R and R the *relation classified by* ρ.

By the definition of small subobjects and small relations there are functors $\mathrm{SSub}_{\mathcal{C}}(-)$ and $\mathrm{SRel}_B(-)$ induced by restricting, for any objects A and B, $\mathrm{Sub}_{\mathcal{C}}(B)$ and $\mathrm{Sub}_{\mathcal{C}}(B \times A)$ to the subposets of small subobjects of B and small relations on $B \times A$, respectively. The content of the small powerobject axiom **(P1)** is then that these functors are representable in the sense that:

$$\begin{aligned} \hom(A, \mathcal{P}_s B) &\cong \mathrm{SRel}_B(A), \text{ and} \\ \hom(1, \mathcal{P}_s B) &\cong \mathrm{SSub}_{\mathcal{C}}(B). \end{aligned}$$

3.2 The Internal Language of Categories with Basic Class Structure

We will now develop some of the properties of the internal language of categories with basic class structure. This approach is influenced by the work of Rummelhoff [14] and will provide a useful stepping stone for deriving further results. In particular, our aim in developing the internal logic explicitly is twofold:

1. By deriving typed versions of the set theoretic axioms with which we are concerned we are able to provide more elegant soundness proofs; for the validity of the untyped axioms ultimately rests on the validity of their typed analogues.

2. Furthermore, we will make some use of the internal language to show that the subcategories of small things have certain category theoretic properties. E.g., if $\mathcal{C}$ is a category with basic class structure, then the subcategory $\mathcal{S}_{\mathcal{C}}$ of small objects is a Heyting pretopos.

More generally, the development of the theory via the internal language allows us to emphasize the contribution of the categorical structure already present in categories with basic class structure and to compare it with the additional structure provided by the move to categories of classes (cf. subsection 3.5 below).

Henceforth we will assume that the ambient category $\mathcal{C}$ is a category with basic class structure. We will denote by π_A the composite:

$$\pi_A : \epsilon_A \rightarrowtail A \times \mathcal{P}_s \longrightarrow \mathcal{P}_s\, A.$$

Throughout we employ infix notation for certain distinguished relations and maps as in the use of $x \epsilon_C y$ for the more cumbersome $\epsilon_C (x, y)$. We abbreviate $\forall x_1 : X_1.\forall x_2 : X_2.\forall \ldots \forall x_n.X_n.\varphi$ by $\forall x_1 : X_1, x_2 : X_2, \ldots, x_n : X_n.\varphi$ and similarly for existential quantifiers. Finally, we write $\forall x \epsilon_C y$ in place of $\forall x : C.x \epsilon_C y$.

Proposition 3.2.1 *1. A relation $R \rightarrowtail C \times D$ is small iff, for some $\rho : D \longrightarrow \mathcal{P}_s\, C$:*

$$\mathcal{C} \models \forall x : C, y : D.R(x,y) \Leftrightarrow x \epsilon_C \rho(y).$$

2. A map $f : C \longrightarrow D$ is small iff, for some $f^{-1} : D \longrightarrow \mathcal{P}_s\, C$:

$$\mathcal{C} \models \forall x : C, y : D.f(x) = y \Leftrightarrow x \epsilon_C f^{-1}(y).$$

Proof Immediate from the definitions of small maps and relations. In particular, the map f^{-1}, which we call the *fiber map*, classifies the graph $\Gamma(f)$ of f. ⊣

Proposition 3.2.2 (Typed Axioms) *The following are true in any category $\mathcal{C}$ with basic class structure:*

Extensionality: *For any object C:*

$$\mathcal{C} \models \forall a, b : \mathcal{P}_s\, C.(\forall x : C.x \in_C a \Leftrightarrow x \in_C b) \Rightarrow a = b.$$

Emptyset: *For each object C there exists a map $\emptyset_C : 1 \longrightarrow \mathcal{P}_s\, C$ such that:*

$$\mathcal{C} \models \forall x : C.x \in_C \emptyset_C \Leftrightarrow \bot.$$

Singleton: *For each object C the singleton map $\{-\}_C$, which is the classifying map for the diagonal $\Delta : C \rightarrowtail C \times C$, is a small monomorphism.*

Binary Union: *For each C there exists a map $\cup_C : \mathcal{P}_s\, C \times \mathcal{P}_s\, C \longrightarrow \mathcal{P}_s\, C$ such that:*

$$\mathcal{C} \models \forall x : C, a, b : \mathcal{P}_s\, C.x \in_C (a \cup_C b) \Leftrightarrow x \in_C a \vee x \in_C b.$$

Product: *For all C and D there exists a map $\times_{C,D} : \mathcal{P}_s C \times \mathcal{P}_s D \longrightarrow \mathcal{P}_s(C \times D)$ such that:*

$$\mathcal{C} \models \forall x : C, y : D, a : \mathcal{P}_s C, b : \mathcal{P}_s D.(x, y) \in_{C \times D} (a \times_{C,D} b) \Leftrightarrow x \in_C a \wedge y \in_D b.$$

Pairing: *For any C there exists a map $\{-,-\}_C : C \times C \longrightarrow \mathcal{P}_s\, C$ such that:*

$$\mathcal{C} \models \forall x, y, z : C.x \in_C \{y, z\}_C \Leftrightarrow x = y \vee x = z.$$

PROOF For Extensionality, let the subobject r be given by the following:

$$[\![a, b : \mathcal{P}_s\, C | (\forall x : C)(x \in_C a \Leftrightarrow x \in_C b)]\!] \overset{r}{\rightarrowtail} \mathcal{P}_s\, C \times \mathcal{P}_s\, C.$$

By **(P1)** there exist subobjects S, S' of $C \times R$ classified by $\pi_1 \circ r$ and $\pi_2 \circ r$, respectively. But by assumption $S = S'$. Notice that r factors through the diagonal Δ iff $\pi_1 \circ r = \pi_2 \circ r$ (recall that Δ is the equalizer of π_2 and π_2). Thus, by **(P1)**, R factors through Δ, as required.

For Emptyset it suffices to notice that $[\![x : C | \bot]\!]$ is small.

For Singleton note that by Proposition 3.2.1 we have that:

$$[\![x, y : C | x \in_C \{y\}]\!] = \Delta,$$

so that if $\mathcal{C} \models \{x\}_C = \{y\}_C$, then $\mathcal{C} \models x = y$. To see that $\{-\}_C$ is small notice that where:

$$\begin{tikzcd} C \arrow[r, "p"] \arrow[d, "\Delta"'] & \in_C \arrow[d, "\in_C"] \\ C \times C \arrow[r, "1_C \times \{-\}_C"'] & C \times \mathcal{P}_s\, C \end{tikzcd}$$

we have $\{-\}_C = \pi_C \circ p$. But p is small since it has a retraction.

Binary Union follows from the fact that, by **(S4)** and **(S5)**, the join of two small subobjects is a small subobject. Product is by **(S2)**. Finally, for Pairing, the map $\{-,-\}_C : C \times C \longrightarrow \mathcal{P}_s\, C$ is the composite $\cup_C \circ (\{-\}_C \times \{-\}_C)$. $\dashv$

The foregoing is a good start, but before we are able to verify that more sophisticated principles (e.g., Replacement) we must first develop several additional properties of the categories in question.

Proposition 3.2.3 *The following are equivalent given* ***(S1)****,* ***(S2)*** *and* ***(P1)*** *(cf. [4]):*

1. ***(S3)****.*
2. *Regular monomorphisms are small.*
3. *If $g \circ f$ is small then f is small.*
4. $\in_C$: $\epsilon_C \rightarrowtail C \times \mathcal{P}_s\, C$ *is a small map.*
5. $[\![x : C, u : \mathcal{P}_s\, C, v : \mathcal{P}_s\, C | x \epsilon_C\, u \wedge x \epsilon_C\, v]\!]$ *is a small relation*
6. *Sections are small.*

Proof For (1)⇒(2) notice that Δ is a regular mono and suppose that $m : A \rightarrowtail B$ is the equalizer of $h, k : B \rightrightarrows C$. Then:

$$\begin{tikzcd} A \arrow[r, "hom=kom"] \arrow[d, tail, "m"'] & C \arrow[d, tail, "\Delta"] \\ B \arrow[r, "\langle h,k \rangle"'] & C \times C \end{tikzcd}$$

is a pullback and m is small by **(S2)**.

To show that (2)⇒(3) suppose regular monos are small and $g \circ f$ is small where:

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

and consider the pullback:

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "g \circ f"'] & C. \end{tikzcd}$$

There is a canonical map $\zeta : A \longrightarrow P$ such that $p_1 \circ \zeta = 1_A$. By **(S1)** f is a small map.

(3)⇒(1) is trivial. Also (3)⇒(4) is trivial. (4)⇒(1) is by **(S2)**. Both (3)⇒(6) and (6)⇒(1) are trivial.

For (4)⇒(5) notice that if $R \rightarrowtail C \times D$ is a small relation and the map $S \rightarrowtail C \times D$ is small, then $R \wedge S$ is a small relation. (5)⇒(1) is by the fact that:

$$\mathcal{C} \models \forall x : C, y : C. x = y \Leftrightarrow \forall z : C. z \,\epsilon_C\, \{x\}_C \wedge z \,\epsilon_C\, \{y\}_C.$$

The reader should be alerted at this point that use of the previous proposition and its corollary will often be made without explicit mention.

Proposition 3.2.4 *$\mathcal{P}_s\,(-)$ is the object part of a covariant endofunctor $\mathcal{P}_s$ on $\mathcal{C}$.*

PROOF As in [10] or [4]. $\dashv$

If $f : C \longrightarrow D$, then we will write $f_! : \mathcal{P}_s\, C \longrightarrow \mathcal{P}_s\, D$ instead of $\mathcal{P}_s\,(f)$.

Proposition 3.2.5 *Where $f : C \longrightarrow D$:*

$$\mathcal{C} \models \forall x : D, a : \mathcal{P}_s\, C.x \;\epsilon_D\; f_!(a) \Leftrightarrow \exists y \;\epsilon_C\; a.f(y) = x.$$

PROOF By Proposition 3.2.4 and the proof of the Yoneda lemma we have:

$$[\![x : D, a : \mathcal{P}_s\, C | x \;\epsilon_D\; f_!(a)]\!] \quad = \quad \exists_{f \times 1_{\mathcal{P}_s\, C}}(\;\epsilon_C\;).$$

Also, in a regular category, given any $\alpha \rightarrowtail X \times Y$ and $f : X \longrightarrow Z$ we have:

$$\exists_{f \times 1_Y}(\alpha) \quad = \quad [\![z : Z, y : Y | \exists x : X.\alpha(x, y) \wedge f(x) = z]\!],$$

as required. $\dashv$

Corollary 3.2.6 *If $m : C \rightarrowtail D$ is monic, then so is $m_! : \mathcal{P}_s\, C \longrightarrow \mathcal{P}_s\, D$. I.e.,*

$$\mathcal{C} \models \forall x, x' : \mathcal{P}_s\, C.m_!(x) = m_!(x') \Rightarrow x = x'.$$

PROOF By Typed Extensionality and the internal language. $\dashv$

Corollary 3.2.7 *If $m : C \rightarrowtail D$ is monic, then:*

$$\begin{tikzcd}
\epsilon_C \arrow[r, tail] \arrow[d, tail] & \epsilon_D \arrow[d, tail] \\
C \times \mathcal{P}_s\, C \arrow[r, tail, "m \times m_!"'] & D \times \mathcal{P}_s\, D
\end{tikzcd}$$

is a pullback.

PROOF Easy. $\dashv$

Proposition 3.2.8 *Every small map $f : C \longrightarrow D$ gives rise to an (internal) inverse image map $f^* : \mathcal{P}_s\, D \longrightarrow \mathcal{P}_s\, C$.*

PROOF As in [10] or [4]. $\dashv$

Proposition 3.2.9 *If $f : C \longrightarrow D$ is a small map, then:*

$$\mathcal{C} \models \forall x : C, a : \mathcal{P}_s\, D.x \;\epsilon_C\; f^*(a) \Leftrightarrow f(x) \;\epsilon_D\; a,$$

where f^* is as above.

PROOF By Proposition 3.2.8 and the proof of the Yoneda lemma f^* corresponds to the small subobject $(f \times 1_{\mathcal{P}_s\, D})^*(\;\epsilon_D\;)$ of $C \times \mathcal{P}_s\, D$, where here the pullback $(f \times 1_{\mathcal{P}_s\, D})^*$ is external. $\dashv$

In the following we write $\subseteq_C$ for the subobject of $\mathcal{P}_s\, C \times \mathcal{P}_s\, C$ given by:

$$\subseteq_C \quad := \quad [\![x : \mathcal{P}_s\, C, y : \mathcal{P}_s\, C | \forall z \in_C x . z \in_C y]\!].$$

From this description of $\subseteq_C$ it easily follows that $\subseteq_C \rightarrowtail \mathcal{P}_s\, C \times \mathcal{P}_s\, C$ is the equalizer of $\pi_1, \cap_C : \mathcal{P}_s\, C \times \mathcal{P}_s\, C \rightrightarrows \mathcal{P}_s\, C$ and that:

$$\mathcal{C} \models \forall x, y : \mathcal{P}_s\, C . x \subseteq_C y \Leftrightarrow x \cap_C y = x.$$

Lemma 3.2.10 *If $f : C \longrightarrow D$ is a small map, then $f_! \dashv f^*$ internally. That is:*

$$\mathcal{C} \models \forall x : \mathcal{P}_s\, C, y : \mathcal{P}_s\, D . f_!(x) \subseteq_D y \Leftrightarrow x \subseteq_C f^*(y).$$

PROOF Easy using the internal language. $\dashv$

Proposition 3.2.11 (Internal Beck–Chevalley Condition) *If $f : C \longrightarrow D$ is a small map and the following diagram is a pullback:*

$$\begin{tikzcd} C' \arrow[r, "g'"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\ D' \arrow[r, "g"'] & D \end{tikzcd}$$

then $f^* \circ g_! = g'_! \circ (f')^*$.

PROOF By the external Beck–Chevalley condition. $\dashv$

3.3 Slicing, Exponentiation and the Subcategory of Small Objects

In this subsection we first show that the structure of categories with basic class structure is preserved under slicing. Next, we show that small objects are exponentiable and introduce the (categorical) *exponentiation* axiom. Finally, we show that the category $\mathcal{S}_\mathcal{C}$ of small objects in a category $\mathcal{C}$ with basic class structure is a Heyting pretopos and, moreover, if $\mathcal{C}$ also satisfies the categorical exponentiation axiom, then $\mathcal{S}_\mathcal{C}$ is a Π–pretopos.

Theorem 3.3.1 *If $\mathcal{C}$ is a category category with basic class structure and D is an object of $\mathcal{C}$, then $\mathcal{C}/D$ has basic class structure.*

PROOF The Heyting category structure of $\mathcal{C}$ is easily seen to be preserved under slicing. Also, the collection $\mathcal{S}_D$ of all maps in $\mathcal{C}/D$ that are small in $\mathcal{C}$ is a system of small maps in $\mathcal{C}/D$.

Where $f : C \longrightarrow D$ is an object in $\mathcal{C}/D$ we define the powerobject $\mathcal{P}_s\,(f : C \longrightarrow D)$ as the composite $p_f : V \rightarrowtail \mathcal{P}_s\, C \times D \longrightarrow D$ where V is defined as follows:

$$V \quad := \quad [\![x : \mathcal{P}_s\, C, y : D | f_!(x) \subseteq_D \{y\}_D]\!].$$

Notice that by the results of the previous section $V = [\![x, y | \forall z \in_C x.f(z) = y]\!]$. Similarly, we define the membership relation ϵ_f as the composite $M \rightarrowtail D \times C \times \mathcal{P}_s\, C \longrightarrow D$ where:

$$M \quad := \quad [\![x : D, y : C, z : \mathcal{P}_s\, C | y \in_C z \wedge \forall x' \in_C z.f(x') = x]\!]. \qquad \dashv$$

We will now show that exponentials D^C exist when C is a small object. We define the exponential in question as a subobject of $\mathcal{P}_s\,(C \times D)$ as follows:

$$D^C \quad := \quad [\![R : \mathcal{P}_s\,(C \times D) | \forall x : C.\exists! y : D.(x, y) \in_{C \times D} R]\!].$$

Lemma 3.3.2 *If C is small, then we have the following special case of the adjunction $- \times C \dashv -^C$:*

$$\frac{C \xrightarrow{f} D}{\overline{\overline{1 \xrightarrow{\ulcorner f \urcorner} D^C}}}\;. \tag{1}$$

PROOF By the internal logic and the fact that $D^C \rightarrowtail \mathcal{P}_s\,(C \times D)$. $\dashv$

Now, using the fact that $\mathcal{C}/E$ has basic class structure and the pullback functor $\Delta_{!_E} : \mathcal{C} \longrightarrow \mathcal{C}/E$ preserves this structure we arrive at the more general lemma:

Lemma 3.3.3 *Where C is a small object we have the following:*

$$\frac{E \times C \xrightarrow{f} D}{\overline{\overline{E \xrightarrow{\ulcorner f \urcorner} D^C}}}\;. \tag{2}$$

Proposition 3.3.4 *Small objects are exponentiable.*

PROOF Using the foregoing lemma one may construct the evaluation map and prove that the object D^C has the requisite universal property. $\dashv$

Proposition 3.3.5 *If $f : C \longrightarrow D$ is a small map, then the pullback functor $\Delta_f : \mathcal{C}/D \longrightarrow \mathcal{C}/C$ has a right adjoint Π_f.*

PROOF Clearly $(f : C \longrightarrow D)$ is a small object in $\mathcal{C}/D$ and, hence, exponentiable there. The existence of the adjoint Π_f then follows as usual. $\dashv$

We may now state the *exponentiation axiom*:

(E) If $f : C \longrightarrow D$ is a small map, then the functor $\Pi_f : \mathcal{C}/C \longrightarrow \mathcal{C}/D$ (which exists by the foregoing proposition) preserves small maps.

Proposition 3.3.6 *In a category with basic class structure satisfying (E) if C and D are both small, then so is D^C.*

PROOF Notice that D^C is $\Pi_{!_C} \circ \Delta_{!_C}(D)$. Moreover, since D is small so is $\Delta_{!_C}(D)$. By **(E)** it follows that $D^C \longrightarrow 1$ is also small. $\dashv$

In the following proposition and theorem we will be concerned with the properties of the full subcategory $\mathcal{S}_\mathcal{C} := \mathcal{S}/1$ of $\mathcal{C}$ consisting of small objects and small maps between them.

Proposition 3.3.7 *Let $\mathcal{C}$ be a category with basic class structure. If $\partial_0, \partial_1 : R \rightrightarrows C \times C$ is an equivalence relation in $\mathcal{S}_\mathcal{C}$, then the coequalizer of ∂_0 and ∂_1 exists in $\mathcal{S}_\mathcal{C}$ and ∂_0, ∂_1 is its kernel pair.*

PROOF We define the coequalizer C/R by:

$$C/R \quad := \quad [\![z : \mathcal{P}_s\, C \mid \exists x : C . \forall y : C . y \in_C z \Leftrightarrow R(x, y)]\!].$$

Notice that since ∂_0 and ∂_1 are small maps so is $\langle \partial_0, \partial_1 \rangle : R \rightarrowtail C \times C$. As such, $\langle \partial_0, \partial_1 \rangle$ is also a small relation and there exists a unique $\alpha : C \longrightarrow \mathcal{P}_s\, C$ such that:

$$\begin{tikzcd} R \arrow[r, "p"] \arrow[d, tail] & \in_C \arrow[d, tail] \\ C \times C \arrow[r, "1 \times \alpha"'] & C \times \mathcal{P}_s\, C \end{tikzcd}$$

is a pullback. That is:

$$\mathcal{C} \models \forall x, y : C . R(x, y) \Leftrightarrow x \in_C \alpha(y). \tag{3}$$

By (3) and Typed Extensionality it follows that C/R is the image of α:

$$\mathrm{im}(\alpha) \quad = \quad [\![z : \mathcal{P}_s\, C \mid \exists x : C . \alpha(x) = z]\!],$$

and, as such, that α factors through $i : C/R \rightarrowtail \mathcal{P}_s\, C$ via a cover $\bar{\alpha}$. Moreover, by **(P1)**, $\bar{\alpha} \circ \partial_0 = \bar{\alpha} \circ \partial_1$ since $\langle \partial_0, \partial_1 \rangle$ is an equivalence relation. Notice that since C is small it follows that $\bar{\alpha}$ is a small map and, by **(S4)**, that C/R is a small object.

Finally, we will show that ∂_0, ∂_1 is the kernel pair of $\bar{\alpha}$; i.e., that:

$$\begin{tikzcd} R \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] & C \arrow[d, "\bar{\alpha}"] \\ C \arrow[r, "\bar{\alpha}"'] & C/R \end{tikzcd}$$

is a pullback. Let an object Z and maps $z_0, z_1 : Z \rightrightarrows C$ be given such that $\bar{\alpha} \circ z_0 = \bar{\alpha} \circ z_1$. Then we also have that $\alpha \circ z_0 = \alpha \circ z_1$. Define a map $\eta : Z \longrightarrow \in_C$ by $\eta := p \circ r \circ z_0$, where r is the 'reflexivity' map. Then we have:

$$\begin{aligned} \in \circ \eta \quad &= \quad \langle \partial_0, \alpha \circ \partial_1 \rangle \circ r \circ z_0 \\ &= \quad \langle z_0, \alpha \circ z_0 \rangle \\ &= \quad (1_C \times \alpha) \circ \langle \partial_0, \partial_1 \rangle. \end{aligned}$$

By the universal property of pullbacks there exists a unique map $\bar{\eta} : Z \longrightarrow R$ with $p \circ \bar{\eta} = \eta$ and $\langle \partial_0, \partial_1 \rangle \circ \bar{\eta} = \langle z_0, z_1 \rangle$. Moreover $\bar{\eta}$ is the unique map from Z to R such that $\partial_0 \circ \bar{\eta} = z_0$ and $\partial_1 \circ \bar{\eta} = z_1$. It follows from the fact that covers coequalize their kernel pairs that $\bar{\alpha}$ is a coequalizer of ∂_0 and ∂_1. It is easily seen that if Z together with z_0 and z_1 are in $\mathcal{S}_\mathcal{C}$, then so is $\bar{\eta}$. $\dashv$

Theorem 3.3.8 *$\mathcal{S}_{\mathcal{C}}$ is a Heyting pretopos. Moreover, if $\mathcal{C}$ satisfies* **(E)**, *then $\mathcal{S}_{\mathcal{C}}$ is a Π–pretopos.*

PROOF By Proposition 3.3.7 $\mathcal{S}_{\mathcal{C}}$ has coequalizers of equivalence relations. It suffices to showt that $\mathcal{S}_{\mathcal{C}}$ is a positive Heyting category. But, this structure is easily seen exist since $\mathcal{C}$ is a positive Heyting category. For instance, to show that $\mathcal{S}_{\mathcal{C}}$ has disjoint finite coproducts note that if C and D are small objects then so is $C + D$ together with the maps $C \longrightarrow C + D$ and $D \longrightarrow C + D$ by **(S5)**. Disjointness and stability are consequences of **(S3)**. Similarly, by the description of $C \times D$ as the pullback of $!_C$ along $!_D$, it follows that $C \times D$ is a small object when C and D are. $\mathcal{S}_{\mathcal{C}}$ is seen to be regular by **(S3)**. Finally, for dual images, let a map $f : C \longrightarrow D$ and a subobject $m : S \rightarrowtail C$ be given in $\mathcal{S}_{\mathcal{C}}$. Consider the subobject $i : \forall_f(m) \rightarrowtail D$. Notice that, in general, if a monomorphism $C \rightarrowtail D$ in a category $\mathcal{C}$ with basic class structure is small, then it is also regular since it is a pullback of the section $\top : 1 \longrightarrow \mathcal{P}_s\, 1$. Moreover since, by Proposition 3.3.5, Π_f exists and is a right adjoint, it follows that i is a small map.

The further result is a consequence of Proposition 3.3.6. $\dashv$

3.4 Typed Union and Replacement

We now show that typed versions of Union and Replacement are valid in categories with basic class structure. To this end, we introduce a typed version of the '$\text{Ƨ}z.\varphi$' notation from above as follows:

$$\text{Ƨ}x : C.\varphi \quad := \quad \exists y : \mathcal{P}_s\, C.\forall x : C.(x \in_C y \Leftrightarrow \varphi),$$

where $y \notin \mathrm{FV}(\varphi)$.

Proposition 3.4.1 *A relation $R \rightarrowtail C \times D$ is small if and only if $\mathcal{C} \models \forall y : D.\text{Ƨ}x : C.R(x, y)$.*

PROOF Suppose $R \rightarrowtail C \times D$ is a small relation and $\rho : D \longrightarrow \mathcal{P}_s\, C$ is the classifying map. Then by Proposition 3.2.1 we have $\mathcal{C} \models \forall y : D.\forall x : C.R(x, y) \Leftrightarrow x \in_C \rho(y)$. The conclusion may be seen to follow from this (use ρ to witness the existential).

For the other direction suppose $\mathcal{C} \models \forall y : D.\text{Ƨ}x : C.R(x, y)$. Then, by Typed Extensionality:

$$\mathcal{C} \models \forall y : D.\exists! z : \mathcal{P}_s\, C.\forall x : C(x \in_C z \Leftrightarrow R(x, y)),$$

and there is a map $\rho : D \longrightarrow \mathcal{P}_s\, C$ with the requisite property. $\dashv$

Proposition 3.4.2 (Typed Union) *For all C:*

$$\mathcal{C} \models \forall a : \mathcal{P}_s\, (\mathcal{P}_s\, C).\text{Ƨ}z : C.\exists x \in_{\mathcal{P}_s\, C} a.z \in_C x.$$

PROOF Let H be defined as:

$$H \quad := \quad [\![x : C, y : \mathcal{P}_s\, C, z : \mathcal{P}_s\, (\mathcal{P}_s\, C) | y \in_{\mathcal{P}_s\, C} z \wedge x \in_C y]\!],$$

and note that the projection:

$$H \rightarrowtail C \times \mathcal{P}_s\, C \times \mathcal{P}_s\, (\mathcal{P}_s\, C) \longrightarrow \mathcal{P}_s\, (\mathcal{P}_s\, C)$$

is small. By **(S4)** it follows that $[\![x : C, z : \mathcal{P}_s\, (\mathcal{P}_s\, C) | \exists y \in_{\mathcal{P}_s\, C} z \wedge x \in_C y]\!]$ is a small relation. We write $\bigcup_C : \mathcal{P}_s\, (\mathcal{P}_s\, C) \longrightarrow \mathcal{P}_s\, C$ for the classifying map. $\dashv$

Proposition 3.4.3 (Typed Replacement) *For all C and D:*

$$\mathcal{C} \models \forall a : \mathcal{P}_s\, C.(\forall x \in_C a.\exists! y : D.\varphi) \Rightarrow (\text{Ɂ}y : D.\exists x \in_C a.\varphi).$$

PROOF Let $a : 1 \longrightarrow \mathcal{P}_s\, C$ be given with $1 \Vdash \forall x \in_C a.\exists! y : D.\varphi$. Let $\alpha \rightarrowtail C$ be the small subobject classified by a. Then the assumption yields a map $f : \alpha \longrightarrow C \longrightarrow D$ such that:

$$\Gamma(f) \quad = \quad [\![x : \alpha, y : D | \varphi(x, y)]\!].$$

Moreover, the image of f is the subobject:

$$I \quad := \quad [\![y : D | \exists x \in_C a.\varphi(x, y)]\!].$$

Since α is a small subobject it follows by **(S4)** that I is also a small subobject. We may now pull the general problem back as usual. $\dashv$

3.5 Summing the Types: Predicative Categories of Classes

A *universal object* in a category $\mathcal{C}$ is an object U of $\mathcal{C}$ such that for any object C there exists a monomorphism $C \rightarrowtail U$ (cf. [16] and [4]). Notice that the monomorphism in question need not be unique. A *basic (predicative) category of classes* is defined to be a category with basic class structure which satisfies:

(U) There exists a universal object U.

Similarly, a *predicative category of classes* is a category of classes which satisfies **(E)**. Categories of classes allow us to interpret untyped first-order theories such as **BCST** which are formulated in the language $\mathcal{L}$ of set theory. This is in contrast to the usual models of set theory in topoi. We will now demonstrate that **BCST** is sound and complete with respect to models in basic categories of classes and that **CST** is sound and complete with respect to models in predicative categories of classes.

In order to interpret the theories in question in basic categories of classes (respectively, predicative categories of classes) we must choose a monomorphism $\iota : \mathcal{P}_s\, U \rightarrowtail U$ (this is because **(U)** is consistent with the existence of multiple monos $\mathcal{P}_s\, U \rightarrowtail U$). An *interpretation* of **BCST** in a basic category of classes $\mathcal{C}$ with is a conventional interpretation $[\![-]\!]$ of the first-order structure $(\in, \mathsf{S})$ extended with the following clauses:

- $[\![\mathsf{S}(x)]\!]$ is defined to be:

$$\mathcal{P}_s\, U \overset{\iota}{\rightarrowtail} U.$$

- $[\![x \in y]\!]$ is interpreted as the subobject:

$$\epsilon_U \overset{\in}{\rightarrowtail} U \times \mathcal{P}_s\, U \overset{1\times\iota}{\rightarrowtail} U \times U.$$

Remark 3.5.1 We write $(\mathcal{C}, U) \models \varphi$ to indicate that φ is satisfied by the interpretation. As above $\mathcal{C} \models \varphi$ indicates that φ is true in the internal language and $Z \Vdash \varphi$ means that Z forces φ.

The following lemma and proposition will allow us to transfer results about the typed internal language to the case of the untyped set theories in question:

Lemma 3.5.2 *If $m : C \rightarrowtail D$ is a small subobject with classifying map $c : 1 \longrightarrow \mathcal{P}_s\, D$ and $r : R \rightarrowtail D \times E$ is a small relation with classifying map $\rho : E \longrightarrow \mathcal{P}_s\, D$ such that $E \Vdash \rho \subseteq_D c$, then there exists a restriction $r' : R' \rightarrowtail C \times E$ of R to C which is a small relation with classifying map $\rho' : E \longrightarrow \mathcal{P}_s\, C$ such that $\rho = m_! \circ \rho'$.*

Proof Let $r' : R' \rightarrowtail C \times E$ be the pullback of $r : R \rightarrowtail D \times E$ along $m \times 1_E$, then, since $!_C$ is small so is $m \times 1_E$. As such, $r' : R' \rightarrowtail C \times E$ is a small relation and there exists a classifying map $\rho' : E \longrightarrow \mathcal{P}_s\, C$.

We use **(P1)** to show that $\rho = m_! \circ \rho'$. In particular, let $p : P \rightarrowtail D \times E$ be the small relation which results by pulling ϵ_D back along $1 \times (m_! \circ \rho')$, then it is a straightforward application of the internal language to show that the following holds:

$$\mathcal{C} \models \forall x : D, y : E.x \,\epsilon_D\, \rho(y) \Leftrightarrow x \,\epsilon_D\, m_! \circ \rho'(y).$$

By **(P1)** it follows that $\rho = m_! \circ \rho'$. ⊣

Proposition 3.5.3 *Suppose $i : \alpha \rightarrowtail C$ is a small subobject with classifying map $a : 1 \longrightarrow \mathcal{P}_s\, C$, then:*

$$\mathcal{P}_s\, \alpha \quad = \quad [\![x : \mathcal{P}_s\, C | x \subseteq_C a]\!].$$

Proof By Corollaries 3.2.6 and 3.2.7 we have $i_! : \mathcal{P}_s\, \alpha \rightarrowtail \mathcal{P}_s\, C$. As such, we need only find a map $\xi : \mathcal{P}_s\, \alpha \longrightarrow \subseteq_C$ such that:

$$\begin{tikzcd} \mathcal{P}_s\, \alpha \arrow[r, "\xi"] \arrow[d, tail, "i_!"'] & \subseteq_C \arrow[d, tail, "j"] \\ \mathcal{P}_s\, C \arrow[r, "1\times a"'] & \mathcal{P}_s\, C \times \mathcal{P}_s\, C \end{tikzcd}$$

is a pullback. We use the description of $\subseteq_C$ as the equalizer of π and $\cap_C$ to show that ξ exists. To this end, notice that, by **(P1)** it suffices to show that:

$$\mathcal{C} \models \forall x : C, y : \mathcal{P}_s\, \alpha.(x \,\epsilon_C\, i_!(y) \wedge x \,\epsilon_C\, a) \Leftrightarrow (x \,\epsilon_C\, i_!(y));$$

for then $\pi_1 \circ (1 \times a) \circ i_! = i_1 = \cap_C \circ (1 \times a) \circ i_!$. This is a straightforward application of the internal language and, by **(P1)** and the universal property of equalizers, there exists a map $\xi : \mathcal{P}_s\, \alpha \longrightarrow \subseteq_C$ such that $j \circ \xi = (1 \times a) \circ i_!$.

To show that the square in question is a pullback suppose given an object Z and maps $l : Z \longrightarrow \mathcal{P}_s\, C$ and $k : Z \longrightarrow \subseteq_C$ such that $j \circ k = (1 \times a) \circ l$. Then l is the classifying map of a small relation $L \rightarrowtail C \times Z$. By Lemma 3.5.2 the restriction of L to α exists with classifying map $\lambda : Z \longrightarrow \mathcal{P}_s\, \alpha$ such that $\lambda : Z \longrightarrow \mathcal{P}_s\, \alpha$. Moreover, λ is the unique map $Z \longrightarrow \mathcal{P}_s\, \alpha$ with this property since $i_!$ is monic. Finally, we have that $\pi_1 \circ (1 \times a) \circ l = \cap_C \circ (1 \times a) \circ l$ so that there exists a unique map $\mu : Z \longrightarrow \subseteq_C$ with $j \circ \mu = (1 \times a) \circ l$. Hence $\mu = k$. Moreover $j \circ (\xi \circ \lambda) = (1 \times a) \circ l$. Therefore, $\xi \circ \lambda = k$, as required. $\dashv$

Corollary 3.5.4 *If $i : \alpha \rightarrowtail C$ and $h : \beta \rightarrowtail D$ are small subobjects with classifying maps $a : 1 \longrightarrow \mathcal{P}_s\, C$ and $b : 1 \longrightarrow \mathcal{P}_s\, D$, respectively, then:*

$$\mathcal{P}_s\, (\alpha \times \beta) \quad = \quad [\![x : \mathcal{P}_s\, (C \times D) | x \subseteq_{C \times D} a \times_{C,D} b]\!].$$

Proof This is a straightforward generalization of the proof of the foregoing proposition using Typed Product. $\dashv$

Lemma 3.5.5 *If $a : 1 \longrightarrow U$ and $1 \Vdash \mathsf{S}(a)$ via some map $\bar{a} : 1 \longrightarrow \mathcal{P}_s\, U$, then:*

$$[\![x | \mathsf{S}(x) \wedge (\forall y)(y \in x \Rightarrow y \in a)]\!] \quad \cong \quad \mathcal{P}_s\, \alpha,$$

where $i : \alpha \rightarrowtail U$ is the small subobject classified by $\bar{a}$ and $\mathcal{P}_s\, \alpha$ is regarded as a subobject of U via $\iota \circ i_!$.

Proof Note that $[\![x, z | (\forall y)(y \in x \Rightarrow y \in z)]\!]$ is the composite:

$$\subseteq_U \overset{j}{\rightarrowtail} \mathcal{P}_s\, U \times \mathcal{P}_s\, U \overset{\iota \times \iota}{\rightarrowtail} U \times U.$$

Using Proposition 3.5.3 the proof is by the following diagram:

$$\begin{tikzcd}
\mathcal{P}_s\, \alpha \arrow[r, tail, "1"] \arrow[d, tail, "i_!"'] & \mathcal{P}_s\, \alpha \arrow[r] \arrow[d, tail, "i_!"'] & \subseteq_U \arrow[d, tail, "j"] \\
\mathcal{P}_s\, U \arrow[r, tail, "1"] \arrow[d, tail, "1"'] & \mathcal{P}_s\, U \arrow[r, "1 \times \bar{a}"] \arrow[d, tail, "\iota"'] & \mathcal{P}_s\, U \times \mathcal{P}_s\, U \arrow[d, tail, "\iota \times \iota"] \\
\mathcal{P}_s\, U \arrow[r, tail, "\iota"'] & U \arrow[r, "1 \times a"'] & U \times U.
\end{tikzcd}$$

Corollary 3.5.6 *If $a, b : 1 \rightrightarrows U$ with $1 \Vdash \mathsf{S}(a) \wedge \mathsf{S}(b)$ via maps $\bar{a}$ and $\bar{b}$, respectively, then:*

$$[\![x | \mathsf{S}(x) \wedge (\forall y)(y \in x \Rightarrow y \in a \times b)]\!] \quad \cong \quad \mathcal{P}_s\, (\alpha \times \beta),$$

where α and β are the small subobjects classified by $\bar{a}$ and $\bar{b}$, respectively, and $\mathcal{P}_s\, (\alpha \times \beta)$ is regarded as a subobject of U.

Theorem 3.5.7 (Soundness of BCST) BCST *is sound with respect to models in basic categories of classes.*

PROOF The Membership axiom is trivial and all of the other axioms follow from the previous results contained in this subsection and the fact that their typed analogues are valid in the internal languages of categories with basic class structure. $\dashv$

Theorem 3.5.8 (Soundness of CST) CST *is sound with respect to models in predicative categories of classes.*

PROOF All that remains to be checked is that $(\mathcal{C}, U) \models$ Exponentiation where $\mathcal{C}$ is a predicative category of classes. To this end, the reader should note that it is possible to eliminate the defined terms (e.g., $a \times b$) which occur in the Exponentiation axiom:

$$\mathsf{S}(a) \wedge \mathsf{S}(b) \Rightarrow \partial z.\, \mathrm{func}(z, a, b)$$

using a combination of the internal language and the fact that **BCST** has been shown to be sound with respect to models in categories of classes.

We will first show that for any $a, b : 1 \longrightarrow U$ factoring through $\iota : \mathcal{P}_s U \rightarrowtail U$ via maps $\bar{a}$ and $\bar{b}$, respectively, the subobject $[\![z|\, \mathrm{func}(z, a, b)]\!]$ is small. By definition there exist small subobjects α and β of U corresponding to $\bar{a}$ and $\bar{b}$.

Since these subobjects are small so is the exponential β^{α} by Proposition 3.3.6 and, by the foregoing lemma and Proposition 3.5.3, it follows that:

$$\beta^{\alpha} \quad = \quad [\![z|z \subseteq a \times b \wedge \forall x \in a.\exists! y \in b.\langle x, y\rangle \in z]\!]. \tag{4}$$

The general result follows from the fact that, given $a, b : Z \rightrightarrows U$ such that $Z \Vdash \mathsf{S}(a) \wedge \mathsf{S}(b)$, we may pull the problem back to $\mathcal{C}/Z$ along $\Delta_{!_Z}$. $\dashv$

Moreover, by constructing the *syntactic categories* $\mathcal{C}_{\mathbf{BCST}}$ and $\mathcal{C}_{\mathbf{CST}}$ as in [16] and [4] we have:

Theorem 3.5.9 (Completeness) *For any formula φ of $\mathcal{L}$, if $(\mathcal{C}, U) \models \varphi$ for all models $(\mathcal{C}, U)$ with $\mathcal{C}$ a category of classes, then* **BCST** $\vdash \varphi$*. Similarly, if $(\mathcal{C}, U) \models \varphi$ for all models with $\mathcal{C}$ a predicative category of classes, then* **CST** $\vdash \varphi$.

4 The Ideal Completion of a Pretopos

We begin by reviewing the category theoretic background necessary to understand the method for constructing models of **PIST** 'over' Π–pretopoi $\mathcal{R}$ which is introduced in the next section. We will be interested in several categories in this section; two of which should be familiar: sheaves (for the coherent topology consisting of finite epi families) and presheaves. We will be concerned with two subcategories of sheaves $\mathbf{Sh}(\mathcal{C})$; namely, the *inductive completion* $\mathbf{Ind}(\mathcal{C})$ of $\mathcal{C}$ and the *ideal completion* $\mathbf{Idl}(\mathcal{C})$ of $\mathcal{C}$. Although the ideal completion is more significant from our perspective it will be useful to keep in mind that the inductive completion has a more illustrious history in category theory (originating, as far as the authors are aware, in SGA4 [2]). The basic picture is this:

$$\mathcal{C} \hookrightarrow \mathbf{Idl}(\mathcal{C}) \hookrightarrow \mathbf{Ind}(\mathcal{C}) \hookrightarrow \mathbf{Sh}(\mathcal{C}) \hookrightarrow \widehat{\mathcal{C}},$$

where $\widehat{\mathcal{C}}$ is presheaves on $\mathcal{C}$. References for this section include [4], [14] and [5].

4.1 Some Useful Properties

Definition 4.1.1 A diagram $D : \mathcal{I} \longrightarrow \mathcal{C}$ is an *ideal diagram* on $\mathcal{C}$ provided that $\mathcal{I}$ is a small filtered category such that for every map $\alpha : i \longrightarrow j$ in $\mathcal{I}$ the map $D(\alpha)$ is a monic. An *ideal* I on a category $\mathcal{C}$ is an object of $\widehat{\mathcal{C}}$ which is a colimit of an ideal diagram of representables. $\lhd$

Using this definition, the *ideal completion* $\mathbf{Idl}(\mathcal{C})$ of a category $\mathcal{C}$ is the full subcategory of $\widehat{\mathcal{C}}$ consisting of ideals. Indeed, since every ideal is a sheaf (cf. [5]), $\mathbf{Idl}(\mathcal{C})$ is also a subcategory of $\mathbf{Sh}(\mathcal{C})$. We state the following theorem from [5] for the record:

Theorem 4.1.2 *If $\mathcal{R}$ is a Heyting pretopos, then $\mathbf{Idl}(\mathcal{R})$ is a Heyting category and $y : \mathcal{R} \longrightarrow \mathbf{Idl}(\mathcal{R})$ preserves all limits existing in $\mathcal{R}$.*

The following theorem is a highly useful tool for the study of ideal categories:

Theorem 4.1.3 (Representable Compactness) *In $\widehat{\mathcal{C}}$, where X is a colimit $\varinjlim_i yD_i$ of representables, any map $f : yC \longrightarrow X$ factors through at least one of the canonical maps $l_i : yD_i \longrightarrow X$.*

Proof Let X, yC and f be given as in the statement of the theorem and let $P := \coprod_i yD_i$ be the coproduct of the yD_i. Then there is a canonical map $\xi : P \longrightarrow X$ such that each $l_j : yD_j \longrightarrow X$ factors as $l_j = \xi \circ \iota_j$, where $\iota_j : yD_j \longrightarrow P$ is the coproduct inclusion. To show that ξ is an epimorphism suppose that there is a map m such that:

$$\begin{tikzcd} yD_j \arrow[rr, "m"] \arrow[dr, "l_j"'] & & yD_{j'} \arrow[dl, "l_{j'}"] \\ & X & \end{tikzcd}$$

commutes and assume that there are maps $h, k : X \longrightarrow Z$ such that $h \circ \xi = k \circ \xi$. We have trivially that $h \circ l_{j'} \circ m = h \circ l_j$ (and similarly for k) and:

$$\begin{aligned} h \circ l_{j'} \circ m &= h \circ \xi \circ \iota_{j'} \circ m \\ &= k \circ \xi \circ \iota_{j'} \circ m \\ &= k \circ l_{j'} \circ m. \end{aligned}$$

Therefore, by the universal property of colimits, $h = k$.

Since representables are projective it follows that there is a map $\zeta : yC \longrightarrow P$ such that $\xi \circ \zeta = f$. It follows that ζ factors through some $\iota_j : yD_j \longrightarrow P$ via a map η. But then:

$$\begin{aligned} f &= \xi \circ \zeta \\ &= \xi \circ \iota_j \circ \eta \\ &= l_j \circ \eta, \end{aligned}$$

as required. ⊣

Of course, when X is an ideal any such factorization will occur also in $\mathbf{Idl}(\mathcal{R})$ since $\mathbf{Idl}(\mathcal{R})$ is a full subcategory of $\widehat{\mathcal{R}}$.

Where $\mathcal{C}$ and $\mathcal{D}$ are categories with colimits of ideal diagrams, a functor $F : \mathcal{D} \longrightarrow \mathcal{C}$ is said to be *continuous* provided that it preserves colimits of ideal diagrams.

Proposition 4.1.4 *If $\mathcal{C}$ is a category with colimits of ideal diagrams and $\mathcal{R}$ is any category, then any functor $F : \mathcal{R} \longrightarrow \mathcal{C}$ which preserves monomorphisms extends to a functor $\bar{F} : \mathbf{Idl}(\mathcal{R}) \longrightarrow \mathcal{C}$ which is continuous and unique up to natural isomorphism. $\mathbf{Idl}(\mathcal{R})$ is the free completion of $\mathcal{R}$ with colimits of ideal diagrams in this sense:*

$$\begin{tikzcd} \mathcal{R} \arrow[r, hook, "y"] \arrow[dr, "F"'] & \mathbf{Idl}(\mathcal{R}) \arrow[d, dotted, "\bar{F}"] \\ & \mathcal{C}. \end{tikzcd}$$

Proof Let $\bar{F}(\varinjlim_{i \in \mathcal{I}} yC_i) := \varinjlim_{i \in \mathcal{I}} F(C_i)$. Notice that the assumption that F preserves monomorphisms is necessary so that the colimit $\varinjlim_i F(C_i)$ exists in $\mathcal{C}$. ⊣

4.2 Class Structure in $\mathbf{Idl}(\mathcal{R})$

Let $\mathcal{R}$ be a pretopos, then based on the intuition of small maps as those maps with small fibers we adopt the following definition:

Definition 4.2.1 A map $f : X \longrightarrow Y$ in $\mathbf{Idl}(\mathcal{R})$ is *small* provided that it pulls representables back to representables. I.e., f is small provided that, for every $yC \longrightarrow Y$ the object P in the following pullback diagram is a representable:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & yC \arrow[d] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

As such, an object is *small* if and only if it is a representable since the terminal object of $\mathbf{Idl}(\mathcal{R})$ is $y1$. ◁

We will make use a characterization proposed by André Joyal of the objects of $\mathbf{Idl}(\mathcal{R})$. Call an object X of $\mathbf{Sh}(\mathcal{R})$ *separated* it its diagonal $\Delta : X \longrightarrow X \times X$ is a small map.

Theorem 4.2.2 (The Joyal Condition) *Let $\mathcal{R}$ be a pretopos, then, for any sheaf F in $\mathbf{Sh}(\mathcal{R})$, the following are equivalent:*

1. *F is an ideal.*
2. *F is separated.*
3. *For all arrows $f : yC \longrightarrow F$ with representable domain, the image of f is representable; i.e., $f : yC \longrightarrow\!\!\triangleright yD \rightarrowtail F$ for some yD.*

PROOF See [5]. $\dashv$

As shown in [5], if $\mathcal{R}$ is a pretopos, then the small map axioms from 3.1 are satisfied:

Proposition 4.2.3 *Let $\mathcal{R}$ be a pretopos, then $\mathbf{Idl}(\mathcal{R})$ satisfies axioms (S1)–(S5).*

PROOF **(S1)** and **(S2)** are easy. **(S3)** is by the Joyal Condition.

For **(S4)** suppose we have:

$$\begin{tikzcd} X \arrow[rr, "e"] \arrow[dr, "f\circ e"'] & & Y \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$

with e a cover and $f \circ e$ small. Let $i : yC \longrightarrow Z$ be given and consider the diagram:

$$\begin{tikzcd} yC' \arrow[r, "e'"] \arrow[d] & P \arrow[r, "f'"] \arrow[d] & yC \arrow[d] \\ X \arrow[r, "e"'] & Y \arrow[r, "f"'] & Z \end{tikzcd}$$

where both squares are pullbacks (as is the outer rectangle). Then it follows that P is representable.

For **(S5)** notice that the pullback of $yC \longrightarrow Z$ along $[f, g] : X + Y \longrightarrow Z$ is the coproduct $f^*(yC) + g^*(yC)$ which is representable since both $f^*(yC)$ and $g^*(yC)$ are representable. $\dashv$

We will strengthen this result by showing that, where $\mathcal{R}$ is a Heyting pretopos, the category $\mathbf{Idl}(\mathcal{R})$ is a category with basic class structure. In order to motivate the definition of the (predicative) powerobjects $\mathcal{P}_s\,(X)$ in $\mathbf{Idl}(\mathcal{R})$ suppose that the indexing category is a topos $\mathcal{E}$ and consider a provisional definition of the powerobject of an object yC in $\widehat{\mathcal{E}}$ as follows:

$$\mathcal{P}_s\,(yC) \quad := \quad y(\mathcal{P}(C)),$$

where $\mathcal{P}(C)$ is the usual (topos) powerobject of C in $\mathcal{E}$. Then we have $\mathcal{P}_s\,(yC) \cong y(\Omega^C)$ and at an object E in $\mathcal{E}$:

$$\begin{aligned} \mathcal{P}_s\,(yC)(E) &\cong y(\Omega^C)(E) \\ &= \hom_{\mathcal{E}}(E, \Omega^C) \\ &\cong \hom_{\mathcal{E}}(E \times C, \Omega) \\ &\cong \mathrm{Sub}_{\mathcal{E}}(E \times C). \end{aligned}$$

Dropping both the assumption that the indexing category is a topos and that we are working in presheaves, we adopt the following provisional definition of the small powerobject of yC in $\mathbf{Idl}(\mathcal{R})$:

$$\mathcal{P}_s\,(yC) \quad := \quad \mathrm{Sub}_{\mathcal{R}}(- \times C).$$

We then extend $\mathcal{P}_s\,(-)$ continuously to ideals; i.e. where $X = \varinjlim_i yC_i$ is an arbitrary ideal:

$$\mathcal{P}_s\,(X) \quad := \quad \varinjlim_i \mathcal{P}_s\,(yC_i).$$

We will show that this definition of $\mathcal{P}_s\,(X)$ is justified by first showing that there is a functor $\mathrm{Sub}^{\mathcal{I}}_{\mathcal{R}} : \mathcal{R} \longrightarrow \mathbf{Idl}(\mathcal{R})$ which takes C to $\mathrm{Sub}_{\mathcal{R}}(- \times C)$ and which preserves monomorphisms. Then it will be possible to apply Proposition 4.1.4 to arrive at an extension $\mathcal{P}_s\ : \mathbf{Idl}(\mathcal{R}) \longrightarrow \mathbf{Idl}(\mathcal{R})$ which will be seen to be a powerobject functor in the sense of satisfying **(P1)**.

4.3 $\mathcal{P}_s$ IS AN IDEAL

Lemma 4.3.1 *If $\mathcal{R}$ is a pretopos and C is an object of $\mathcal{R}$, then the purported powerobject functor $\mathcal{P}_s\,(yC) := \mathrm{Sub}_{\mathcal{R}}(- \times C)$ is a sheaf.*

PROOF Notice that $\mathcal{P}_s\,(yC)(0) \cong \{*\}$ and, since coproducts in $\mathcal{R}$ are stable, $\mathcal{P}_s(yC)(A{+}B) \cong \mathcal{P}_s(yC)(A){\times}\mathcal{P}_s(yC)(B)$. Suppose $f : A \longrightarrow\!\!\triangleright B$ is a cover and let $h, k : Z \rightrightarrows \mathrm{Sub}_{\mathcal{R}}(B{\times}C)$ be given such that $\mathrm{Sub}_{\mathcal{R}}(f{\times}C){\circ}h = \mathrm{Sub}_{\mathcal{R}}(f{\times}C){\circ}k$. Then, for any $z \in Z$, $h(z), k(z) \in \mathrm{Sub}_{\mathcal{R}}(B{\times}C)$ and the pullback P of $h(z)$ along $f \times 1_C$ is also the pullback of $k(z)$ along $f \times 1_C$. But covers are preserved under pullback in $\mathcal{R}$ so that $h(z) = k(z)$ by the uniqueness of image factorizations. $\dashv$

Proposition 4.3.2 *If $\mathcal{R}$ is a Heyting pretopos and C is an object of $\mathcal{R}$, then the small powerobject functor $\mathcal{P}_s\,(yC)$ is an ideal.*

PROOF Since $\mathcal{R}$ is effective it suffices to show that $\mathcal{P}_s\,(yC)$ is separated. To that end let $yD \longrightarrow \mathcal{P}_s\,(yC) \times \mathcal{P}_s\,(yC)$ be given and consider the following diagram:

$$\begin{tikzcd}
 & yD \arrow[d, "i"] & \\
\mathcal{P}_s\,(yC) \arrow[r, tail, "\Delta"] & \mathcal{P}_s\,(yC) \times \mathcal{P}_s\,(yC) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathcal{P}_s\,(yC)
\end{tikzcd}$$

We will show that the equalizer of $\pi_1 \circ i$ and $\pi_2 \circ i$ is representable.

By the Yoneda lemma there are subobjects $\alpha, \beta \in \mathrm{Sub}_{\mathcal{R}}(D \times C)$ corresponding to $\pi_1 \circ i$ and $\pi_2 \circ i$, respectively. We want to find some H and $h : H \longrightarrow D$ in $\mathcal{R}$ such that the result of pulling α back along $h \times 1_C$ is

the same as the result of pulling β back along $h \times 1_C$. Define a subobjects G and H of $D \times C$ and D, respectively, as follows:

$$G \quad := \quad [\![x, y | \alpha(x, y) \Leftrightarrow \beta(x, y)]\!],$$

and:

$$\begin{aligned} H &:= \forall_{\pi_D}(G) \\ &= [\![x|(\forall z)(\alpha(x,z) \Leftrightarrow \beta(x,z))]\!], \end{aligned}$$

where π_D is the projection $D \times C \longrightarrow D$. Finally, let $h : H \rightarrowtail D$.

To see that α and β both pull back to the same thing along $h \times 1_C$ notice that, where $\bar{\alpha}$ is the pullback of α along $h \times 1_C$ and $\bar{\beta}$ is similarly defined:

$$\begin{aligned} \bar{\alpha} &= [\![x,y|\alpha(x,y) \wedge (\forall z)(\alpha(x,z) \Leftrightarrow \beta(x,z))]\!] \\ &= [\![x,y|\beta(x,y) \wedge (\forall z)(\alpha(x,z) \Leftrightarrow \beta(x,z))]\!] \\ &= \bar{\beta}. \end{aligned}$$

So, $\pi_1 \circ i \circ yh = \pi_2 \circ i \circ yh$.

To see that yH is the equalizer suppose given some $\eta : X \longrightarrow yD$ with $\pi_1 \circ i \circ \eta = \pi_2 \circ i \circ \eta$. It suffices to assume that X is representable, so suppose $X \cong yE$. Consider the image factorization yE' of η:

$$\begin{tikzcd} yE \arrow[rr, "ye'"] \arrow[dr, "\eta"'] & & yE' \arrow[dl, "ye"] \\ & yD & \end{tikzcd}$$

Notice that $\pi_1 \circ i \circ ye = \pi_2 \circ i \circ ye$ since ye' is a cover. That is, it suffices to consider monomorphisms m into yD with $\pi_1 \circ i \circ m = \pi_2 \circ i \circ m$. In particular, if α and β pull back to the same thing along $\eta \times 1_C$, then they already are the same when pulled back along $e \times 1_C$. Let ϵ denote the result of pulling α, β back along $e \times 1_C$.

We will now show that $E' \overset{e}{\rightarrowtail} D$ factors through $H \overset{h}{\rightarrowtail} D$ in $\mathcal{R}$. Note that:

$$\begin{aligned} E' \leq H \text{ in } \mathrm{Sub}_{\mathcal{R}}(D) \quad &\text{iff} \quad \pi_D^*(E') \leq G \text{ in } \mathrm{Sub}_{\mathcal{R}}(D \times C), \\ &\text{iff} \quad \pi_D^*(E') \leq \alpha \Rightarrow \beta \text{ and } \leq \beta \Rightarrow \alpha, \\ &\text{iff} \quad \alpha \wedge \pi_D^*(E') \leq \beta \text{ and } \beta \wedge \pi_D^*(E') \leq \alpha. \end{aligned}$$

But $\alpha \wedge \pi_D^*(E') = \epsilon = \beta \wedge \pi_D^*(E')$ is $\leq \alpha$ and $\leq \beta$ by definition.

So there exists a map $\bar{e} : E' \longrightarrow H$ such that $h \circ \bar{e} = e$. To show $\bar{e} \circ e'$ is the unique map from E making η factor through H suppose that $f : E \longrightarrow H$ and $h \circ f = \eta$. By the uniqueness of image factorizations it follows that $f = \bar{e} \circ e'$.$\dashv$

Lemma 4.3.3 *The functor* $\mathrm{Sub}^r_{\mathcal{R}} : \mathcal{R} \longrightarrow \mathbf{Idl}(\mathcal{R})$ *defined by* $\mathrm{Sub}^r_{\mathcal{R}}(C) := \mathrm{Sub}_{\mathcal{R}}(- \times C)$ *preserves monomorphisms.*

Proof A map $f : D \longrightarrow C$ induces a natural transformation $\varphi : \mathrm{Sub}_{\mathcal{R}}(- \times D) \longrightarrow \mathrm{Sub}_{\mathcal{R}}(- \times C)$ given at an object E of $\mathcal{R}$ by:

$$\begin{aligned} S \in \mathrm{Sub}_{\mathcal{R}}(E \times D) \quad &\overset{\varphi_E}{\longmapsto} \quad S' \in \mathrm{Sub}_{\mathcal{R}}(E \times C), \textit{where} \\ S' \quad &:= \quad (1_E \times f)_!(S). \end{aligned}$$

As such, we define $\mathrm{Sub}^r_{\mathcal{R}}(f) := \varphi$. Notice that φ is natural since $\mathcal{R}$ satisfies the Beck–Chevalley condition.

If f is monic, then each component φ_E is monic and, by the Yoneda lemma, φ is monic (since the monomorphisms, like other limits, in $\mathbf{Idl}(\mathcal{R})$ agree with those in $\widehat{\mathcal{R}}$). ⊣

Definition 4.3.4 For any object $X = \varinjlim_i yC_i$ of $\mathbf{Idl}(\mathcal{R})$, where $\mathcal{R}$ is a Heyting pretopos, we have by Proposition 4.1.4 that there is a unique functor $\mathcal{P}_s : \mathbf{Idl}(\mathcal{R}) \longrightarrow \mathbf{Idl}(\mathcal{R})$ with:

$$\begin{aligned} \mathcal{P}_s\,(X) &\cong \mathcal{P}_s\,(\varinjlim_i yC_i) \\ &\cong \varinjlim_i \mathrm{Sub}^r_{\mathcal{R}}(C_i) \\ &= \varinjlim_i \mathrm{Sub}_{\mathcal{R}}(- \times C_i). \end{aligned}$$ ◁

4.4 $\mathcal{P}_s\,(X)$ IS A POWEROBJECT

We will now show that the axiom **(P1)** holds in $\mathbf{Idl}(\mathcal{R})$ where $\mathcal{R}$ is a Heyting pretopos. It will be more efficient to break the proof into several steps. Also, notice that we write $\in_X$ for the membership relation in $\widehat{\mathcal{R}}$ and ϵ_X for the membership relation in $\mathbf{Idl}(\mathcal{R})$. Similarly, we write $\mathcal{P}X$ for the power object in $\widehat{\mathcal{R}}$ and $\mathcal{P}_s\,X$ for the small power object in $\mathbf{Idl}(\mathcal{R})$.

Lemma 4.4.1 *Given any small relation $R \overset{r}{\rightarrowtail} X \times Y$ in $\mathbf{Idl}(\mathcal{R})$ there exists a unique classifying map $\hat{r} : Y \longrightarrow \mathcal{P}_s\,X$.*

PROOF First consider the case where $R \rightarrowtail yC \times yD$. Then in $\widehat{\mathcal{R}}$ both of the following squares (and the outer rectangle):

$$\begin{tikzcd} \epsilon_{yC} \arrow[r, tail] \arrow[d, tail] & \in_{yC} \arrow[d, tail] \\ yC \times \mathcal{P}_s\,yC \arrow[r, tail, "1\times i"'] \arrow[d] & yC \times \mathcal{P}yC \arrow[d] \\ \mathcal{P}_s\,yC \arrow[r, tail, "i"'] & \mathcal{P}yC \end{tikzcd}$$

are pullbacks where $\in_{yC}$ and $\mathcal{P}yC$ are the presheaf membership and powerobject relations and i is the inclusion of $\mathcal{P}_s\,yC$ into $\mathcal{P}yC$ ($\mathcal{P}_s\,yC$ is, by definition, a subfunctor of $\mathcal{P}_s\,yC$). Notice that R is representable since r is a small relation. In particular, $R = yE$ for some object E of $\mathcal{R}$ and $r = ye$. So, using the 'twist' isomorphism $\tilde{}: C \times D \cong D \times C$, we have a relation $\tilde{e} : E \rightarrowtail D \times C$. By the Yoneda lemma such an element corresponds to a map $\hat{r} : yD \longrightarrow \mathcal{P}_s\,yC$.

We will now show that the canonical classifying map $\rho : yD \longrightarrow \mathcal{P}yC$ in $\widehat{\mathcal{R}}$

factors through $\hat{r}$. I.e., we show that:

$$\begin{tikzcd} yD \arrow[rr, "\hat{r}"] \arrow[dr, "\rho"'] & & \mathcal{P}_s\, yC \\ & \mathcal{P} yC \arrow[ur, tail, "i"'] & \end{tikzcd}$$

commutes. Notice that, by the two pullbacks lemma, this will suffice to show that $\hat{r}$ is a classifying map for R in $\mathbf{Idl}(\mathcal{R})$. By the proof of the Yoneda lemma the action of $\hat{r}$ on a given member f of $yD(F)$ is:

$$f \longmapsto \mathcal{P}_s\,(yC)(f)(\tilde{e}).$$

But, $\rho_F(f) = (yf \times 1_{yC})^*(y\tilde{e}) = i(\mathcal{P}_s\,(yC)(f)(\tilde{e}))$.

For uniqueness suppose that $q : yD \longrightarrow \mathcal{P}_s\, yC$ such that:

$$\begin{tikzcd} yE \arrow[r] \arrow[d, tail] & \epsilon_{yC} \arrow[d, tail] \\ yC \times yD \arrow[r, "1\times q"'] & yC \times \mathcal{P}_s\, yC \end{tikzcd}$$

is a pullback. Then, in $\widehat{\mathcal{R}}$, ye is the pullback of $\in_{yC}$ along $i \circ q$ and along $i \circ \hat{r} = \rho$. Since ρ is unique with this property it follows that $i \circ \hat{r} = i \circ q$ and, since i is monic, $q = \hat{r}$.

Now, for any ideal $X \cong \varinjlim_i yC_i$ and small relation $r : R \rightarrowtail X \times yD$, R must be representable since the projection:

$$R \rightarrowtail X \times yD \longrightarrow yD$$

is small. I.e., $R \cong yE$ for some E. By Representable Compactness there exists then a factorization of r:

$$R \rightarrowtail yC_i \times yD \rightarrowtail X \times yD$$

for some i. Thus indeed $\mathrm{SRel}_X \cong \varinjlim_i \mathrm{SRel}_{yC_i}$. ⊣

Lemma 4.4.2 *For any ideal X, ϵ_X is a small relation.*

Proof We verify this for the case where X is a representable yC. Let $yD \rightarrowtail \mathcal{P}_s\, yC$ be given. Then there is a $r : R \rightarrowtail C \times D$ in $\mathcal{R}$ such that:

$$\begin{tikzcd} yR \arrow[r, "\pi \circ yr"] \arrow[d] & yD \arrow[d] \\ \epsilon_{yC} \arrow[r, "\pi_{yC}"'] & \mathcal{P}_s\, yC \end{tikzcd}$$

is a pullback, as required. ⊣

Corollary 4.4.3 *Any relation $R \rightarrowtail X \times Y$ such that there exists a unique classifying map $\rho : Y \longrightarrow \mathcal{P}_s\, X$ is a small relation.*

PROOF By **(S2)** and the fact that ϵ_X is a small relation. $\dashv$

Putting the foregoing together we have the following proposition:

Proposition 4.4.4 *If $\mathcal{R}$ is a Heyting pretopos and $X \cong \varinjlim_i yC_i$ is an object of* $\mathbf{Idl}(\mathcal{R})$, *then $\mathcal{P}_s(X) = \varinjlim_i \mathrm{Sub}_{\mathcal{R}}(- \times C_i)$ is a small powerobject.*

Moreover, when combined with the fact that axioms **(S1)**–**(S5)** are satisfied in pretopoi we have shown the following:

Theorem 4.4.5 *If $\mathcal{R}$ is a Heyting pretopos, then* $\mathbf{Idl}(\mathcal{R})$ *is a category with basic class structure.*

Remark 4.4.6 It should be mentioned that Alex Simpson was the first to realize that it is sufficient for $\mathcal{R}$ to be a Heyting pretopos in order for $\mathbf{Idl}(\mathcal{R})$ to have basic class structure. Moreover, Simpson also was the first to give a proof of Proposition 4.3.2 (the main difference between his proof and the one given in this paper is that his proof does not make use of the Joyal Condition).

4.5 EXPONENTIATION

We now extend the results of the preceding subsection by showing that if $\mathcal{R}$ is a Π–pretopos, then $\mathbf{Idl}(\mathcal{R})$ satisfies **(E)**. First we need the following useful fact:

Proposition 4.5.1 *If $\mathcal{C}$ is a small category and P is an object of* $\mathbf{Idl}(\mathcal{C})$, *then:*

$$\mathbf{Idl}(\mathcal{C})/P \;\simeq\; \mathbf{Idl}(\int_{\mathcal{C}} \mathcal{P}).$$

PROOF It is well known (cf. exercise 8 on p. 157 of [11]) that $\widehat{\mathcal{C}}/P \simeq \widehat{\int_{\mathcal{C}} \mathcal{P}}$. In particular, there are two functors $R : \widehat{\mathcal{C}}/P \longrightarrow \widehat{\int_{\mathcal{C}} \mathcal{P}}$ and $L : \widehat{\int_{\mathcal{C}} \mathcal{P}} \longrightarrow \widehat{\mathcal{C}}/P$ such that $L \dashv R$ and the two maps are pseudo–inverse to one another. These functors are defined as follows:

- $R(\eta : F \longrightarrow P)$ is a functor given by:

  $$(c, C) \;\longmapsto\; \hom_{\widehat{\mathcal{C}}/P}(\tilde{c} : yC \longrightarrow P, \eta : F \longrightarrow P),$$

  where $\tilde{c}$ is the map in $\widehat{\mathcal{C}}$ corresponding to the element $c \in P(C)$ by the Yoneda lemma.

- $L(F) := \varinjlim_{\mathcal{J}} \pi \circ i$ where $\mathcal{J} := \int_{\int_{\mathcal{C}} P}$, $i : \int_{\mathcal{C}} P \longrightarrow \widehat{\mathcal{C}}/P$ is the map taking an object (c, C) to the corresponding $\tilde{c} : yC \longrightarrow P$ as above and π is the projection from the category of elements.

We begin by showing that if $(\eta : F \longrightarrow P)$ is an object of $\mathbf{Idl}(\mathcal{C})/P$, then $R(P)$ is isomorphic to an object of $\mathbf{Idl}(\int_{\mathcal{C}} \mathcal{P})$. Let η be given as mentioned. Then, since F is an ideal we have $F \cong \varinjlim_{\mathcal{I}} yD_i$ with maps $\mu_i : yD_i \longrightarrow F$ making up the cocone.

We define a functor $G : \mathcal{I} \longrightarrow \int_C P$ such that $\varinjlim_{\mathcal{I}} yG_i \cong R(\eta)$ and $\varinjlim_{\mathcal{I}} yG_i$ is an object of $\mathbf{Idl}(\int_C \mathcal{P})$. Let $G(i) := \widetilde{\eta \circ \mu_i}$ be the object corresponding via the Yoneda lemma to $\eta \circ \mu_i$. Given $f : i \longrightarrow j$ in $\mathcal{I}$, let $G(f) := D(f)$. G is easily seen to be functorial.

Next, let $T := \varinjlim_{\mathcal{I}} yG$. We now define an isomorphism $\varphi : R(\eta) \longrightarrow T$. If $f \in R(\eta)(c, C)$ then we have $f : yC \longrightarrow F$. But using Representable Compactness there exists an i together with a map $yl : yC \longrightarrow yD_i$ such that $\mu_i \circ yl = f$. Now, an element of $T(c, C)$ is an equivalence class $[g : C \longrightarrow D_i]_\sim$ where $g : C \longrightarrow D_i \sim g' : C \longrightarrow D_{i'}$ if and only if there exists an object i'' of $\mathcal{I}$ together with maps $h : i \longrightarrow i''$ and $h' : i' \longrightarrow i''$ such that $D(h) \circ g = D(h') \circ g'$. So we define $\varphi_{(c,C)}(f) := [l]_\sim$. The naturality of φ follows from the fact that $\mathcal{I}$ is filtered and the maps $\mu_k : yD_k \longrightarrow F$ are monic.

Now we need an inverse map $\psi : T \longrightarrow R(\eta)$. If $[g : C \longrightarrow D_i]_\sim \in T(c, C)$, then let $\psi_{(c,C)}([g]_\sim) := \mu_i \circ yg$. This definition is independent of choice of representative by the fact that $\mathcal{I}$ is filtered and naturality is straightforward.

Finally, it is straightforward to verify, using the fact that $\mathcal{I}$ is filtered, that $\varphi \circ \psi = 1_T$. Moreover, $\psi \circ \varphi = 1_{R(\eta)}$ is trivial. Furthermore, G is easily seen to preserve monomorphisms. As such, we have shown that $R(\eta)$ is an ideal in $\mathbf{Idl}(\int_C \mathcal{P})$.

Similarly, given an object F of $\mathbf{Idl}(\int_C \mathcal{P})$ it follows from the fact that $\pi : \int_{\int_C P} \longrightarrow \int_C P$ and $i : \int_C P \longrightarrow \widehat{\mathcal{C}}/P$ both preserve monomorphisms that $L(F)$ is an object of $\mathbf{Idl}(\mathcal{C})/P$. ⊣

Proposition 4.5.2 *If $\mathcal{R}$ is a Π–pretopos, then* $\mathbf{Idl}(\mathcal{R})$ *satisfies* ***(E)****.*

Proof First, we show that given $!_{yC} : yC \longrightarrow 1$ and $f : X \longrightarrow yC$ the map $\Pi_{!_{yC}}(f) \longrightarrow 1$ is small. By definition we have the following pullback square:

$$\begin{tikzcd}
\Pi_{!_{yC}}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{yC} \arrow[d, "f^{yC}"] \\
1 \arrow[r, "\widetilde{\pi_{yC}}"'] & yC^{yC}
\end{tikzcd}$$

where $\widetilde{\pi_{yC}}$ is the transpose of 1_{yC}. However, since f is small it follows that X is representable. I.e., $X \cong yE$ for some E. But since $\mathcal{R}$ is a Π–pretopos it follows that:

$$\begin{aligned} yC^{yC} &\cong y(C^C), \text{ and} \\ yE^{yC} &\cong y(E^C). \end{aligned}$$

Therefore f^{yC} is a small map and by **(S2)** so is the map $\pi_{!_{yC}}(f) \longrightarrow 1$.

The general case then follows from the foregoing proposition. ⊣

4.6 Universes and Additional Topics

Recall from [16] and [3] that a universe U in a category with basic class structure $\mathcal{C}$ is an object U such that $\mathcal{P}_s U \rightarrowtail U$. Where $\mathcal{R}$ is a Heyting pretopos, we may construct universes U in $\mathbf{Idl}(\mathcal{R})$ as fixedpoints for endofunctors (cf. [14] or [5]). Moreover, given a universe U in $\mathbf{Idl}(\mathcal{R})$, the full subcategory $\downarrow(U)$ of $\mathbf{Idl}(\mathcal{R})$ consisting of those objects X of $\mathbf{Idl}(\mathcal{R})$ which are subobjects of U is a category of classes with U as the universal object (cf. [16]). Putting this fact together with the results of the foregoing subsections we have:

Theorem 4.6.1 *If $\mathcal{R}$ is a Heyting pretopos, then there exists a universe U in $\mathbf{Idl}(\mathcal{R})$ such that $\downarrow(U)$ is a basic category of classes in which $\mathcal{R}$ is equivalent to the category of small objects:*

$$\mathcal{R} \simeq \mathcal{S}_{\mathbf{Idl}(\mathcal{R})}.$$

Moreover, if $\mathcal{R}$ is a Π–pretopos, then $\downarrow(U)$ is a predicative category of classes.

PROOF Let $A := \coprod_{C\in\mathcal{R}} C$ and U a fixed point of $F(X) = A + \mathcal{P}_s(X)$:

$$U \cong A + \mathcal{P}_s(U).$$

$\dashv$

Ideal categories actually have some additional properties which are worth briefly mentioning. First, another axiom in which we will be interested is the *strong collection axiom*:

(S6) For any cover $p : D \twoheadrightarrow C$ and $f : C \to A$ in $\mathcal{S}$ there exists a quasi–pullback square:

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "f'"'] & D \arrow[r, "p"] & C \arrow[d, "f"] \\ A' \arrow[rr, "h"'] & & A \end{tikzcd}$$

such that h is a cover and f' is in $\mathcal{S}$.

Proposition 4.6.2 (Typed Strong Collection) *If a category $\mathcal{C}$ with basic class structure satisfies **(S6)**, then:*

$$\mathcal{C} \models \forall a : \mathcal{P}_s\, C.(\forall x \in_C a.\exists y : D.\varphi(x,y) \Rightarrow \exists b : \mathcal{P}_s\, D.\,\mathrm{coll}(x \in_C a, y \in_D b, \varphi(x,y)),$$

where φ is any relation on $C \times D$.

PROOF A routine but fairly lengthy exercise in the internal language. $\dashv$

Proposition 4.6.3 (Small Covers) *In a category $\mathcal{C}$ with basic class struture if, given any cover $e : E \twoheadrightarrow C$ with C a small object, there exists a small subobject $i : D \rightarrowtail E$ such that $e \circ i$ is also a cover, then $\mathcal{C}$ satisfies **(S6)**.*

PROOF By Theorem 3.3.1, it suffices to show consider the case where we are given a cover $e : E \twoheadrightarrow C$ with $!_C : C \to 1$ a small map. By (2) there exists a small subobject $m : B \rightarrowtail E$ and the following is easily seen to be a quasi–pullback:

$$\begin{tikzcd} B \arrow[r, tail] \arrow[d] & E \arrow[r, "e"] & C \arrow[d] \\ 1 \arrow[rr, "1"'] & & 1 \end{tikzcd}$$

Proposition 4.6.4 *If $\mathcal{R}$ is a pretopos, then* $\mathbf{Idl}(\mathcal{R})$ *has small covers.*

PROOF Using the fact that representables are projective in $\widehat{\mathcal{R}}$ and split epis are the same in $\mathbf{Idl}(\mathcal{R})$ as they are in $\widehat{\mathcal{R}}$. $\dashv$

References

[1] Peter Aczel and Michael Rathjen. Notes on constructive set theory. Technical Report 40, Institut Mittag–Leffler (Royal Swedish Academy of Sciences), 2001. http://www.ml.kva.se/preprints/archive2000-2001.php.

[2] M. Artin, A. Grothendieck, and J.L. Verdier. *Théorie des topos et cohomologie étal des schémas.* (SGA4) Springer LNM 269 and 270. Springer-Verlag, Berlin, 1972.

[3] Steve Awodey, Carsten Butz, Alex Simpson, and Thomas Streicher. Relating topos theory and set theory via categories of classes. Department of Philosophy Technical Report CMU-PHIL-146, Carnegie Mellon University, Pittsburgh, Pennsylvania, June 2003.

[4] Steven Awodey, Carsten Butz, Alex Simpson, and Thomas Streicher. Relating topos theory and set theory (tentative). *In preparation*, 2004 (tentative). A preliminary version is the tech report [3].

[5] Steven Awodey and Henrik Forssell. Algebraic models of intuitionistic theories of sets and classes. Unpublished, 2004.

[6] Carsten Butz. Bernays–Gödel type theory. *Journal of Pure and Applied Algebra*, 178(1):1–23, 2003.

[7] Bart Jacobs. *Categorical Logic and Type Theory.* Elsevier, Amsterdam, 1999.

[8] Peter T. Johnstone. *Sketches of an Elephant*, volume 1. Oxford University Press, Oxford, 2003.

[9] Peter T. Johnstone. *Sketches of an Elephant*, volume 2. Oxford University Press, Oxford, 2003.

[10] A. Joyal and I. Moerdijk. *Algebraic Set Theory.* Cambridge University Press, Cambridge, 1995.

[11] Saunders MacLane and Ieke Moerdijk. *Sheaves in Geometry and Logic.* Springer–Verlag, Berlin, 1992.

[12] Ieke Moerdijk and Erik Palmgren. Wellfounded trees in categories. *Annals of Pure and Applied Logic*, 104:189–218, 2000.

[13] Ieke Moerdijk and Erik Palmgren. Type theories, toposes and constructive set theory: Predicateive aspects of ast. *Annals of Pure and Applied Logic*, 114:155–201, 2002.

[14] Ivar Rummelhoff. *Algebraic Set Theory.* PhD thesis, University of Oslo, 2004. Forthcoming.

[15] Alex Simpson. Constructive set theories and their category–theoretic models. Draft., 2004.

[16] Alex K. Simpson. Elementary axioms for categories of classes. In *Proceedings of the 14th Annual Symposium on Logic in Computer Science*, pages 77–85, 1999.