Dedekind's Analysis of Number (1)

Systems and Axioms

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Treated philosophically,
it [mathematics] becomes a part of philosophy.*

Introduction. In 1888 Hilbert made his *Rundreise* from Königsberg to other German university towns. He arrived in Berlin just as Dedekind's *Was sind und was sollen die Zahlen?* had been published. Hilbert reports that in mathematical circles everyone, young and old, talked about Dedekind's essay, but mostly in an opposing or even hostile sense.[1] Helmholtz and Kronecker had published, a year earlier, articles on the concept of number in a *Festschrift* for Eduard Zeller. When reading those essays in parallel to Dedekind's and assuming that they reflect accurately more standard contemporaneous views on the foundations of number theory, it is easy to understand how difficult it must have been to grasp and appreciate Dedekind's remarkably novel and thoroughly abstract approach. This is true even for people sympathetic with Dedekind's ways. Consider, for example, the remarks Frobenius made in a letter to Dedekind's collaborator and friend Heinrich Weber who was planning to write a book on algebra:

I hope you often walk on the paths of Dedekind, but avoid the too abstract corners, which he now likes so much to visit. His newest edition contains so many beauties, § 173 is highly ingenious, but his permutations are too disembodied, and it is also unnecessary to push abstraction so far.[2]

This remark was made, on 23 December 1893, by someone who refers to Dedekind as "our admired friend and master" (unser verehrter Freund und Meister); the use of permutations, i.e., isomorphisms, in Dedekind's algebraic

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* Herbart, as quoted in *Scholz 1982*, p. 437.

The translations in this paper are mostly our own; if not, we refer explicitly to the English edition from which they are taken – sometimes with minor corrections.

We thank Dr. Helmut Rohlfing from the Niedersächsische Staats- und Universitäts-Bibliothek in Göttingen for access to the unpublished Dedekind manuscripts.

[1] In *Hilbert 1931*: "Im Jahre 1888 machte ich als junger Privatdozent von Königsberg aus eine Rundreise an die deutschen Universitäten. Auf meiner ersten Station, in Berlin, hörte ich in allen mathematischen Kreisen bei jung und alt von der damals eben erschienenen Arbeit Dedekinds *Was sind und was sollen die Zahlen?* sprechen – meist in gegnerischem Sinne. Die Abhandlung ist neben der Untersuchung von Frege der wichtigste erste tiefgreifende Versuch einer Begründung der elementaren Zahlenlehre." (p. 487) On this trip Hilbert visited also Paul du Bois-Reymond who told Hilbert "die dedekindsche Arbeit 'Was sollen Zahlen' sei ihm grässlich" (related in Hilbert's report, Cod. Ms. 741, 1/5 and also mentioned in *Dugac*, p. 203).

In the preface to the second edition of his *1888*, Dedekind reports on p. IX: "Die vorliegende Schrift hat bald nach ihrem Erscheinen neben günstigen auch ungünstige Beurteilungen gefunden, ja es sind ihr arge Fehler vorgeworfen. Ich habe mich von der Richtigkeit dieser Vorwürfe nicht überzeugen können und lasse jetzt die seit kurzem vergriffene Schrift, zu deren öffentlicher Verteidigung es mir an Zeit fehlt, ohne jede Änderung wieder abdrucken, indem ich nur folgende Bemerkungen dem ersten Vorwort hinzufüge." This preface was written in August 1893.

[2] "Hoffentlich gehen Sie vielfach die Wege von Dedekind, vermeiden aber die gar zu abstrakten Winkel, die er jetzt so gern aufsucht. Seine neueste Auflage enthält so viele Schönheiten, der § 173 ist hochgenial, aber seine Permutationen sind zu körperlos, und es ist doch auch unnöthig, die Abstraktion so weit zu treiben." *Dugac 1976*, p. 269, Appendix XLVIII. Frobenius refers to Dedekind's investigations in Supplement XI to the fourth edition of Dirichlet's *Vorlesungen über Zahlentheorie*.

investigations is systematically related to, indeed, mirrored in *Was sind und was sollen die Zahlen?*.

Dedekind was well aware that such difficulties would arise. In the preface to the first edition of *1888* he writes that anyone with sound common sense can understand his essay; philosophical or mathematical school knowledge is not needed in the least. He continues, however, as if anticipating the reproach of having pushed mathematical abstraction and systematic logical analysis too far:

> But I know very well that many a reader will hardly recognize his numbers, which have accompanied him as faithful and familiar friends all his life, in the shadowy figures I present to him; he will be frightened by the long series of simple inferences corresponding to our step-by-step understanding, by the sober analysis of the sequence of thoughts on which the laws of numbers depend, and he will become impatient at having to follow proofs for truths which to his supposed inner intuition seem evident and certain from the very beginning.[3]

Dedekind arrived at his approach only after protracted labor as he emphasized in his letter to Keferstein dated 27 February 1890; in the letter, Dedekind defended his essay against Keferstein's critical review *1890*. Indeed, Dedekind had started to develop his views concerning numbers in a manuscript, or rather a sequence of manuscripts, written during the period between 1872 and 1878.[4] These intellectual developments are not isolated foundational ruminations, but have to be seen in parallel to Dedekind's contemporaneous work on algebraic number theory; cf. section **B1** below.

The publication of the essays by Helmholtz and Kronecker moved him finally to complete, sharpen, and publish his considerations. He characterized his views as "being in some respects similar [to those of Helmholtz and Kronecker], but through their grounding essentially different."[5] This is a gentle

[3] *Ewald 1996*, 791. "Aber ich weiß sehr wohl, daß gar mancher in den schattenhaften Gestalten, die ich ihm vorführe, seine Zahlen, die ihn als treue und vertraute Freunde durch das ganze Leben begleitet haben, kaum wiedererkennen mag; er wird durch die lange, der Beschaffenheit unseres Treppenverstandes entsprechende Reihe von einfachen Schlüssen, durch die nüchterne Zergliederung der Gedankenreihen, auf denen die Gesetze der Zahlen beruhen, abgeschreckt und ungeduldig darüber werden, Beweise für Wahrheiten verfolgen zu sollen, die ihm nach seiner vermeintlichen inneren Anschauung von vornherein einleuchtend und gewiß erscheinen." Dedekind expressed such sentiments also in a letter to Klein written on 6 April 1888; the letter is contained in Appendix XXV of *Dugac*, on pp. 188-9. A review by F. Meyer is partially contained in Appendix A.

[4] These manuscripts are analyzed in section **B** below; their dating is Dedekind's own.

[5] These observations are made in the first note to the preface of the first edition: "Das Erscheinen dieser Abhandlungen [i.e., the essays by Helmholtz and Kronecker] ist die Veranlassung, die mich bewogen hat, nun auch mit meiner, in mancher Beziehung ähnlichen, aber durch ihre Begründung doch wesentlich verschiedenen Auffassung hervorzutreten, die ich mir seit vielen Jahren und ohne jede Beeinflussung von irgendwelcher Seite gebildet habe."

formulation of sharp mathematical and philosophical differences. The differences emerged slowly in Dedekind's reflections over a long period of time, but they ultimately resulted in a dramatic shift. The latter can be understood, or so we will argue, as articulating the *axiomatic* method and joining it with the *genetic* one in a methodologically coherent way. Hilbert distinguished sharply between these two methods in his essay *Über den Zahlbegriff* and did not recognize the complementary roles they play for the foundations of arithmetic. To our knowledge, he also did not view Dedekind as having used the axiomatic method. Dedekind's approach is associated with a novel *structuralist* perspective on mathematics and is grounded in *logic broadly conceived*. These investigations should be seen against the backdrop of the "arithmetization of analysis," that is, of attempting to reduce (a part of) analysis to number theory.

In tracing Dedekind's development, we will provide a view of Dedekind's foundational perspective that apparently differs from Hilbert's and that is definitely in conflict with that of contemporary writers like Ferreirós, Corry, and McCarty. According to Ferreirós, Dedekind is non-modern in logical matters, as he is "anti-axiomatic;" according to Corry, Dedekind is non-modern in mathematical matters, as he can't be taken to be a mathematical structuralist; finally, according to McCarty, Dedekind is non-modern in philosophical matters, as he is a thorough-going Kantian. The views of, in particular, Belna, Dugac, Gray, Mehrtens, Noether, Parsons, Stein, and Tait have informed our understanding. Most importantly, however, our perspective has been shaped by a close reading of manuscripts in Dedekind's Nachlass; it is fair to say that these manuscripts – including *Arithmetische Grundlagen*, the drafts listed in section **A2**, *1871/72*, *1872/78*, and also *1887* – have not yet been taken into account properly, when analyzing the development of Dedekind's foundational views and its intimate connection to the evolution of his mathematical work.

There remains a great deal of important historico-analytical work that can and should be done. We mention three broad and temporally distinct directions for such work, namely, (i) an exploration of the mathematical and philosophical context for Dedekind's work, in particular, the impact of Dirichlet, Riemann, and Herbart; (ii) a detailed examination of the deep interaction between Dedekind's mathematical and foundational work, in particular in the 1870's, and (iii) a thorough investigation of Dedekind's influence on Hilbert's mathematical work, in particular, on the *Zahlbericht*.

Our essay continues and deepens earlier work in*Sieg 1990, 2000* and *Schlimm 2000*. We decided to publish it in two distinct parts. The first part, consisting of sections **A** and **B**, focuses on the development of Dedekind's approach to the foundations of the theory of numbers; in the second part, consisting of sections **C** and **D**, general methodological concerns take center stage. Section **A** presents Dedekind's early treatment of natural numbers and the extensions to integers and rationals; the central demands underlying such extensions together with a quite new aspect are emphasized, when considering briefly the creation of the irrational numbers. Dedekind's theory of chains (Kettentheorie) and its metamathematical investigation are analyzed in **B** together with related foundational reflections; this central part of Dedekind's work is in accord with the spirit of Herbart's remark we quoted as the motto for our essay.

In section **C** we argue that Dedekind's *Stetigkeit und irrationale Zahlen* (1872) is a significant stepping-stone in this development. The essay is commonly, and certainly by Hilbert, viewed as providing the final step in a *genetic* presentation of the reals via cuts. However, from the perspective of *1888* and the (unpublished) work that contains all its central notions already before 1878, it can be seen as containing a thoroughly axiomatic characterization of the reals as a complete ordered field together with a semantic consistency proof for these axioms. Dedekind's investigation concerning the correspondence between the geometric line and the systems of all cuts contains the crucial elements of a proof of the categoricity of the axioms; what is missing at this stage of his foundational reflections in 1872 is, we argue, the *general concept of mapping* (Abbildung). To support our claim, we trace in section **D** the evolution of that notion in his mathematical and foundational work. In the *Concluding Remarks*, we connect our analysis to the contemporary discussion of *structuralism* in (the philosophy of) mathematics.

Our systematic discussion is supplemented by a more detailed presentation of three episodes in Göttingen in 1854, 1899, and 1932. The first episode is concerned with Dedekind's *Habilitationsrede*: it provides a glimpse of

Dedekind's early perspective on the classical number systems and a distinctive view on some broad methodological issues. The second deals with Hilbert's first publication on the foundations of arithmetic, *Über den Zahlbegriff.* That paper is motivated by Cantor's discovery of contradictions in Dedekind's logical framework; it formulates, as mentioned already, the strong opposition of the genetic with the axiomatic method, but also an axiom system for the real numbers and the programmatic suggestion to guarantee the existence of the set of real numbers via a consistency proof for the axiom system. Finally, the third episode surrounds Emmy Noether's edition of Dedekind's *Gesammelte mathematische Werke*. *Göttingen 1854* opens this part of our essay, *Göttingen 1899* is the beginning of the second part, and *Göttingen 1932* is incorporated into our *Concluding Remarks*.

Göttingen 1854: extending operations. Richard Dedekind, born in 1831 as a citizen of Braunschweig, finished his dissertation under Gauss in 1852 and, only two years later, had to give a talk on the occasion of his Habilitation. That talk was entitled *Über die Einführung neuer Funktionen in der Mathematik* and was presented on 30 June 1854 to an audience that included Gauss, the classical philologist Hoeck, the historian Waitz, and the physicist Weber. Dedekind had chosen to talk about the general way, "in which new functions, or, as one might also want to say, new *operations*, are added to the chain of already existing ones in the progressive development of this science [i.e., mathematics]" (wie in der fortschreitenden Entwicklung dieser Wissenschaft [i.e., der Mathematik] neue Funktionen, oder, wie man ebensowohl sagen kann, neue *Operationen* zu der Kette der bisherigen hinzugefügt werden).[6] In the preface to *1888* Dedekind mentions with some satisfaction that the *purpose* (Absicht) of his talk was approved by Gauss; he characterizes it then and there as defending the claim that the most significant and most fruitful advances in mathematics and other sciences have been made "by the creation and introduction of new concepts, rendered necessary by the frequent recurrence of complex phenomena, which could be controlled only with difficulty by the old ones" (durch die Schöpfung

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[6] *Dedekind 1854*, p. 428.

und Einführung neuer Begriffe ..., nachdem die häufige Wiederkehr zusammengesetzter Erscheinungen, welche von den alten Begriffen nur mühselig beherrscht werden, dazu gedrängt hat).[7]

The need to introduce new and more appropriate notions arises for Dedekind from the fact that human intellectual powers are imperfect; the limitation of these powers leads us to frame the object of a science in different forms or systems. To introduce a concept, "as a motive for shaping the system," means in a certain sense to formulate an hypothesis concerning the inner nature of a science, and it is only the further development that determines the real value of such a notion by its *efficacy* (Wirksamkeit) in recognizing general truths. These truths, in turn, affect the formulation of definitions. Dedekind summarizes his considerations in a most revealing way:

> So it may very well happen that the concepts, introduced for whatever motive, have to be modified, because they were initially conceived either too narrowly or too broadly; they will require modification so that their efficacy, their import, can be extended to a larger domain. The greatest art of the systematizer lies in carefully turning over definitions for the sake of the discovered laws or truths in which they play a role.[8]

After these general reflections Dedekind turns his attention to mathematics. The definitions of mathematics are initially of a restricted form, but their generalizations are determined without arbitrariness. Indeed, Dedekind asserts, "they follow with compelling necessity from the earlier narrower ones" (sie folgen mit zwingender Notwendigkeit aus den früheren beschränkten). I.e., they do follow with necessity, if one applies the principle that some laws holding for the initial definitions are viewed as *generally valid* (allgemeingültig). These laws become consequently the source of the generalized definitions, when one asks, "How must the general definition be formulated such that it is always satisfies the found characteristic law?" (Wie muß die allgemeine Definition gefaßt werden, damit dem gefundenen charakteristischen Gesetze stets Genüge

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[7] *Dedekind 1888*, p. VI.

[8] *Ewald 1996*, p. 756 [4]. "So zeigt sich wohl, daß die aus irgendeinem Motive eingeführten Begriffe, weil sie anfangs zu beschränkt oder zu weit gefaßt waren, einer Abänderung bedürfen, um ihre Wirksamkeit, ihre Tragweite auf ein größeres Gebiet erstrecken zu können. Dieses Drehen und Wenden der Definitionen, den aufgefundenen Gesetzen oder Wahrheiten zuliebe, in denen sie eine Rolle spielen, bildet die größte Kunst des Systematikers." *Dedekind 1854*, p. 430. – In the Introduction to the second edition of *Dirichlet* he emphasized this general aspect for the particular mathematical work. He presented in the tenth supplement his general theory of ideals in order, as he put it, "to cast, from a higher standpoint, a new light on the main subject of the whole book." He continues, "hierbei habe ich mich freilich auf die Darstellung der Grundlagen beschränken müssen, doch hoffe ich, daß das Streben nach charakteristischen Grundbegriffen,

geschieht?) Dedekind views this as the distinctive feature of mathematical definitions, and the feature by which mathematics is distinguished from the other sciences. This claim will be taken up below; here we just note that in mathematics the *creation* (Schaffung) of new objects may be involved, whereas the objects of the other sciences are presumably given. In order to illustrate this general point, we consider one of Dedekind's mathematical examples - an example that, furthermore, provides insight into his contemporaneous understanding of the classical number systems.

Dedekind describes elementary arithmetic as being "based on the formation of ordinal and cardinal numbers" and continues, "the successive progress from one member of the sequence of positive integers to the next is the first and simplest operation of arithmetic; all other operations rest on it."[9] Dedekind shows a sophisticated grasp of the other arithmetic operations: addition, multiplication, and exponentiation are obtained through joining into one single act the iteration of the "first and simplest operation," addition, and multiplication, respectively. For the further development of arithmetic these particular definitions of the basic operations are insufficient as they are restricted to *a very small domain* (ein sehr kleines Gebiet). The demand that one should be able to carry out without any restrictions also the indirect or inverted operations of subtraction, division etc. leads to the creation of new classes of numbers, namely, the classes of "the negative, fractional, irrational and finally also the so-called imaginary numbers."

Indeed, Dedekind views this last demand as another formulation of the demand "to create anew by each of these operations the whole given number domain" (Zahlgebiet). Having expanded the domain of numbers in order to guarantee the unrestricted invertibility of addition, the question arises (in a subtly "circular" way), how to extend the definitions of the fundamental operations, so that they are applicable also to the newly created numbers. Here

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welches in anderen Teilen der Mathematik mit so schönem Erfolg gekrönt ist, mir nicht ganz mißglückt sein möge."

[9] *Ewald 1996*, p. 757 ff. The German text (*Dedekind 1932*, pp. 430-1) is as follows: "Die Elementararithmetik geht aus von der Bildung der Ordinal- und Kardinalzahlen; der sukzessive Fortschritt von einem Gliede der Reihe der absoluten ganzen Zahlen zu dem nächstfolgenden ist die erste und einfachste Operation der Arithmetik; auf ihr fußen alle andern. Faßt man die mehrere Male hintereinander wiederholte Ausführung dieser Elementaroperation in einem einzigen Akt zusammen, so gelangt man zum Begriff der Addition. Aus diesem bildet sich auf ähnliche Weise der der Multiplikation, aus diesem der der Potenzierung."

Dedekind joins the general reflections above and illustrates them by considering the extension of multiplication from the natural numbers to all integers. The extension of the definition of multiplication is non-arbitrary, Dedekind asserts, if one follows his principle of the general validity of laws as the source for deriving "the meaning of the operations for the new number domains" (die Bedeutung der Operationen für die neuen Zahlengebiete).

How is this to be achieved here? – As noted already, Dedekind defines multiplication for the natural numbers as joining the iteration of addition into a single act, and it is of course assumed that addition is already available for all integers. This definition of multiplication makes sense only, if the multiplicator is positive; the multiplicator is the number which indicates, how often one has to iterate the addition of the multiplicand. (The multiplicand may now be positive or negative.) Dedekind asserts:

> A special definition is therefore needed in order to admit negative multipliers [we use Dedekind's term, multiplicator] as well, and to liberate in this way the operation from the initial restriction; but such a definition involves an *a priori* complete arbitrariness, and it would only later be determined whether this arbitrarily chosen definition would bring any real advantage to arithmetic; and even if this succeeded, one could only call it a lucky guess, a happy coincidence—the sort of thing a scientific method ought to avoid.[10]

What consideration might provide the grounds for the principled definition of extended multiplication? - "One has to investigate," Dedekind demands, "which laws govern the product, if the multiplicator is successively subjected to the same changes by which the series of negative numbers is generated, in the first place, from the absolute series of whole numbers." (Man muß untersuchen welchen Gesetzen das Produkt unterwofen ist, wenn der Multiplikator sukzessive dieselben Veränderungen erleidet, durch welche überhaupt aus der absoluten ganzen Zahlenreihe die der negativen erzeugt wurde.) The starting point is the observation that $a \times (m+1) = a \times m + a$. That yields the "addition theorem for the multiplicator," i.e., $a \times (m+n) = a \times m + a \times n$. From this follows a "subtraction theorem" $a \times (m-n) = a \times m - a \times n$ as long as the minuend m is greater than the subtrahend n. Taking this law as generally valid also for the case that the

[10] *Ewald 1996*, p. 758 [8]. "Es bedarf daher einer besonderen Definition, um auch negative Multiplikatoren zuzulassen, und auf diese Weise die Operation von der anfänglichen Beschränkung zu befreien; eine solche involviert aber eine a priori vollständige Willkürlichkeit, und es würde sich erst später entscheiden, ob denn die so beliebig gewählte Definition der Arithmetik einen wesentlichen Nutzen brächte; und glückte es auch, so könnte man dies doch immer nur ein zufälliges Erraten, ein glückliches Zutreffen nennen, von welchem eine wissenschaftliche Methode sich frei halten soll." *Dedekind 1854*, pp. 431-2.

difference representing the multiplicator is negative, one obtains from it the definition of multiplication with negative multiplicators. Thus, Dedekind concludes, "it is no longer an accident that the general law for multiplication is in both cases exactly the same."

Next, Dedekind obtains in a similar way the generalized definition of exponentiation for rational numbers. The case of the extension to the real and imaginary numbers is only alluded to by saying, "these advances [obtained by creating these new classes of numbers] are so immense that it is difficult to decide which of the many paths that are opened up here one should follow first." (Diese Fortschritte sind so unermeßlich, daß es schwer zu entscheiden ist, welche der vielen verschiedenen Bahnen, die sich hier auftun, man zuerst betreten soll.) So much is clear, however, that the operations of arithmetic have to be extended to these new classes and that here, "at least with the treatment of the imaginary numbers," the main difficulties for the systematic development of arithmetic begin. Dedekind ends the discussion of the number systems in a surprising way:

> However, one might well hope that a truly solid edifice of arithmetic will be attained by persistently applying the principle not to permit ourselves any arbitrariness, but always to be led on by the discovered laws. Everybody knows that until now, an unobjectionable theory of the imaginary numbers, not to mention those newly invented by Hamilton, does not exist, or at any rate has not been published yet.[11]

Only four years later, in the fall of 1858, when lecturing on the infinitesimal calculus at the "Eidgenössisches Polytechnikum" in Zürich Dedekind was motivated - by the "overwhelming feeling of dissatisfaction" with the need to appeal to geometric evidences when discussing certain limit considerations - to search for "a purely arithmetic and completely rigorous foundation of the principles of infinitesimal analysis." He found it in his examination of continuity and the resulting definition of real numbers as, or through, cuts of rationals. He discussed the solution with his friend Heinrich Durège at the time and presented the material to the "Wissenschaftlicher Verein" in Braunschweig on 11 January

[11] *Ewald 1996*, p. 759 [9]. "Indessen ist wohl zu hoffen, daß man durch beharrliche Anwendung des Grundsatzes, sich auch hier keine Willkürlichkeit zu erlauben, sondern immer durch die gefundenen Gesetze selbst sich weiterleiten zu lassen, zu einem wirklich festen Gebäude der Arithmetik gelangen wird. Bis jetzt ist bekanntlich eine vorwurfsfreie Theorie der imaginären, geschweige denn der neuerdings von Hamilton erdachten Zahlen entweder nicht vorhanden, oder doch wenigstens noch nicht publiziert." *Dedekind 1854*, p. 434. To see why Gauss's geometric interpretation of complex numbers did not satisfy Dedekind's purely arithmetic ambitions, it is instructive to read Gauss's defense of the use of complex numbers in his *1831*, in particular, pp. 310-3.

1864, but also in some of his lectures on the differential and integral calculus.[12] Already in 1870 he had the intention of publishing his theory of continuity, according to a letter from his friend Adolf Dauber.[13] We have an extended draft *1871/72* of the essay *Stetigkeit und irrationale Zahlen* that was seemingly written in late 1871 and early 1872; the draft is preserved in Göttingen as Cod. Ms. Dedekind III, 17 and published as Appendix XXXII in *Dugac*. Before discussing the central issues of the draft and the essay itself, we want to explore Dedekind's view of natural numbers and the extensions of the simple number system around 1872.

A. Natural numbers and beyond. Assuming that the difficulty mentioned explicitly in the Habilitationsrede - to obtain an "irreproachable theory of imaginary numbers" (vorwurfsfreie Theorie der imaginären Zahlen) - has been addressed[14], and that the definition of real numbers as cuts answers Dedekind's concerns for a rigorous foundation of analysis, two questions remain for Dedekind in 1872: (**1**) What are (the principles for) natural numbers? and (**2**) How are the integers and rational numbers obtained or, in the terminology of *1854*, how are they created starting with the natural numbers? - In the latter work Dedekind took for granted that the mathematical objects (negative and fractional numbers) had been obtained already from the natural numbers; the only real issue was there, how to extend the basic arithmetic operations to the wider number systems. Question (**2**) is addressed in a sequence of manuscripts contained in Cod. Ms. Dedekind III, 4, and it seems that the issues were settled to Dedekind's satisfaction before the essay on continuity and irrational numbers was completed. Question (**1**) was not settled at that time, however. On the contrary, Dedekind struggled with it intermittently over the next six years; the intense work is reflected in the manuscripts *1872/78* that are contained in Cod.

[12] See, for example, the outline for such a course in the winter semester of 1862/3, published in *Dugac* as Appendix IV.

[13] Dauber asks Dedekind in a letter of 20 June 1871, whether Dedekind had come closer to realizing his plans for publishing his theory of continuity, and remarks that Dedekind had written him about such plans a year earlier. The letter is part of Appendix XXVI in *Dugac*; the remark can be found on p. 192.

[14] Ferreirós reports on p. 220 of his *1999* that Dedekind borrowed in 1857 Hamilton's *Lectures on Quaternions* from the Göttingen Library. Hamilton gives in the *Preface* to his book the definition of complex numbers as pairs of reals. Pairs are viewed there as genuine mathematical objects for which operations can be defined appropriately; see *Hamilton 1853*, pp. 381—85. "Thus," Ferreirós concludes convincingly, "Dedekind could regard the problem of complex numbers as satisfactorily solved, ..." Indeed, Dedekind uses Hamilton's way later; cf. section **A3** below.

Ms. Dedekind III, 1, I; they served as the first draft for the 1888 essay on the nature and meaning of numbers, and are published as Appendix LVI in *Dugac* with the title *Gedanken über die Zahlen.*

A1. Analyzing, naïvely. The manuscripts *1872/78* have the subtitle *Attempt to analyze the number concept from the naïve point of view* (Versuch einer Analyse des Zahlbegriffs vom naiven Standpuncte aus). Perhaps it is in the "naïve" approach to the topic that Dedekind sees, as he does in *1888*, a certain similarity between his view and that of Helmholtz and Kronecker. What Dedekind had in mind when calling his approach naïve is possibly captured in his letter to Keferstein by a remark that answers the rhetorical question, "How did my essay come into being?" (Wie ist meine Schrift entstanden?):

> Surely not all at once, but it is a synthesis constructed after protracted labor, which is based on a preceding analysis of the sequence of natural numbers as it presents itself, in experience so to speak, to our consideration.[15]

Basic is for Dedekind a thoroughgoing analysis of the data of ordinary mathematical experience, free from philosophical preconceptions. Such an analysis, as Dedekind demanded already in *1854*, should lead to notions that reflect the nature of the subject and prove their efficacy in its development. The independence from philosophical preconceptions is brought out clearly, when Dedekind at the very beginning of *1872/78* writes that the notions he uses for the foundation of the number concept "remain necessary for arithmetic even when the notion of cardinal number is assumed as immediately evident ('inner intuition')" (bleiben auch dann für die Arithmetik unentbehrlich, selbst wenn man den Begriff der Anzahl als unmittelbar evident ('innere Anschauung') voraussetzen wollte). We focus on this "naïve" analysis, before isolating the crucial elements of Dedekind's synthesis as the background for the mathematically central and philosophically distinctive features of *1888*.

Recall that in *1854* elementary arithmetic begins with the formation of ordinal and cardinal numbers. The "successive progress from one member of the sequence of positive integers to the next" is viewed by Dedekind as "the first and simplest operation of arithmetic" on which all other operations rest. We also saw,

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[15] *Van Heijenoort 1967, p. 99.* "Gewiss nicht in einem Zuge, sondern sie ist eine nach langer Arbeit aufgebaute Synthesis, die sich auf eine vorausgehende Analyse der Reihe der natürlichen Zahlen stützt, so wie diese sich, gewissermassen erfahrungsmässig, unserer Betrachtung darbietet."

how addition is obtained by joining iterations of this "first and simplest operation" into a single act; one obtains, in completely parallel ways, multiplication from addition and exponentiation from multiplication. This standpoint concerning the character of natural numbers is hardly changed, when Dedekind expresses his views in section 1 of *1872*. There he uses *chain* (Kette), the central notion of *1888*, not yet in the precise mathematical sense of that later work, but rather as a fitting informal notion to capture the structural character of the domain that has been obtained by successively generating its objects through the "simplest arithmetical act:"

I regard the whole of arithmetic as a necessary, or at least natural, consequence of the simplest arithmetical act, that of counting, and counting itself is nothing other than the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding; the simplest act is to pass from an already-created individual to its successor that is to be newly created. The chain [Kette] of these numbers already forms in itself an exceedingly useful instrument for the human mind; it presents an inexhaustible wealth of remarkable laws, which one obtains by introducing the four fundamental operations of arithmetic.[16]

One should notice that Dedekind speaks of counting as "nothing other" than the *successive creation* (sukzessive Schöpfung) of the individual positive integers.

In *1888* there is, of course, a systematic development of number theory, based on the proof principle of induction and the definition principle of recursion, both justified in the broad logical setting of that essay. An elementary development of arithmetic is already given in an earlier manuscript, entitled *Arithmetische Grundlagen;* this manuscript is found in three distinct versions in Dedekind's Nachlass (Cod. Ms. Dedekind III, 4, II).[17] The first version starts out in the following way:

§1

Act of creation 1; 1+1=2; 2+1=3; 3+1=4 ... numbers (ordinal).

§2

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[16] *Ewald 1996*, p. 768. The German text is as follows: "Ich sehe die ganze Arithmetik als eine notwendige oder wenigstens natürliche Folge des einfachsten arithmetischen Aktes, des Zählens, an, und das Zählen selbst ist nichts anderes als die sukzessive Schöpfung der unendlichen Reihe der positiven ganzen Zahlen, in welcher jedes Individuum durch das unmittelbar vorhergehende definiert ist; der einfachste Akt ist der Übergang von einem schon erschaffenen Individuum zu dem darauffolgenden neu zu erschaffenden. Die Kette dieser Zahlen bildet an sich schon ein überaus nützliches Hilfsmittel für den menschlichen Geist, und sie bietet einen unerschöpflichen Reichtum an merkwürdigen Gesetzen dar, zu welchen man durch die Einführung der vier arithmetischen Grundoperationen gelangt." (*Dedekind 1872*, pp. 5-6)

[17] Ferreirós discusses *Arithmetische Grundlagen* on p. 218 and more extensively on pp. 222-4. Our perspectives are different on the dating of the manuscripts and on the "rational reconstruction" of the mathematical content (and context). Our reasons for differing are presented with the detailed discussion of the manuscripts below. On the crucial issue we do agree with Ferreirós, namely, that the introduction of the successor operation in (what we take to be) the third version of the manuscript is of utmost significance and a central result of the informal analysis.

Definition of addition by a+(b+1)=(a+b)+1. After this, consequences are — according to the nature of the subject — always to be deduced by complete induction.[18]

This is only slightly modified in the second version that reads:

§1

Creation of the numbers: 1; 1+1=2; 2+1=3; 3+1=4 … from each number *a* the *following* number *a*+1 is formed by the act +1. — Therefore, everything by complete induction.

§2

Definition of addition: a+(b+1)=(a+b)+1.[19]

In both versions elementary arithmetic is then briefly developed - employing definitions by recursion and proofs by induction; this is done very thoroughly, establishing (in different ways) associativity and commutativity of addition and multiplication and ending with a proof of the distributive law in the form a×(b+c) = a×b+a×c. In the second version, Dedekind remarks on the margin that the distributive law can be obtained much more directly from the definition of multiplication (and the associativity of addition). Such a more direct argument is indeed presented in the third version. However, most remarkable about this version is that Dedekind separates the generating "successor operation" from addition, i.e., the sequence of numbers is indicated by 1, φ(1)=2, φ(2)=3, φ(3)=4, ..., and the recursive definition of addition is given by the two equations a+φ(b) = φ(a+b) and a+1 = φ(a) instead of just by the single equation a+(b+1) = (a+b)+1. With this notational change to the unary successor operation begins a quite dramatic conceptual shift that finds its systematic expression in the manuscripts *1872/78* and provides one solid reason for thinking that *Arithmetische Grundlagen* was written in (early) 1872.

The third version makes also clear that Dedekind is trying to use these foundations for constructing the extended number systems, here, of all integers. Dedekind defines subtraction by a-b = c, in case a = b+c; this is taken, implicitly, as the motivation for considering an extension of the positive integers that contains zero 0 and the negative numbers 1*, 2*, 3* etc.. The successor operation is suitably extended by setting, in particular, 0+1 = 1, 1*+1 = 0, 2*+1 = 1*, 3*+1 =

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18 "§1 Schöpfungsakt 1; 1+1=2; 2+1=3; 3+1=4 … Zahlen (Ordinal). §2 Erklärung der Addition durch a+(b+1)=(a+b)+1. Hiernach Folgerungen, der Natur der Sache nach [,] immer durch die vollständige Induktion abzuleiten."

19 "§1 *Erschaffung der Zahlen*: 1; 1+1=2; 2+1=3; 3+1=4 … aus jeder Zahl a wird durch den Act +1 die *folgende* Zahl a+1 gebildet. – Deshalb Alles durch vollständige Induction. §2 Erklärung der Addition: a+(b+1)=(a+b)+1."

2* etc.. Having defined the predecessor operation b = a-1, in case b+1 = a, he considers 1-1, (1-1)-1, etc. as the *new numbers* (neue Zahlen). This - together with the systematic development up to the distributive law (central for restricting the possible extensions in *1854* and called there the "addition theorem for the multiplicator") - sets the stage, it seems, for a development along the lines suggested in his *Habilitationsrede*. That was not uncommon at the time; indeed, Heine pursues a similar route in his *Elemente der Functionenlehre.*[20] Though Heine's is a natural way of proceeding, Dedekind must have found it quite unsatisfactory at this time. As we just saw, he first assumes a domain containing zero and the negative numbers in order to define the extended successor operation; that allows him, in turn, to define the general predecessor operation and to describe the desired (minimal) extension by the *new numbers*. But of what objects does the first extension consist? What are negative numbers? Heine answers the general question "*What are numbers?*" not by a conceptual definition, but rather by taking a purely formal standpoint (so acerbically criticized by Frege): "In the definition [of numbers] I adopt the purely formal standpoint, *by calling certain tangible marks numbers,* such that the existence of these numbers is not in question." (Ich stelle mich bei der Definition [der Zahlen] auf den rein formalen Standpunkt, *indem ich gewisse greifbare Zeichen Zahlen nenne,* so dass die Existenz dieser Zahlen also nicht in Frage steht.)[21] Dedekind will take some time to arrive at an answer that is philosophically convincing and based on penetrating metamathematical reflection.

There is no indication on the manuscript itself, when *Arithmetische Grundlagen* was written. We conjecture, for three reasons, that it was written in early 1872. The first reason is simply the fact that the beginnings of the various versions are fully in accord with the informal description of *1872*. The second reason was mentioned already, when we looked at the third version and noticed an important overlap with (the beginning of the "first layer" of) the manuscripts *1872/78*. Finally, the third reason is provided by the systematic context to be elaborated in the next subsection, namely, the context of creating the system of

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[20] Dedekind received Heine's paper, when working on the draft of *1872*; Heine describes his way of introducing the negative numbers on pp. 173-4 of his essay. As to possible precedents of Dedekind's way of proceeding cf. *Ferreirós*, p. 219, note 1.

[21] *Heine 1872*, p. 173.

rational numbers – on these arithmetic foundations – in the other manuscripts contained in the folder Cod. Ms. Dedekind III, 4. The significance of this context will be even more apparent, when we notice at the beginning of **A2** a subtle shift in its description by Dedekind.

A2. Naive analysis, continued. In the manuscript *1871/72* Dedekind emphasizes that the rational numbers are *a free human creation* (eine freie Schöpfung des Menschen). Indeed, the rational numbers are viewed as the necessary consequence of counting which "itself is the creation of the positive whole numbers." However, the "instrument the mathematicians have constructed by creating the rational numbers" has to be refined – Dedekind claims – by the creation of the irrational numbers. Though it can't be denied that the reason for this extension is provided by non-arithmetic considerations, the latter should not become a part of arithmetic, the science of numbers. Dedekind continues:

> Just as negative and fractional rational numbers are formed by a free creation, and just as the laws of operating with these numbers are reduced to the laws of operating with positive integers (at least it *should* be done in this way), in the same way the irrational numbers must also be defined by means of the rational numbers.[22]

This long sentence is repeated almost verbatim in the publication *1872*. Here it is (and we want the reader to notice, in particular, the underlined replacement for the parenthetical remark in the above quotation):

> Just as negative and fractional rational numbers are formed by a free creation, and just as the laws of operating with these numbers must and can be reduced to the laws of operating with positive integers, in the same way the irrational numbers must also be completely defined by means of the rational numbers alone.[23]

We conjecture that the material contained in Cod. Ms. Dedekind III, 4 provides the reason for this change: having established proper arithmetic foundations, Dedekind convinces himself in detail that the system of rational numbers can be created, and that the laws for calculating with these numbers can be reduced to those for calculating with the positive whole numbers. This point is certainly reemphasized in the introduction to *1888*, where Dedekind situates his treatment of the natural numbers into the general context of providing a completely clear

[22] "So wie die negativen und gebrochenen rationalen Zahlen durch eine freie Schöpfung hergestellt, und wie die Gesetze der Rechnungen mit diesen Zahlen auf die Gesetze der Rechnungen mit ganzen positiven Zahlen zurückgeführt werden (so *sollte* es wenigstens geschehen), ebenso müssen auch die irrationalen Zahlen durch die rationalen Zahlen definiert werden." *Dugac*, p. 205. The emphasis of "sollte" is Dedekind's.

[23] *Ewald 1996*, p. 771. "So wie die negativen und gebrochenen rationalen Zahlen durch eine freie Schöpfung hergestellt, und wie die Gesetze der Rechnungen mit diesen Zahlen auf die Gesetze der Rechnungen mit

picture of the science of numbers. He refers to the example of the real numbers presented in *1872* and remarks that the other classes of numbers can be treated easily in a *quite similar fashion* (in ganz ähnlicher Weise).

What is the mathematical substance that allows us to understand the shift from *"should"* to *"must and can"*? As we mentioned already, in Dedekind's *Nachlass* there are several manuscripts concerning the extension of the natural numbers to the integers and rational numbers. The further extension to the real numbers is, of course, the central topic of *1872*. Particular ways of extending the number concept are pursued in the following manuscripts:

- Cod. Ms. Dedekind III, 4, I, Blatt 1-4, entitled *Die Schöpfung der Null und der negativen ganzen Zahlen;*
- Cod. Ms. Dedekind III 4, I, Blatt 5-7, without title, but we will refer to it as *Ganze und rationale Zahlen;*
- Cod. Ms. Dedekind III, 2, I, entitled *Die Erweiterung des Zahlbegriffs auf Grund der Reihe der natürlichen Zahlen.*

The first two manuscripts, we conjecture, were written in 1872.[24] The third one was definitely written after *1888*, as the latter essay is explicitly referred to, and gives an altogether modern approach. In this subsection, we give a detailed account of the first manuscript.

The description in **A1** of how to extend the system of natural numbers to that of the integers is based on very brief remarks in the third version of *Arithmetische Grundlagen*. In *Die Schöpfung der Null und der negativen ganzen Zahlen* a beautifully detailed presentation of the first step of those considerations is given. The manuscript starts out by stating basic facts regarding the series of natural numbers N: (1) N is closed under addition; addition is (2) commutative and (3) associative; (4) if a > b, then there exists one and only one natural number c, such that b+c = a, whereas in the opposite case, when a ≤ b, no such number c exists. Dedekind notes that the fourth condition states a certain *irregularity*

ganzen positiven Zahlen zurückgeführt werden müssen und können, ebenso hat man dahin zu streben, daß auch die irrationalen Zahlen durch die rationalen Zahlen allein vollständig definiert werden."

[24] To be more precise, we conjecture that the first manuscript was written in 1872, whereas the second one was written much later, but that its essential content goes back to 1872. (The evidence for the latter conjecture is twofold: (i) the manuscript is written in Roman, not German script as all the earlier manuscripts; (ii) one part of the detailed calculations is written on the back of a receipt for a journal subscription - from 1907.)

(Unregelmäßigkeit), and he raises the crucial question, whether it is possible to extend the sequence N to a system M (by the addition of elements or numbers to be newly created) in such a way that M satisfies conditions (1) to (3) and also (4'), i.e., for any two elements a and b from M, there exists exactly one element c, such that b+c = a? And, how rich must the *smallest* such system M be?

In the following *Investigation* (Untersuchung), which is also called *Analysis*, Dedekind assumes the existence of such a system M. He reasons that M must contain a unique element 0 (called zero), such that a+0 = a; furthermore, for every element a in N there must be a new element a* in M, such that a+a* = 0. Thus, any system M satisfying (1)-(4') must contain in addition to the elements of N the new element zero and all the different new elements a*. Dedekind considers now the system P consisting of just N together with these new elements and shows that P has already the *completeness* (Vollständigkeit) expressed by conditions (1)-(4'); P is obviously the smallest such system, as it must be contained in any complete system M. The investigation is carried out in exemplary mathematical clarity, but it assumes quite explicitly the existence of a suitable M. This methodologically crucial issue is addressed in the second, and unfortunately incomplete, section of the manuscript that is entitled *Synthesis*. Here is the full text of that section:

> From the sequence N of natural numbers *a* is to be created a system P, which contains in addition to the elements *a* also an element 0 and for each *a* a corresponding element *a**, with the stipulation that all these elements in P are *different* from each other (easy to formulate more precisely; on the *possibility* of such a creation, see below).[25]

There is no "farther below" and thus no discussion of the *possibility of such a creation*. The manuscript ends abruptly on page 4 with the remark just quoted. The folder contains, however, additional material that was written at a later date (as argued above), but its substance was undoubtedly clear to Dedekind in 1872 and can be understood as realizing such a creation.

A3. Creating numbers. The systematic considerations are continued in *Ganze und Rationale Zahlen*. The manuscript has two main parts: the first deals with the extension of the domain of all natural numbers to that of all integers; the second

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[25] "Man erschaffe aus der Reihe *N* der natürlichen Zahlen *a* ein System P, welches außer den Elementen *a* noch ein Element 0, und zu jedem *a* ein entsprechendes Element *a** enthält, mit der Festlegung, daß alle diese Elemente in P von einander *verschieden* sind (leicht genauer auszudrücken; über die *Möglichkeit* einer solchen Schöpfung weiter unten)."

is concerned with the "transition from the domain G of all whole numbers to the field R of all rational numbers" (Übergang vom Gebiet G aller ganzen Zahlen zum Körper R aller rationalen Zahlen). The first part consists of three handwritten pages together with a few further "Zettel" filled with detailed calculations concerning integers; the second part sketches very briefly similar considerations for the rationals on just one page. We describe the first part in detail, despite the fact that the steps are routine - for a modern reader.

Dedekind starts out with the domain N of all natural numbers together with the operations of addition and multiplication. Both operations satisfy the commutative and associative laws, and the distributive law connects them. The domain G of *all* whole numbers is then formed from N, as Dedekind puts it, *by extension* (durch Erweiterung): "Any two numbers m,n in N generate a number (m,n) in G." (Je zwei Zahlen m, n in N erzeugen eine Zahl (m,n) in G.) Dedekind defines when two pairs of numbers are *identical*, namely, (m,n)=(m',n') if and only if m+n'=m'+n, and verifies that the relation is symmetric and transitive. (It is obviously also reflexive, thus an equivalence relation.) Then he defines the operation of *addition* on pairs by letting the sum (m,n)+(m',n') be identical to the pair (m+m',n+n'). Having checked that the defined addition yields identical results when applied to identical pairs, he verifies easily the associative and commutative laws. *Multiplication* (m,n)(m',n') for pairs is given by (mm'+nn',mn'+m'n) and is treated in a completely parallel way: uniqueness is checked (that is actually a quite lengthy argument and spills over onto the "Zettel") and laws are verified; the final step is the verification of the distributive law. In contemporary terms, the "identity" is shown to be a congruence relation with respect to the two operations.

This is the central part of constructing the integers as pairs of natural numbers that represent positive and negative numbers, but of course also zero. It is reminiscent of the very early considerations in *1854*, when Dedekind extends subtraction from the natural numbers to the integers and, in essence, uses "differences" between natural numbers to *represent* negative numbers. Thinking of the pairs as "differences" and using the ordinary calculation rules, the operations are obtained in a direct way and obey the standard laws. A parallel construction is sketched in the second part of the manuscript to obtain the rationals R from the integers G: for pairs (m,n) and (m',n') - where m, n, m' and n' are in G, but n and n' are different from zero - "identity" is defined by mn'=m'n; this is again an "identity" that satisfies symmetry and transitivity. Thinking of pairs (m,n) as fractions m/n, addition and multiplication are defined via the ordinary calculation rules as (mn'+nm',nn'), respectively (mm',nn'). The various laws can be verified. It is also clear, though Dedekind does not prove it, that the inverted operations can be performed without any restriction.

We emphasize that this manuscript is in very rough form and indicates only the bare minimum of the needed considerations. But even so, it does provide a quite novel way in which to ensure the permanence of laws. Dedekind does not create – out of thin air – new individual elements: he rather obtains by pairing natural numbers, respectively integers, new systems of genuine mathematical objects. The arithmetic operations are then defined in terms of the operations on natural numbers, respectively integers. These systems satisfy the laws (axiomatic conditions) for integers and rationals, i.e., Dedekind exhibits models for these laws. In fact, the models he presents are exactly the ones that are still being employed today: except that in a modern exposition one would deal with equivalence classes of pairs. That is done very beautifully in the final and later manuscript concerned with the extension of the number systems, *Die Erweiterung des Zahlbegriffs auf Grund der Reihe der natürlichen Zahlen*. However, it should be noticed clearly that Dedekind could have taken this step in 1872; there was no ideological reason for avoiding infinite mathematical objects: he had used such objects in the ideal theoretic investigations of Supplement X for the second edition of Dirichlet's *Zahlentheorie* of 1871, but also in the 1872 essay on continuity and irrational numbers. Yet there is one question that is left open (and, as we will see next, an analogous question is answered in *1872* concerning the reals): The rational numbers, "are" they these specific infinite objects?

In his considerations Dedekind heeds, first of all, his own later warning in a letter to Lipschitz of 27 July 1876 that "nothing is more dangerous in mathematics than to make existence assumptions without sufficient proof." This refers to the definition of the system of real numbers: that system is to allow us to pursue all phenomena of the geometric line in a purely arithmetic way; thus, it has to be defined by means of rational numbers and (the laws for) the arithmetic operations have to be reduced to (those for) the operations on rational numbers. Clearly, the construction has to be done in such a way that the resulting system of reals has the same kind of continuity or completeness as the geometric line. We discuss the underlying methodological issues in greater detail in section C. Here we emphasize, on the one hand, the considerations involved in extending the system of rationals to that of the reals and bring out, on the other hand, very clearly the new answer to the question that parallels the above for rationals: "Are" the constructed objects, cuts, the real numbers? (The central issues are discussed in almost identical ways in *1871/72* and *1872*.)

Cuts are partitions (A_1,A_2) of the system of rationals with the property that all a_1 in A_1 are less than all a_2 in A_2; they are viewed extensionally: $(A_1,A_2)=(B_1,B_2)$ if and only if A_1 and A_2 have the same members as B_1 and B_2, respectively. If A_2 contains a smallest element a', then the cut (A_1, A_2) is said to have been *engendered* (hervorgebracht) by a'; the fact that not all cuts are engendered by rationals constitutes the *incompleteness* (Unvollständigkeit) or *discontinuity* (Unstetigkeit) of the domain of rationals.[26] Dedekind continues, in the section entitled *Creation of irrational numbers* (Schöpfung der irrationalen Zahlen):

Thus, whenever we have a cut (A_1,A_2) produced by no rational number, we *create* a new number, an *irrational* number α, which we regard as completely defined by this cut (A_1,A_2); we shall say that the number α corresponds to this cut, or that it produces this cut. From now on, therefore, to every definite cut there corresponds a definite rational or irrational number, and we regard two numbers as *different* or *unequal* if and only if they correspond to essentially different cuts.[27]

The system of real numbers consists thus of all rational numbers (corresponding of course to the cuts engendered by them) together with these newly *created* irrational ones or, to put it in other words, the system of rationals has been extended by these irrational numbers. The crucial point here is this: reals are not identified with cuts, but rather "correspond" to cuts; the latter are for Dedekind

[26] There is a simple issue of whether the partition (A',B') that is exactly like (A,B) except that b' is no longer the smallest element of B but the largest element in A' should also be a cut or not; Dedekind discusses these matters in *1871/72* on p. 11, i.e., on p. 207 in *Dugac*. For his own presentation, he decides, to consider such cuts as not *essentially different*.

[27] *Ewald 1996*, p. 773. "Jedesmal nun, wenn ein Schnitt (A_1,A_2) vorliegt, welcher durch keine rationale Zahl hervorgebracht wird, so *erschaffen* wir eine neue, eine *irrationale* Zahl α, welche wir als durch diesen Schnitt (A_1,A_2) vollständig definiert ansehen; wie werden sagen, daß die Zahl α diesem Schnitt entspricht, oder daß sie diesen Schnitt hervorbringt. Es entspricht also von jetzt ab jedem bestimmten Schnitt eine und nur eine rationale oder irrationale Zahl und wir sehen zwei Zahlen stets und nur dann als *verschieden* oder *ungleich* an, wenn sie wesentlich verschiedenen Schnitten entsprechen."

genuine mathematical objects, and the relations between reals and operations on them are defined in terms of the corresponding cuts.

The ordering between two reals α and β corresponding to the cuts (A_1,A_2) and (B_1,B_2) is defined as follows: $\alpha < \beta$ if and only if $A_1 \subset B_1$ (if, for any rational cut, the rational that engenders the cut is always, say, in the right part of the cut). Addition and multiplication of reals is defined in terms of the corresponding operations for the rationals. Consider two reals α and β that correspond to the cuts (A_1,A_2) and (B_1,B_2); the sum $\alpha+\beta$ (the product $\alpha\beta$) corresponds to the cut (C_1,C_2), where C_1 consists of all c that are smaller or equal to a_1+b_1 (a_1b_1) for some a_1 in A_1 and b_1 in B_1, and C_2 consists of the remaining rational numbers. It is not difficult to verify the arithmetic laws for a field; Dedekind verifies the order laws and proves that the system of reals is continuous. In modern terminology, the system of reals or, more directly, the system of all cuts is recognized to be a complete ordered field.

As we will see below, Dedekind had excellent reasons for not identifying the real numbers with cuts of rationals; he articulated them in his correspondence with Lipschitz in 1876. The correspondence was partially stimulated by the preparation of Dedekind's essay *Sur la Théorie des Nombres entiers algébriques*, published in 1877 in the Bulletin des sciences mathématiques. Lipschitz had actually suggested that Dedekind be invited to report on his work in algebraic number theory. The resulting attempt of Dedekind's to present his work (essentially contained in Supplement X of the second edition of Dirichlet's *Zahlentheorie*) in a new, possibly more accessible way, contains in the *Introduction* a long methodological note attached to remarks about Kummer's ideal numbers and his own ideals. In the note he points to *1872* as making even more evident - in the case of introducing the irrational numbers and defining the arithmetic operations on them - the "legitimacy, or rather necessity, of such demands, which must always be imposed with the introduction or creation of new arithmetic elements," i.e., demands concerning the precise definition of new mathematical objects in terms of already existing ones and the general definition of operations on them in terms of the given ones. In more contemporary language, the structures of pairs or cuts provide models of the axioms for integers (rationals) and reals; the particular elements of these structures are not identified with the respective numbers, but the latter are specifically obtained by an *abstracting* "free creation." One last remark: if we think of the "genetic method" as underlying the construction of mathematical objects, systems of which are models of appropriate axiom systems, we can see very clearly already now, how it complements the axiomatic method.

B. Natural numbers: foundational investigations. Given the goal of arithmetizing analysis and Dedekind's methodological demands for creating the irrational numbers, it is essential to characterize the very basis of the construction, i.e., the natural numbers or, as Dedekind calls them also, the *absolute integers*. First steps are taken in distinct layers of an early manuscript for *1888* that was written, modified, and rewritten between 1872 and 1878 (Cod. Ms. Dedekind III, 1, I). At the end of this period Dedekind must have thought about publishing a booklet with the very title of *1888*, as his friend Heinrich Weber writes in a letter of 13 November 1878:[28] "I am awaiting your book *Was sind und was sollen die Zahlen* with great anticipation." (Deinem Buch *Was sind und was sollen die Zahlen* sehe ich mit grosser Spannung entgegen.)

The "great anticipation" was more than justified already then, as Dedekind's reflections had led him to a novel conceptualization of natural numbers within, what he viewed as, a *logical* framework using the fundamental concepts of *system* and *mapping*. Indeed, in the manuscript Dedekind writes:

> If one accurately tracks what we are doing when we count a set or a number of things, one is necessarily led to the concept of correspondence or mapping.
> The concepts of system, of mapping, which shall be introduced in the following in order to ground the concept of number, cardinal number, remain indispensable for arithmetic even if one wants to assume the concept of cardinal number as being immediately evident ("inner intuition").[29]

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[28] The letter is found in Appendix L of *Dugac*, on p. 272. - Dedekind responded on 19 November 1878, saying: Du fragst auch nach meiner Untersuchung über den Uranfang der Arithmetik: "Was sind und was sollen die Zahlen?" Sie ruht und ich zweifle, ob ich sie je publiciren werde; sie ist auch nur in rohem Entwurfe aufgeschrieben, mit dem Motto: "Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden." Die Hauptsache ist die Unterscheidung des Zählbaren vom Unzählbaren, und der Begriff der Anzahl, und die Begründung der sog. vollständigen Induction .... *GW III*, p. 486.

[29] "Verfolgt man genau, was wir beim Abzählen der Menge oder Anzahl von Dingen thun, so wird man nothwendig auf den Begriff der Correspondenz oder Abbildung geführt.
Die Begriffe des Systems, der Abbildung, welche im Folgenden eingeführt werden, um den Begriff der Zahl, der Anzahl zu begründen, bleiben auch dann für die Arithmetik unentbehrlich, selbst wenn man den Begriff der Anzahl als unmittelbar evident ("innere Anschauung") voraussetzen wollte." *Dugac*, p. 293.

This is the basis for the *radical break* with the considerations in *1854* and the description of the positive integers in *1872*, a break that was hinted at by the notational change from the creative act +1 to the successor operation φ in (the third version of) *Arithmetische Grundlagen*. The facts one is forced to accept from an informal analysis of number using these new conceptual tools "are still far from being adequate for completely characterizing the nature of the number sequence N";[30] for that the general notion of the *chain of a system A* is introduced. Its specialization to the chain of the system {1} leads then to the "complete" characterization of N as a simply infinite system. We will turn to these issues in subsection **B1** where we analyze the manuscripts *1872/78* and notice in **B2**, how this material is presented in the published version of *Was sind und was sollen die Zahlen?*.

In the Introduction to *1888* on page IV, the earlier manuscript is said to contain "all essential basic thoughts of my present essay" (alle wesentlichen Grundgedanken meiner vorliegenden Schrift); Dedekind mentions as some of the main points the "sharp distinction between the finite and the infinite", the concept of cardinal (Anzahl von Dingen), the justification of proof by induction and definition by recursion. The emphasis in the drafts is, however, almost exclusively on the proof principle; there are some very brief, almost cryptic hints concerning definition by recursion. From a modern perspective there is much more to the essay; for one, Dedekind does not mention the detailed metamathematical considerations and his reflections based on them. In the letter to Keferstein they are properly emphasized, and we will discuss some of them in section **B3**, namely, the existence and uniqueness (up to isomorphism) of simply infinite systems. That will be the background for discussing, again very briefly, the creation of numbers – with a new, more systematically founded perspective. In section **C3** we return to Dedekind's discussion of what the natural numbers are and what they are good for; then we will also look at his way of extending the absolute integers to all integers as he presented it after 1888 in the manuscript *Die Erweiterung des Zahlbegriffs auf Grund der Reihe der natürlichen Zahlen.*

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[30] Dedekind in his letter to Keferstein, in *van Heijenoort*, p. 100.

B1. Thoughts on numbers (between 2 and 3). We do not mean to discuss numbers between the integers two and three, but rather thoughts on numbers that Dedekind formulated between the publication of the second and third edition of Dirichlet's *Zahlentheorie* in 1871, respectively 1879. This was a fruitful and important period in Dedekind's work on algebraic number theory; in this period he published the French essay *Sur la Théorie des Nombres entiers algébriques*, and he worked intermittently, but strenuously, on a proper formulation of his *Gedanken über Zahlen*. The broad considerations, which were central for the mathematical and the foundational work, are highlighted in the *announcement* (Ankündigung) of the third edition and in a footnote to that very work.[31] Indeed, Dedekind refers back to these considerations in (a note to §161 of) the fourth edition of 1894 indicating very clearly, how important those reflections were for him:

> It is stated already in the third edition of the present work (1879, footnote on p. 470) that the entire science of numbers is also based on this intellectual ability to compare a thing *a* with a thing *a'*, or to relate *a* to *a'*, or to let *a* correspond to *a'*, without which no thinking at all is possible. The development of this thought has meanwhile been published in my essay "Was sind und was sollen die Zahlen?" (Braunschweig, 1888); ..."[32]

This remark is attached to a discussion of the general notion of mapping; the evolution of that notion in Dedekind's work is the focus of section **D**, but the material from the manuscript *Gedanken über Zahlen* reveals already some crucial aspects of this development.

The manuscript contains three quite distinct layers.[33] The first layer uses - in its initial attempt of characterizing natural numbers via *chains* – the notions *mappable, corresponding, image* (abbildbar, correspondirend, Bild), etc.; that matches in terminology and outlook *1871* (section I of §159 in Supplement X) as well as *1872*. In its second attempt, calling a chain a *group* (sic), the manuscript

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[31] See subsection **D2**, where both texts are quoted.

[32] "Schon in der dritten Auflage dieses Werkes (1879, Anmerkung auf S. 470) ist ausgesprochen, daß auf dieser Fähigkeit des Geistes, ein Ding a mit einem Ding a' zu vergleichen, oder a auf a' zu beziehen, oder dem a ein a' entsprechen zu lassen, ohne welche überhaupt kein Denken möglich ist, auch die gesamte Wissenschaft der Zahlen beruht. Die Durchführung dieses Gedankens ist seitdem veröffentlicht in meiner Schrift "Was sind und was sollen die Zahlen?" (Braunschweig 1888); ..."

[33] The first layer extends in *Dugac* from p. 293 to p. 297, the second from p. 297 to p. 304, and the third from p. 304 to p. 309. – The order of the layers reflects, quite clearly, the temporal evolution of Dedekind's ideas – with one exception: much of the material in the right-hand columns on pp. 293-4 must have been added later. In particular, we conjecture that the remarks quoted above from p. 293 of the manuscript (at the very beginning of this part of our paper) are from a later date; they fit systematically best with the beginning of

introduces on p. 296 for the first time in Dedekind's writings the term *mapping* (Abbildung); Dedekind distinguishes there, without any further explanation, between *injective* (deutliche) and *non-injective* (undeutliche) mappings. The second layer is the longest and most intricate one; it alone discusses finite cardinals. The third layer is closest to the eventual presentation of this material in *1888* and takes mappings officially as objects of study; it matches the remarks and note in *1879* mentioned above. Let us indicate briefly the *common* content. In each layer Dedekind considers a system S and a(n arbitray) mapping φ from S to S.[34] If φ is an *injective mapping* (deutliche Abbildung), the system S is called *infinite* just in case there is a proper *subset* (Theil) U of S, such that the system of *images* (Bilder) φ(S) is a subset of U. The other notions are defined relative to S and φ. A subset K of S is called a *chain* (Kette) if and only if it is closed under φ. A subset B of S is called *dependent on* (abhängig von) A if and only if B is a subset of any chain that contains A; or, more formally, for every chain K, $A \subseteq K$ implies $B \subseteq K$. The system of all things dependent on A is denoted by (A), and it is finally shown, as the central claim, that (A) *is a chain*.

These considerations provide the foundation for the proof principle of complete induction; in the third layer, on p. 307, it is formulated in a general way for any subset A of S and any chain K:

$$\text{If } A \subseteq K \text{ and } K \subseteq (A), \text{ then } (A)=K.$$

As a justification for induction it is easily established that, given two subsets A and K of S,

$$\text{If } A \subseteq K \text{ and } \varphi(K) \subseteq K, \text{ then } (A) \subseteq K.$$

Assume $A \subseteq K$ and $\varphi(K) \subseteq K$, consider an arbitrary a in (A), and distinguish two cases. In the first case a is in A, then – by the first assumption – a is in K. In the second case {a} is dependent on A, but not in A, i.e., contained in any chain that contains A. But K is such a chain; thus {a} is a subset of K, and a is an element of K. – The sequence of steps here anticipates that in *1888*, except for the definition

the third layer. The material on p. 294 uses notations that are introduced and explained only on p. 308, respectively on p. 301.

[34] It should be emphasized that the (in our view, original part of the) first layer does not have the explicit notation φ for a mapping, though §159 of *1871* does – for special functions, namely substitutions or, in our terminology, isomorphisms between fields.

of (A) via the dependency relation. It is only in a remark to the third layer that Dedekind recognizes (A) explicitly as the "smallest chain" containing A.

The second layer defines the relation "is dependent on" just for elements and calls the system (a) - of all elements dependent on a - the *sequence of* a (die Reihe von a). Specializing the considerations to injective mappings and infinite systems S, Dedekind establishes as a theorem that (1) *is an infinite system*. Every element of (1) is called a *number* (Zahl); proof by induction is justified as above, but the issue of definition by recursion is raised here, briefly. Dedekind notes on the margin:

The proof of the correctness of the method of proof from *n* to *n+1* is correct; in contrast, the proof (completeness) of the definition of concepts by the method from *n* to *n+1* is not yet sufficient at this point; the existence (consistent) of the concept remains in doubt. This will become possible only by *injectivity* [Deutlichkeit], by the consideration of the system [*n*]!!!!!! Foundation.[35]

This is a pregnant remark and points, together with theorems established on pp. 300-304, ahead to central issues in *1888*. To explain that claim, we have to define first of all the notation [n]. Informally, [n] is the system of all numbers less than or equal to n, for any n in (1). Systematically, [n] is defined (p. 300) as the system of numbers not contained in (n'), and it is shown to be finite. (In *1888* the systems [n] are denoted by Z_n.) Dedekind formulates as a theorem that *a system* B *is infinite, if every system* [n] *can be mapped injectively into* B. He remarks, on the margin, "Umständlich, aber möglich zu beweisen." This is, of course, the central and deep fact used to establish in §14 of *1888* that Dedekind's definition of (in)finite is equivalent to the standard one.[36] But that requires definition by recursion (and a form of the axiom of choice[37]): to show generally the existence of a mapping satisfying recursion equations the systems [n] are invoked, and Dedekind shows in Remark 130 of *1888* that the *injectivity* (Deutlichkeit) of the mapping φ is needed for that existence proof. All of this seems to be hinted at in the remark quoted above; it is a dramatic step for gaining a proper perspective.

The third layer is a very polished version of the considerations leading up to theorem 31 that states, (A) *is a chain*. But this time there is, as we mentioned

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[35] "Der Beweis der Richtigkeit der Beweismethode von n auf n+1 ist richtig; dagegen ist der Beweis (Vollständigkeit) der Begriffserklärung durch die Methode von n auf n+1 an dieser Stelle noch nicht genügend; die Existenz (widerspruchsfrei) des Begriffs bleibt zweifelhaft. Dies wird erst möglich durch die *Deutlichkeit*, durch die Betrachtung des Systems [n]!!!!!! Fundament." *Dugac*, p. 300.
[36] Such a standard definition is given, for example, in Bolzano's *Paradoxien des Unendlichen*, §§8-9.
[37] *Tarski 1924*; mentioned in *Belna* on p. 41.

already, a most interesting and important remark next to the statement of the theorem: "(A) is the 'smallest' chain that contains the system A" (die 'kleinste' das System A enthaltende Kette). The layer ends with brief remarks on the "direct treatment of the system Z of natural (i.e., whole positive rational) numbers." We quote those in full and note that Dedekind wrote next to the sentence just quoted "better N than Z" (Besser N statt Z):

Characteristic of the system Z. There is an injective [deutliche] mapping from Z—if *T* is a part of Z, then the image of *T* is denoted by *T'*—, which has the following property.

I. *Z'* is a part of Z.

II. There is a number (i.e., a thing contained in Z), which is not contained in *Z'*. This number shall be called "one" and is denoted by 1.

III. A number chain (i.e., each part *T* of Z, whose image *T'* is a part of *T*) that contains the number 1 is identical with Z.[38]

This "characteristic" corresponds perfectly to the axiomatic conditions for a simply infinite system in *1888*, i.e., we have here the very first formulation of the so-called Peano Axioms.[39]

B2. Axioms for numbers. In our systematic analysis of *1888* we repeatedly use Dedekind's private letter to Keferstein and also his official reply *1890** to Keferstein's review of *1888*. In these documents, Dedekind makes his methodological considerations much more explicit than in the essay itself. Indeed, in the letter Dedekind poses these general questions:

What are the mutually independent fundamental properties of the sequence N, that is, those properties that are not derivable from one another but from which all others follow? And how should we divest these properties of their specifically arithmetic character so that they are subsumed under more general notions and under activities of the understanding *without* which

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38 "*Charakteristik des Systems* Z. Es giebt eine deutliche Abbildung von Z - ist T ein Theil von Z, so soll das Bild von T mit T' bezeichnet werden -, welche folgende Eigenschaft besitzt.

I. Z' ist Theil von Z.

II. Es giebt eine Zahl (d.h. ein in Z enthaltenes Ding), welche nicht in Z' enthalten ist. Diese Zahl soll "Eins" heissen und mit 1 bezeichnet werden.

III. Eine Zahlkette (d.h. jeder Theil T von Z, dessen Bild T' ein Theil von T ist), welche die Zahl 1 enthält, ist identisch mit Z."

39 Peano mentions in the Introduction of his *1889*: "In this paper I have used the research of others." In particular, he states later in the paragraph that begins with the sentence just quoted, "Also quite useful to me was the recent work by R. Dedekind, *Was sind und was sollen die Zahlen* (Braunschweig, 1888), in which questions pertaining to the foundations of numbers are acutely examined." (p. 103). *Belna 1996*, on p.60, refers to a text from 1891, in which "Peano recognizes that his axioms 'are due to Dedekind' and drawn from #71 of the latter's book." Stein remarks in his *2000a* that "Giuseppe Peano directly borrowed his axioms for arithmetic" from Dedekind's characterization of the system of natural numbers as a simply infinite system. C.S. Peirce made priority claims at a number of occasions; they are discussed very well, and accorded their proper place, in *Belna* on pp. 57-9. - It is quite clear from the above discussion that Dedekind gives an analysis of natural numbers in *1872/78* that culminates in their axiomatic characterization. However, the further claim - as found in *Belna* on p. 58 and *Stein 2000a* - that there is no essential difference (except by the absence of the theorem concerning the existence of infinite systems) between the 1872/78 manuscript and *1888* is not correct; for example, none of the metamathematical results and broader conceptual reflections discussed in the next subsection are contained in *1872/78*.

no thinking is possible at all but *with* which a foundation is provided for the reliability and completeness of proofs and for the formulation of consistent definitions of concepts?[40]

When one poses the problem in this way, Dedekind continues, then one is "forced to accept the following facts" that are presented as points (1) through (5) in the letter, namely: the number sequence N is a system of elements or individuals, called *numbers*; the relation between these elements is given by a mapping φ from N to N; φ must be *similar* (ähnlich - the term replacing "deutlich" used in the earlier discussion); the image of N under φ is a proper part of N, and 1 is the only element not in the image. The central methodological problem, Dedekind emphasizes, is the precise characterization of just those individuals that are obtained by iterated application of φ to 1; this is to be achieved in general logical terms, not presupposing arithmetic notions. Before addressing this central problem, Dedekind introduces as above — relative to a system S and an arbitrary mapping φ from S to S — the general concept of a chain. Then he defines directly, using the insight hinted at in the third layer of *1872/78*, the *chain* A_0 *of a system* A as the *intersection* (Gemeinheit) of all chains containing A. A_0 has the obvious properties

$$A \subseteq A_0$$

$$\varphi(A_0) \subseteq A_0$$

$$A \subseteq K \ \& \ \varphi(K) \subseteq K \ \Rightarrow \ A_0 \subseteq K$$

These properties characterize A_0 uniquely. From the last property, expressing the minimality of A_0 among all the chains that contain A, it is easy to prove a general induction principle in the form:

$$(*) \quad A \subseteq \Sigma \ \& \ \varphi(A_0 \cap \Sigma) \subseteq \Sigma \ \Rightarrow \ A_0 \subseteq \Sigma$$

Σ denotes the extension of any property E pertaining to the elements of S.

After this preparatory step, Dedekind specializes the consideration to the chain N of the system {1} for the similar mapping φ, i.e., his *simply infinite system*

[40] *Van Heijenoort*, pp. 99-100, except for a correction in the very last sentence, where "formulation of consistent definitions of concepts" replaces "construction of consistent notions and definitions." The German text, also reprinted in *Sinaceur 1974* p. 272, is as follows: Welches sind die von einander unabhängigen Grundeigenschaften dieser Reihe N, d.h. diejenigen Eigenschaften, welche sich nicht aus einander ableiten lassen, aus denen aber alle anderen folgen? Und wie muss man diese Eigenschaften ihres spezifisch arithmetischen Characters entkleiden, der Art, dass sie sich allgemeinen Begriffen und solchen Tätigkeiten des Verstandes unterordnen, *ohne* welche überhaupt kein Denken möglich ist, *mit* welchen aber auch die Grundlage gegeben ist für die Sicherheit und Vollständigkeit der Beweise, wie für die Bildung widerspuchsfreier Begriffs-Erklärungen?

(N, φ, 1). The essence (Wesen) of such a system is consequently given by the axiomatic conditions α, β, γ, and δ as formulated by Dedekind in *Erklärung 71*:

$$\varphi(N) \subseteq N$$
$$N = 1_0$$
$$1 \notin \varphi\,(N)$$
$$\varphi \text{ is a similar mapping}$$

Condition β expresses in Dedekind's notation that N is the chain $\{1\}_0$ of the system {1}; it is the basis for the usual induction principle for natural numbers formulated now as follows:

$$(**)\quad \{1\} \subseteq \Sigma \ \&\ \varphi(N \cap \Sigma) \subseteq \Sigma \ \Rightarrow\ N \subseteq \Sigma$$

The considerations leading to (**) are completely parallel to those for (*) above. Indeed, reordering conditions α, β, γ, and δ, reformulating them a little, and using (**) as the induction principle yields:

$$1 \in N$$
$$(\forall\, n \in N)\ \ \varphi(n) \in N$$
$$(\forall\, n, m \in N)\ \ (\varphi(n) = \varphi(m)\ \Rightarrow\ n = m)$$
$$(\forall\, n \in N)\ \varphi(n) \neq 1$$
$$(1 \in \Sigma\ \ \&\ \ (\forall\, n \in N)\ (n \in \Sigma \Rightarrow \varphi(n) \in \Sigma))\ \Rightarrow\ (\forall\, n \in N)\ n \in \Sigma$$

These statements make explicit the principles underlying Dedekind's earlier "characteristic of the system Z" and are mere notational variants of the five axioms for the positive integers formulated in Peano's *1889*.

Hilbert's axiomatization for N in his *1905* also uses these axioms, clearly extracted from Dedekind's characterization of simply infinite systems.[41] Hilbert's syntactic consistency proof in that paper was to guarantee the existence of the "smallest infinite." Thus, his proof was to serve the dual purpose of Dedekind's argument for the existence of a simply infinite system. Already in his *1900a* and *1900b* Hilbert intended to insure the existence of a set, here the set of real numbers, by a "direct" proof of the consistency of an appropriate axiomatic theory. The theory was formulated in Dedekind's style: one considers a system of objects satisfying certain "axiomatic" conditions. In contrast to Dedekind,

[41] For details, see *Sieg 2002*, pp. 366-371. Hilbert does not formulate the induction principle formally; he just claims that it can be so formulated.

Hilbert called a theory consistent (widerspruchsfrei) if it does not allow establishing – in finitely many steps – a contradiction; note that this is only a quasi-syntactic specification of consistency, as no characterization is given of the steps that are allowed in proofs. We turn our attention now to Dedekind's way of thinking about, and addressing, this issue; for more details on Hilbert's approach cf. the section *Göttingen 1899* (in Part 2 of our essay).

B3. Existence and uniqueness. The number sequence N is characterized completely as (the abstract type of) a simply infinite system, Dedekind writes to Keferstein; how is this to be understood? The answer to the question will evolve through a sequence of detailed metamathematical, reflective steps concerning simply infinite systems. We discuss the existence (proof) of such systems and two senses of uniqueness, namely, the literal uniqueness of "the chain of {1}" and then the completely new sense of uniqeness "up to isomorphism."

The very notion of a simply infinite system is defined as a triple (N, φ, 1), or in contemporary model-theoretic terminology as a structure, that satisfies the conditions α, β, γ, and δ from Dedekind's *Erklärung* 71. These conditions correspond, as we saw, to the so-called Peano Axioms. It is perfectly natural for Dedekind to ask: "Does such a system exist at all in our realm of thoughts?" (Existiert überhaupt ein solches System in unserer Gedankenwelt?)[42] The answer to this question is given by a *logical existence proof*, and Dedekind explains to Keferstein that without such a proof "it would remain always doubtful, whether the concept of such a system does not perhaps contain internal contradictions."[43] In his official response *1890** to Keferstein's article, Dedekind asserts more strongly, "as long as such a proof has not been given one must fear that the above definition of the system N contains an internal contradiction, whereby the certainty of arithmetic would be lost."[44] That is the reason, he emphasizes in his letter, why the proofs in #66 and #72 of his essay are necessary.

The crucial considerations are really presented in the proof of theorem 66; theorem 72 just states that every infinite system S contains a simply infinite one

[42] In *van Heijenoort* "Gedankenwelt" is (misleadingly) translated as "realm of ideas."
[43] The German text is: "Ohne den logischen Existenz-Beweis würde es immer zweifelhaft bleiben, ob nicht der Begriff eines solchen Systems vielleicht innere Widersprüche enthält."

as a part, and that claim can be established straightforwardly. In the proof of the claim "There are infinite systems," Dedekind starts out by formulating the claim for a specific system, namely, for his *Gedankenwelt*. Dedekind's *Gedankenwelt* is defined as "the totality S of all things that can be an object of my thinking" (die Gesamtheit S aller Dinge, welche Gegenstand meines Denkens sein können); the thought s′ that "s can be an object of my thinking" is, for an arbitrary element s of S, itself an element of S. The associated operation φ that leads from s to s′ is injective; in addition, the set of images S′ is a proper part of S, as Dedekind's own self, for example, is not in S′. Thus, S together with φ is an infinite system. In *1890** Dedekind reproduces the proof of #66, asserts on p. 262 that he considers it as "streng richtig," and explicates it in an informative way without – he claims – adding anything new. Indeed, the explication consists in expanding the above specification of φ by a parenthetical remark; instead of considering "the thought s′ that …," Dedekind considers here "the thought s′ (expressible in the form of a sentence or judgement) that … ." This seems to indicate directly that Dedekind's thoughts are not to be viewed as psychological ideas. There is also indirect evidence: Frege, on p. 147 of his manuscript *Logik*, says that he himself uses the word "Gedanke" in an unusual way and asserts: "Dedekind's usage agrees with mine." (Dedekinds Gebrauchsweise stimmt mit meiner überein.)[45] Such a Fregean understanding is reinforced, when Dedekind continues his explication by claiming that this thought s′ can be an object of his thinking – "I may think, e.g. of this thought s′, that it is obvious, that it has a subject and a predicate, etc." (ich darf z.B. von diesem Gedanken s′ denken, dass er selbstverständlich ist, dass er ein Subjekt und ein Prädikat besitzt u.s.w.) – and that it is consequently an element of S.

The need for a logical existence proof of an infinite system is not discussed at all in the manuscripts *1872/78*; such a proof is given in the manuscript from

[44] *Sinaceur 1974*, p. 266. The German text is: "…so lange ein solcher Beweis nicht geliefert ist, darf man befürchten, dass die obige Definition des Systems N einen inneren Widerspruch enthält, womit dann die Gewissheit der Arithmetik hinfällig würde."

[45] Quoted from p. 138 of *Frege 1969*. – McCarty asserts in his *1995* that section 66 distinguishes itself "as the most blatantly psychologistic." To support this claim in note 5, p. 93, and also to bolster his contention of a strong connection between Kant and Dedekind on p. 71, McCarty relies on the mistranslation of "Gedankenwelt" as "realm of ideas" in *van Heijenoort*. McCarty writes on p. 71: "…we will find the mathematical objects of Dedekind among the pure ideas of Kant. Dedekind did, after all, write to Keferstein

1887 that precedes the final writing of *1888*. The manuscript *1887* was begun, according to Dedekind's annotation, on 10 June 1887 and presents an "outline of the investigation" (Gerippe der Untersuchung). Its fourth section is entitled "The finite and infinite" (Das Endliche und Unendliche) and begins with a definition.

40. Definition. S is called an <u>infinite</u> system, if there is an <u>injective</u> mapping from S, such that the image of S is a <u>proper</u> part of S; in the opposite case S is called a finite system.[46]

This is immediately followed by the remark "all hitherto known definitions of the finite and the infinite are completely useless, to be rejected by all means."[47] This is in turn followed by a proposition, numbered 41, with a very detailed proof.

41. Proposition: If S=M(*a*,T), with *a* an element of S, and T a finite system, then also S is a finite system.[48]

M is the union operation, joining the singleton {*a*} to T.

As in other manuscripts of Dedekind's, the pages of *1887* are vertically divided in half. The main text is written on one half, whereas the other half is reserved for later additions. On this particular page a number of important additions are made. The very first line indicates that the manuscript is still in the process of being reorganized:

See: Remarks to §4 on separate page. <u>Here</u> also the first two propositions from §7 belong.[49]

This is followed by three propositions, numbered 40^{x}, 40^{xx}, and 40^{xxx}; the last in this sequence claims: "There are infinite systems." (Es giebt unendliche Systeme.) Dedekind added parenthetically, "Remarks on separate page" (Bemerkungen auf dem Beiblatt), and mentioned there that the following proposition can be added immediately to the "fundamental definition 40" (fundamentale Erklärung 40):

<u>Proposition</u>: There are infinite systems; the system S of all those things s (- this word understood in the sense given in the introduction -) that <u>can be</u> objects of my thinking, is infinite (my realm of thoughts).[50]

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that he must locate the infinite system of natural numbers 'in the realm of our ideas.' (Dedekind 1967, p. 101)"

[46] *Dedekind 1887*. "40. Erklärung: S heißt ein <u>unendliches</u> System, wenn es eine derartige deutliche Abbildung von S gibt, daß das Bild von S ein <u>echter</u> Teil von S ist; im entgegengesetzten Fall heißt S ein <u>endliches</u> System."

[47] *Dedekind 1887*. "Anmerkung: alle bisher bekannten Definitionen des Endlichen und Unendlichen sind gänzlich unbrauchbar, durchaus zu verwerfen."

[48] *Dedekind 1887*. "41. Satz: Ist S=M(*a*,T), wo *a* ein Element fo S, und T ein endliches System bedeutet, so ist auch S ein endliches System."

[49] *Dedekind 1887*. "Siehe: Bemerkungen zu §4 auf Beiblatt. <u>Hierher</u> gehören aud die ersten Sätze aus §7."

[50] *Dedekind 1887*. "<u>Satz</u>. Es giebt unendliche Systeme; das System S aller derjenigen Dinge s (- dieses Wort in dem in der Einleitung angegebenen Sinne verstanden -), welche Gegenstand meines Denkens <u>sein können</u>, ist unendlich (meine Gedankenwelt)."

The proposition is followed by a proof of roughly the same character as given in the later sources. What is significant, and the reason for our tediously detailed description of this part of *1887*, is the fact that the (proof of the) existence of infinite systems enters Dedekind's available manuscripts only at this late stage. However, this manuscript of 1887 is so remarkably different from those in *1872/78* that one can only conjecture: Dedekind must have had an "intermediate" manuscript. Given his own remarks concerning the connection with Bolzano's, and Cantor's work (quoted in the next paragraph), we speculate that Dedekind must have completed such an intermediate manuscript *before* 1878.

In *1888*, Dedekind writes in the footnote to theorem 66, "A similar consideration is found in §13 of Bolzano's *Paradoxien des Unendlichen* (Leipzig 1851)." (Eine ähnliche Betrachtung findet sich in §13 der *Paradoxien des Unendlichen* von Bolzano (Leipzig 1851).) Indeed, the considerations are similar, in particular, if we compare Bolzano's with Dedekind's in the "explicated" form pertaining to thoughts expressible in sentences. Bolzano argues that "the set of sentences and truths in themselves" (die Menge der Sätze und Wahrheiten an sich) is an infinite multiplicity. This is achieved by considering, first of all, any truth whatsoever and then the construction principle (*the proposition*) A *is true,* leading from any true proposition A to a distinct new and true proposition. Bolzano concludes that this set "enjoys a multiplicity surpassing every individual integer" and is therefore, according to Bolzano's definition, infinite.[51]

Dedekind uses, as we saw and as he emphasizes in the preface to the second edition of *1888*, a different definition of "infinite" that corresponds to a property of infinite sets that had been recognized also by Bolzano and Cantor. However,

> ... neither of these authors made the attempt to use this property for the definition of the infinite and to establish upon this foundation with rigorous logic the science of numbers. But this is precisely the content of my difficult labour, which in all its essentials I had completed several years before the appearance of Cantor's memoir [i.e., *Cantor 1878*] and at a time when the work of Bolzano was completely unknown to me, even by name.[52]

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[51] *Bolzano 1851*, p. 258.

[52] *Ewald 1996*, p. 796. "... keiner der genannten Schriftsteller hat den Versuch gemacht, diese Eigenschaft zur Definition des Unendlichen zu erheben und auf dieser Grundlage die Wissenschaft von den Zahlen streng logisch aufzubauen, und gerade hierin besteht der Inhalt meiner mühsamen Arbeit, die ich in allem Wesentlichen schon mehrere Jahre vor dem Erscheinen der Abhandlung von G. Cantor [i.e., *Cantor 1878*] und zu einer Zeit vollendet hatte, als mir das Werk von Bolzano selbst dem Namen nach gänzlich unbekannt war." *Dedekind 1888*, pp. IX-X.

In October of 1882, Cantor sent Dedekind a copy of Bolzano's booklet - characterized by Cantor as "a strange little work" (ein merkwürdiges Werkchen) of which he happened to have a second copy - and wrote: "Although much, maybe most of it, is mistaken, it still was extremely stimulating for me, in particular by the contradictions, which it provoked in me." (Trotzdem darin Vieles, vielleicht das meiste verfehlt ist, war es für mich doch äusserst anregend, namentlich durch die Widersprüche, die es in mir erregt hat.) Whether and how Dedekind was influenced by this work in formulating his proof of theorem 66 remains a topic of speculation; he himself just points out a "similarity." (But see, for example, *Sinaceur 1974*, p. 254, and *Belna* pp. 37-8 and 54ff.) We point to a central dissimilarity and, without further elaboration, to the fact that Dedekind's formulations are dramatically more rigorous.[53] Bolzano bases his considerations concerning the *objective existence* of the infinite implicitly on the existence of the species of integers and explicitly on the existence of the "set of sentences and truths in themselves" (Menge der Sätze und Wahrheiten an sich), whereas Dedekind uses "only" the universal system, his *Gedankenwelt;* a simply infinite system and then the natural numbers are obtained from it. It seems, Dedekind could have used Bolzano's set; after all, he just needed one "natural" system that is non-empty and closed under a suitable successor operation.

How then are natural numbers obtained in Dedekind's case? Any infinite system whatsoever has as a part a simply infinite one that is unique as a minimal chain, as we observed above. To insist on minimality has a remarkable metamathematical reason emphasized by Dedekind in both *1890* and *1890** (pp. 267 ff), namely, to exclude "intruders." (These intruders are, in modern terminology, non-standard elements.) The minimality captures rigorously, as Dedekind struggled to do, the informal, motivating idea that every element of the chain is obtained by (finite) iteration of the operation φ applied to 1.[54] This is the basis for establishing that simply infinite systems are unique in a quite novel sense. Given the above analysis, it is most direct – to use the general concept of a

[53] We discuss how Dedekind obtains natural numbers below. Compare that approach to Bolzano's quick step in §8, where - after describing the formation of series that start from a particular individual of a species A and proceed by adjoining a fresh individual from that species – says: "Such multitudes I call *finite* [endlich] or *countable* [zählbar], or quite boldly: *numbers;* and more specifically: *whole* numbers ..."
[54] Cf. *Sinaceur 1974*, p. 268.

mapping and – to conceive of a bijection ψ between two arbitrary simply infinite systems based on operations φ and θ that would map the first element of one to the first element of the other; in addition, the bijection would satisfy the recursion equation $\psi(\varphi(n)) = \theta(\psi(n))$. Of course, it is *one thing* to graphically draw such a bijection and *quite another thing* (i) to have the appropriate mathematical (or logical) notions to capture the essence of the situation, and (ii) to provide proofs of the unique existence of such a structure-preserving mapping. The *one thing* is undoubtedly in the back of everybody's mind – Kronecker's, for example, in § 1, *Definition des Zahlbegriffs*, of his *1887*. That applies also to Bolzano. The *other thing* is what Dedekind does in §9 of *1888*! The informal understanding allows Dedekind to say already in *Erklärung* 73:

If in the consideration of a simply infinite system *N* ordered by a mapping φ we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations in which they are placed to one another by the ordering mapping φ, then these elements are called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series N*. With reference to this freeing of the elements from every other content (abstraction) we are justified in calling the numbers a free creation of the human mind.[55]

This remark is considered to be "completely justified" only after the work in §9 and §10; we will return to it in **C3**.

There is another very important reason motivating the detailed investigation of definition by recursion. In his letter to Keferstein Dedekind emphasizes the need "to formulate the *definitions* of operations on numbers without contradictions for *all* numbers n" (die *Definitionen* von Zahlen-Operationen widerspruchsfrei für *alle* Zahlen n aufzustellen). It is one particular aspect of Dedekind's general concern to provide the tools for the actual re-development of arithmetic in his rigorous framework, and thus to achieve a "quasi-empirical completeness;" recall the parallel considerations for operations on the reals in *1872*. Here, in the case of number theory, the principle of ordinary induction is central as a proof principle (and we saw, how it is justified); the principle of definition by recursion is equally crucial as it allows

[55] *Ewald 1996*, p. 809. "Wenn man bei der Betrachtung eines einfach unendlichen, durch eine Abbildung φ geordneten Systems N von der besonderen Beschaffenheit der Elemente gänzlich absieht, lediglich ihre Unterscheidbarkeit festhält und nur die Beziehungen auffaßt, in die sie durch die ordnende Abbildung φ zueinander gesetzt sind, so heißen diese Elemente *natürliche Zahlen* oder *Ordinalzahlen* oder auch schlechthin *Zahlen*, und das Grundelement 1 heißt die *Grundzahl* der *Zahlenreihe* N. In Rücksicht auf diese Befreiung der Elemente von jedem anderen Inhalt (Abstraktion) kann man die Zahlen mit Recht eine freie Schöpfung des menschlichen Geistes nennen."

defining the standard arithmetic operations like addition, multiplication, and exponentiation. Dedekind isolates it in the following form, his proposition 126: *Satz der Definition durch Induktion*: let N be the number-series as discussed above, let θ be an arbitrary mapping from a system Ω to Ω, and let ω be an element of Ω; then there is exactly one mapping ψ from N to Ω that satisfies the conditions

I. $\psi(N) \subseteq \Omega$,

II. $\psi(1) = \omega$,

III. $\psi(n')=\theta(\psi(n))$.

The justification requires subtle metamathematical considerations; i.e., a proof by induction of the existence of suitable approximations to the intended mapping for initial segments of N. The basic idea was used later in axiomatic set theory and extended to transfinite recursion, but Gödel used it also within formal arithmetic.[56]

These reflections concerning induction and recursion have consequently two fundamental goals: to provide the tools for developing ordinary number theory and to serve as the systematic background for Dedekind's answer to the question *Was sind die Zahlen?*. We saw already, how they are needed for the full justification of the abstractionist move described above, when the natural numbers are viewed as a "free creation of the human mind." In the earlier manuscript *1887* one finds at this very spot a more expanded and explicit formulation; he writes there:

> By this abstraction, the originally given elements n of N are turned into new elements *n*, namely into numbers (and N itself is consequently also turned into a new abstract system $\mathcal{N}$). Thus, one is justified in saying that the numbers owe their existence to an act of free creation of the mind. For our mode of expression, however, it is more convenient to speak of the numbers as of the original elements of the system N and to disregard the transition from N to $\mathcal{N}$, which itself is an injective mapping. Thereby, as one can convince oneself using the theorems regarding definition by recursion, nothing essential is changed, nor is anything obtained surreptitiously in illegitimate ways.[57]

[56] In contrast to induction, the recursion principle is not correct for arbitrary chains; that is discussed in *Bemerkung* 130 of *1888*.

[57] The German text is: Da durch diese Abstraction die ursprünglich vorliegenden Elemente n von N (und folglich auch N selbst in ein neues abstraktes System N) in neue Elemente n, nämlich in Zahlen umgewandelt sind, so kann man mit Recht sagen, daß die Zahlen ihr Dasein einem freien Schöpfungsacte des Geistes verdanken. Für die Ausdrucksweise ist es aber bequemer, von den Zahlen wie von den ursprünglichen Elementen des Systems N zu sprechen, und den Übergang von N zu N, welcher selbst eine deutliche Abbildung ist, außer Acht zu lassen, wodurch, wie man sich mit Hilfe der Sätze über Definition durch Recursion ... überzeugt, nichts Wesentliches geändert, auch Nichts auf unerlaubte Weise erschlichen wird.

Kronecker and Helmholtz share the "naïve" starting-point with Dedekind, but it is only Dedekind who builds a remarkable conceptual frame in which he can express sharply the (naïve) analysis and provide a fruitful meta-mathematical investigation. Speaking in 1877, section 107, again about the "creation of the pure natural numbers" (Schöpfung der reinen natürlichen Zahlen), Dedekind emphasizes the consequences of such an investigation:

It follows from the above, that the laws regarding the relations between the numbers are completely independent from the choice of that simply infinite system $\mathcal{N}$, which we called the number sequence, and that they are also independent from the mapping of $\mathcal{N}$ that orders $\mathcal{N}$ as a simple sequence.[58]

These methodological themes will be at the center of our discussions in **Part 2**: *models and mappings*. (A table of contents is provided in Appendix B.)

Appendix A

This is a review of Dedekind's 1888, written by F. Meyer who was professor at Clausthal; the review was published in 1902!

"In der neueren Zeit tritt das Bestreben mehrfach hervor, der Arithmetik eine festere Grundlage zu geben. Auch die vorliegende Schrift verfolgt diesen Zweck in einer eigenartigen Weise, sie geht von vornherein von dem Princip aus, dass die Lehre von den ganzen Zahlen und ihren Verknüpfungen einen Teil der reinen Logik zu bilden habe, führt aber dasselbe weit eingehender und präciser durch, als dies bisher je geschehen sein dürfte.

[...]

Der Verfasser sieht bei seinen Darlegungen von specifischen mathematischen Kenntnissen völlig ab, er wendet sich demgemäss an jeden Gebildeten. Trotzdem lässt sich wohl nicht leugnen, dass er der Abstractionskraft des Lesers im Ganzen mehr zumutet, als irgend eine rein mathematische Schrift. Zum Teil liegt die Schwierigkeit des Verständnisses in der Form der Darstellung, die nach dem classischen Muster der Alten den ganzen Stoff in einer grossen Anzahl ganz allmählich

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[58] The German text is: Aus dem Vorhergehenden ergiebt sich, daß die Gesetze über die Beziehungen zwischen den Zahlen gänzlich unabhängig von der Wahl desjenigen einfach unendlichen Systems $\mathcal{N}$ sind, welches wir die Zahlenreihe genannt haben, sowie auch unabhängig von der Abbildung von $\mathcal{N}$, durch welche $\mathcal{N}$ als einfache Reihe geordnet ist.

fortschreitender Sätze bewältigen will. So gross daher die Deutlichkeit im Einzelnen ist, so ist doch andererseits, da hier jede geometrische Anschauung fehlt, eine grosse Ausdauer nöthig, um die Fortschritte der leitenden Gedanken im Ganzen übersehen zu können. Zum Teil aber ist es auch die grosse Allgemeinheit der Grundauffassung des Autors.

[...]

Für unsere Vorstellung allerdings sinken die gemeinhin Zahlen genannten Dinge vermöge der erwähnten Abstractionen zu blossen Schatten herab, dafür sind sie aber auch aller subjectiven Willkür entzogen, und, strengen rein logischen Regeln unterworfen, bieten sie für den Arithmetiker völligen Ersatz für jene populären Zahlen."

Appendix B

PART 2: models and mappings

Göttingen 1899: axiomatizing the reals

C. Numbers: methodological considerations

- C1. Continuous domains (geometric line; cuts)
- C2. Axiomatic method and correspondences
- C3. Creating number systems

D. Mappings

- D1. Substitutions
- D2. Deutlichkeit and permutations
- D3. Categoricity

Concluding remarks (which incorporate *Göttingen 1932: gauging the impact* and discuss existential axiomatics as well as reductive structuralism)

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