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Dedicated to Saunders Mac Lane on his 90th birthday.

Abstract

We present a complete elementary axiomatization of local maps of toposes.

1 Introduction

We recall the definition of a local map of toposes [9, 10, 7] (see in particular [7, Proposition 1.4]).

Definition 1.1. Let $\mathcal{E}$ and $\mathcal{F}$ be elementary toposes. A geometric morphism $f = (f^*, f_*)\colon \mathcal{E} \to \mathcal{F}$ is **local** if it is bounded and the direct image functor f_* has a right adjoint $f^!$ which is full and faithful.

There are many examples of local maps of toposes, the classical one being (the structure map of sheaves on) the spec of a local ring (arising, *e.g.*, from localization at a point). See, *e.g.*, [7] for many other topological and presheaf examples. See [1] for an example of a (localic) local map between realizability toposes; this example is the one that gave rise to this work.

Suppose $(\Delta, \Gamma)\colon \mathcal{E} \to \mathcal{F}$ is a local map of toposes. Then since the right adjoint, call it ∇, of Γ is full and faithful, it follows easily that the inverse image functor Δ is full and faithful. Moreover, $\Gamma\Delta \cong 1 \cong \Gamma\nabla$. Further note that there is therefore a geometric inclusion $(\Gamma, \nabla)\colon \mathcal{F} \to \mathcal{E}$. Thus there is a Lawvere-Tierney topology j on $\mathcal{E}$ and an equivalence $\mathcal{F} \simeq \mathrm{Sh}_j\mathcal{E}$ such that (Γ, ∇), under this equivalence, is identified with $(\mathbf{a}, i)$, the associated sheaf functor and the inclusion of sheaves. Since Γ has a left exact left adjoint Δ, it follows that $\mathbf{a}$ has the same (namely $\Delta\Gamma i$). Summarizing, a local map from $\mathcal{E}$ is essentially a sheaf subtopos with a left exact left adjoint to sheafification.

Next, recall that a sheaf subtopos $\mathrm{Sh}_j\mathcal{E}$ of $\mathcal{E}$ can be characterized as the full subcategory of objects orthogonal to all morphisms inverted by the associated sheaf functor $\mathbf{a}$ [4, 6]. Dually, define an object $D \in \mathcal{E}$ to be **discrete** iff D

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is *coorthogonal* to all morphisms inverted by $\mathbf{a}$. (Recall that an object X is coorthogonal to a morphism $f: A \to B$ in a category $\mathbb{C}$, written $f \top X$, if, for all $b: X \to B$, there exists a unique $a: X \to A$ such that the diagram

$$\begin{tikzcd} & A \arrow[d, "f"] \\ X \arrow[ur, dashed, "a"] \arrow[r, "b"'] & B \end{tikzcd}$$

commutes.) We let $\mathrm{D}_j\mathcal{E}$ denote the full subcategory of $\mathcal{E}$ on the discrete objects. By Theorem 2.4 of Kelly and Lawvere [8] it follows that $\mathrm{D}_j\mathcal{E}$ is equivalent to $\mathrm{Sh}_j\mathcal{E}$ just in case $\mathrm{D}_j\mathcal{E}$ is coreflective in $\mathcal{E}$, making $\mathrm{Sh}_j\mathcal{E}$ an essential localization. Hence to show that there is a local map from $\mathcal{E}$ to $\mathrm{Sh}_j\mathcal{E}$ it suffices to show that the inclusion of $\mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$ of the discrete objects has a right adjoint and is itself left exact. This, finally, is the approach we shall take to axiomatizing local maps—we assume given a topos $\mathcal{E}$ with a topology j and find conditions on $\mathcal{E}$ and j such that the inclusion of $\mathrm{D}_j\mathcal{E}$ into $\mathcal{E}$ is left exact and has a right adjoint.

The final section of the paper is devoted to analysing the "internal logic" of a local map of toposes. This is determined to be a modal logic with two propositional operations, one of which is an S4 box operation and the other, its *right* adjoint.

Acknowledgements

Some of the work presented here forms part of the second author's PhD thesis [2], written under the guidance of Prof. Dana Scott. We refer the reader to *loc. cit.* for more details than we can include here. We are both grateful for useful discussions with Dana Scott, Andrej Bauer, Martin Hyland, Jaap van Oosten, Pino Rosolini, and Peter Johnstone. We also thank the organizers of CT'99 in Coimbra, Portugal, for a nice conference.

2 Preliminaries

Throughout this section, let $\mathcal{E}$ be an elementary topos with a Lawvere-Tierney topology j, and write $\mathrm{Sh}_j\mathcal{E}$ for the subcategory of sheaves, with associated sheaf functor $\mathbf{a}: \mathcal{E} \to \mathrm{Sh}_j\mathcal{E}$. Write $\mathrm{D}_j\mathcal{E}$ for the subcategory of discrete objects as defined above.

Observe that since $\mathrm{D}_j\mathcal{E}$ is defined by coorthogonality conditions, the category $\mathrm{D}_j\mathcal{E}$ is closed under colimits in $\mathcal{E}$ and the inclusion functor $\mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$ preserves them.

We write $V \mapsto \overline{V}$ for the j-closure operation on subobjects $V \rightarrowtail X$.

Definition 2.1. We say j is **principal** if, for all $X \in \mathcal{E}$, the closure operation on $\mathrm{Sub}(X)$ has a left adjoint $U \mapsto \mathring{U}$, called **interior**; that is,

$$\mathring{U} \leq V \iff U \leq \overline{V} \qquad \text{in } \mathrm{Sub}(X). \tag{1}$$

Remark 2.2. The interior operation is *not* assumed to commute with pullback. It follows that in general, unlike closure, the interior operation is not induced by an internal map on the subobject classifier Ω in the topos $\mathcal{E}$.

Lemma 2.3. *A topology j in a topos $\mathcal{E}$ is principal iff, for all $X \in \mathcal{E}$, there exists a* least dense *subobject U_X of X.*

Proof. Given a principal topology, the least dense subobject U_X of X is $\mathring{X}$. Conversely, given least dense subobjects U_X, define $\mathring{V} = U_V \rightarrowtail V \rightarrowtail X$. The condition (1) then follows easily. □

Observe that, if the topology j is principal, then for all $X \in \mathcal{E}$ and all $V \in \mathrm{Sub}(X)$, $\overline{\mathring{V}} = \overline{V}$ and $\mathring{\overline{V}} = \mathring{V}$ in $\mathrm{Sub}(X)$.

The interior operation $X \mapsto \mathring{X}$ extends to a functor on $\mathcal{E}$ as follows: given $f\colon X \to Y$, consider the diagram

$$\begin{tikzcd}
\mathring{X} \arrow[r, dashed] \arrow[dr] & f^*(\mathring{Y}) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathring{Y} \arrow[d, tail] \\
& X \arrow[r, "f"'] & Y,
\end{tikzcd}$$

where the right hand square is a pullback. Since closure preserves dense subobjects, we have that $f^*(\mathring{Y})$ is dense in X; hence $\mathring{X} \leq f^*(\mathring{Y})$ as shown in the diagram. Letting $\mathring{f}$ be the composite morphism across the top of the diagram, we clearly get a functor on $\mathcal{E}$. We refer to this functor as the **interior** functor; it clearly preserves monomorphisms.

For $f\colon X \to Y$ in $\mathcal{E}$ we write $\exists_f$ for the left adjoint to the pullback functor $f^*\colon \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$. Since closure commutes with pullback, by taking left adjoints we see that, when j is principal, $\exists_f(\mathring{V}) \cong (\mathring{\exists_f V})$, for all $X, Y \in \mathcal{E}$, $V \in \mathrm{Sub}(X)$, and $f\colon X \to Y$ in $\mathcal{E}$ (this is why the interior $\mathring{V} \subseteq X$ does not depend on the superobject X). Thus:

Lemma 2.4. *If j is principal, then the interior functor $X \mapsto \mathring{X}\colon \mathcal{E} \to \mathcal{E}$ preserves epis.*

Proof. If $f\colon X \twoheadrightarrow Y$, then $\exists_f(X) = Y$, so $\exists_f(\mathring{X}) = (\mathring{\exists_f X}) = \mathring{Y}$. Thus

$$\begin{tikzcd}
\mathring{X} \arrow[r, two heads, "\mathring{f}"'] & \mathring{Y} \arrow[r, tail] & Y
\end{tikzcd}$$

is the epi-mono factorization of

$$\begin{tikzcd}
\mathring{X} \arrow[r, tail] & X \arrow[r, two heads, "f"'] & Y.
\end{tikzcd}$$

□

Definition 2.5. An object $X \in \mathcal{E}$ is **open** if $\mathring{X} \cong X$.

Lemma 2.6. *Every discrete object $C \in \mathrm{D}_j\mathcal{E}$ is open.*

Proof. Since $U_X \rightarrowtail X$ is inverted by $\mathbf{a}$, if X is discrete, then $id_X : X \to X$ must factor through U_X. $\square$

Lemma 2.7. *Suppose j is principal. Then a quotient of an open object is open.*

Proof. Suppose X is open and $e \colon X \twoheadrightarrow Y$. Then we have

$$Y \cong \mathrm{Im}(e) = \exists_e X \cong \exists_e(\mathring{X}) \cong \overset{\circ}{(\exists_e X)} \cong \overset{\circ}{(\exists_e(\mathring{X}))},$$

so $\exists_e(\mathring{X})$ is open, so Y is open. $\square$

Suppose j is principal. We then define $\mathrm{O}_j\mathcal{E}$ to be the full subcategory of $\mathcal{E}$ of open objects. Note that $\mathrm{O}_j\mathcal{E}$ is a coreflective subcategory of $\mathcal{E}$, the coreflector being, of course, the interior functor.

To determine whether an object is a sheaf, one does not need to consider orthogonality with respect to *all* morphisms inverted by $\mathbf{a}$, but can restrict attention to dense monos, as in the usual definition of a sheaf. We next show that in the case of discrete objects, we need not require coorthogonality with respect to *all* morphisms inverted by $\mathbf{a}$, but just with respect to the smaller class of what we shall call *codense epis.*

Definition 2.8. Suppose j is principal and let $e \colon X \twoheadrightarrow Y$ be an epi. Write $\Delta_X \rightarrowtail X \times X$ for the diagonal and write K_e for the kernel of e, viewed as a subobject of $X \times X$. We say that e is **codense** if $\mathring{K}_e = \mathring{\Delta}_X$ in $\mathrm{Sub}(X \times X)$.

Lemma 2.9. *Suppose j is principal and let $e \colon X \twoheadrightarrow Y$ be an epi. Then e is codense iff $\mathbf{a}(e)$ is iso, iff e is bidense (the latter by [6]).*

Proposition 2.10. *Let j be principal. Then C is discrete if and only if C is coorthogonal to all codense epis in $\mathcal{E}$.*

To prove the proposition we shall make use of the following lemma.

Lemma 2.11. *Suppose j is principal and $C \in \mathcal{E}$ is coorthogonal to all codense epis in $\mathcal{E}$. Then C is coorthogonal to all dense monos.*

Proof. Let C, $m \colon Y \rightarrowtail X$, and $f : C \to X$ be as in the diagram:

$$\begin{tikzcd} & Y \arrow[d, tail, "m"] \\ C \arrow[ur, dashed, "f'"] \arrow[r, "f"'] & X. \end{tikzcd}$$

Consider the following diagrams

$$
\begin{tikzcd}
 & Y \arrow[d, tail, "m"] \\
C \arrow[ur, dashed, "f'"] \arrow[r, "f"'] & X \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] \\
 & W \arrow[d, two heads, "e"] \\
 & P
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{a}\,Y \arrow[d, tail, "\mathbf{a}\,m,\ \cong"] \\
\mathbf{a}\,X \arrow[d, shift right, "\mathbf{a}\,u"'] \arrow[d, shift left, "\mathbf{a}\,v"] \\
\mathbf{a}\,W \arrow[d, two heads, "\mathbf{a}\,e"] \\
\mathbf{a}\,P,
\end{tikzcd}
$$

where u, v is the cokernel pair of m and e is the coequalizer of u, v. Since $\mathbf{a}$ is a left adjoint, it preserves cokernel pairs and coequalizers, so $\mathbf{a}\,u, \mathbf{a}\,v$ is the cokernel pair of $\mathbf{a}\,m$, which is an iso by assumption that m is dense. Hence $\mathbf{a}\,u = \mathbf{a}\,v$. Therefore $\mathbf{a}\,e$ is an iso and thus, by Lemma 2.9 e is codense. Since $euf = evf \colon C \to P$ and since $C \top e$ by assumption, we get that $uf = vf$ by uniqueness. Hence f factors uniquely through the equalizer of u, v. But m is the equalizer of u, v as every mono in a topos is the equalizer of its cokernel pair, so f factors uniquely through m via an f' as shown in the diagram. □

Proof of Proposition 2.10. Suppose $C \in \mathcal{E}$ and that $C \top e$ for all codense epis e. Let $h \colon X \to Y$ such that $\mathbf{a}(h)$ is iso be given and let $f \colon C \to Y$ be arbitrary. Consider the following diagrams

$$
\begin{tikzcd}
 & X \arrow[d, two heads, "e"] \arrow[dd, bend left=50, "h"] \\
 & I \arrow[d, tail, "m"] \\
C \arrow[uur, dashed, "f''"] \arrow[ur, dashed, "f'"'] \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{a}\,X \arrow[d, two heads, "\mathbf{a}\,e"'] \arrow[dd, bend left=50, "\mathbf{a}\,h,\ \cong"] \\
\mathbf{a}\,I \arrow[d, tail, "\mathbf{a}\,m"'] \\
\mathbf{a}\,Y,
\end{tikzcd}
$$

where me is the image factorization of h and the diagram on the right is $\mathbf{a}$ applied to it. Since $\mathbf{a}$ preserves image factorizations and $\mathbf{a}\,h$ is iso by assumption, we have that $\mathbf{a}\,m$ and $\mathbf{a}\,e$ are iso. Hence m is dense and, by Lemma 2.9, e is codense. Thus by Lemma 2.11, there exists a unique $f' \colon C \to I$ such that $mf' = f$. By assumption $C \top e$, so there exists a unique $f'' \colon C \to X$ such that $ef'' = f'$. Thus $C \top h$, as required. □

We now define an *exterior* operation on quotients, which one can think of as dual to the closure operation on subobjects.

Definition 2.12. Suppose j is principal. For an epi $e \colon X \twoheadrightarrow Y$, we define the **exterior** of e, written $\widetilde{e} \colon X \twoheadrightarrow \widetilde{Y}$, to be the coequalizer of the interior $\mathring{K}_e$ of the kernel pair K_e of e as indicated in the following diagram:

$$\begin{tikzcd}
K_e \arrow[dr, shift left, "k"] \arrow[dr, shift right, "k'"'] & & \widetilde{Y} = \mathrm{CoEq}(km, k'm) \arrow[dd, dashed, "h"] \\
& X \arrow[ur, "\widetilde{e}"] \arrow[dr, "e"'] & \\
\overset{\circ}{K_e} \arrow[uu, tail, "m"] \arrow[ur, shift left, "km"] \arrow[ur, shift right, "k'm"'] & & Y
\end{tikzcd} \tag{2}$$

By the universal property of the coequalizer, since $ekm = ek'm$, there is a unique map $h\colon \widetilde{Y} \to Y$ such that $h\widetilde{e} = e$, as shown in the diagram. Since e is epic, h is also epic.

Lemma 2.13. *Referring to the diagram* (2) *above, the epi h is codense.*

Proof. By Lemma 2.9 it suffices to show that $\mathbf{a}\,h$ is iso. Apply $\mathbf{a}$ to the diagram (2): since $m\colon \overset{\circ}{K_e} \rightarrowtail K_e$ is dense, $\mathbf{a}(m)$ is iso. Hence, since $\mathbf{a}$ preserves kernel pairs and coequalizers, $\mathbf{a}\,h$ is iso. □

3 Axioms for Local Maps

We can now state conditions under which the category of discrete objects is "lex coreflective." For simplicity, and because its an important special case, we first consider the axioms for *localic* local maps. We then briefly mention how the axioms can be relaxed for arbitrary (bounded) local maps.

Let $\mathcal{E}$ an elementary topos with a topology j.

Axiom 1 j is principal.

Axiom 2 For all $X \in \mathcal{E}$, there exists a discrete object D and a diagram

$$\begin{tikzcd}
S \arrow[r, tail] \arrow[d, two heads] & D \\
X &
\end{tikzcd}$$

in $\mathcal{E}$, presenting X as a subquotient of D.

Axiom 3 For all discrete $D \in \mathcal{E}$, if $X \rightarrowtail D$ is open, then X is also discrete.

Axiom 4 For all discrete $D, D' \in \mathcal{E}$, $D \times D'$ is discrete.

Note that Axiom 2 essentially says that $\mathcal{E}$ is localic over $\mathrm{D}_j\mathcal{E}$.

Theorem 3.1 (Completeness). *Let $\mathcal{E}$ be a topos with a topology j satisfying Axioms 1–4. Then there is a localic local map from $\mathcal{E}$ to $\mathrm{D}_j\mathcal{E} \simeq \mathrm{Sh}_j\mathcal{E}$.*

We break the proof down into two steps, designated Propositions 3.2 and 3.3 below.

Proposition 3.2. *Let $\mathcal{E}$ be a topos with a topology j satisfying Axioms 1–4. Then the category of discrete objects $\mathrm{D}_j\mathcal{E}$ is coreflective in $\mathcal{E}$.*

Proof. We show how to construct an associated discrete object for any object $X \in \mathcal{E}$. By Axiom 2, we have a diagram

$$\begin{tikzcd} S \arrow[r, tail, "m"] \arrow[d, two heads, "e"] & D_X \\ X & \end{tikzcd}$$

in $\mathcal{E}$ presenting X as a subquotient of a discrete object D_X. Now consider the following diagram

$$\begin{tikzcd} \mathring{K}_{\mathring{e}} \arrow[r, tail] & K_{\mathring{e}} \arrow[d, shift left] \arrow[d, shift right] & & \\ & \mathring{S} \arrow[r, tail, "m"] \arrow[dl] \arrow[d, two heads, "\mathring{e}"] & S \arrow[r, tail] \arrow[d, two heads, "e"] & D_X \\ \tilde{\mathring{X}} \arrow[r, two heads, "h"'] & \mathring{X} \arrow[r, tail] & X & \end{tikzcd}$$

Since interior preserves epimorphisms by Lemma 2.4, $\mathring{e}\colon \mathring{S} \to \mathring{X}$ is epic. The exterior $\tilde{\mathring{X}}$ of the interior $\mathring{X}$ of X is obtained as in Definition 2.12, as the coequalizer of the interior $\mathring{K}_{\mathring{e}}$ of the kernel pair $K_{\mathring{e}}$ of $\mathring{e}$. By Axiom 3, $\mathring{S}$ is discrete and thus also $\mathring{K}_{\mathring{e}}$ is discrete by Axioms 3 and 4. Hence $\tilde{\mathring{X}}$ is obtained as the coequalizer of a diagram of discrete objects, namely:

$$\mathring{K}_{\mathring{e}} \rightrightarrows \mathring{S} \twoheadrightarrow \tilde{\mathring{X}}$$

Thus $\tilde{\mathring{X}}$ is also discrete. We claim that $\tilde{\mathring{X}} \twoheadrightarrow \mathring{X} \rightarrowtail X$ is universal among arrows from discrete objects into X, thus establishing the existence of a right adjoint to the inclusion $\mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$. Indeed, let D be any discrete object and let $f : D \to X$ be arbitrary. Consider the following diagram

$$\begin{tikzcd} \tilde{\mathring{X}} \arrow[r, dashed, "h"] & \mathring{X} \arrow[r, tail] & X \\ D. \arrow[u, dashed, "f''"] \arrow[ur, dashed, "f'"] \arrow[urr, "f"'] & & \end{tikzcd}$$

Since D is open by Lemma 2.6 and the interior functor $\overset{\circ}{-}: \mathcal{E} \to \mathrm{O}_j\mathcal{E}$ is right adjoint to the inclusion of open objects into $\mathcal{E}$, as already noted, there is a unique morphism f' making the right triangle commute. Then since h is a codense epi by Lemma 2.13 and D is discrete, we have by Proposition 2.10 that D is coorthogonal to h, so there exists a unique f'' making the left triangle commute. This shows the required universality. □

Proposition 3.3. *Let $\mathcal{E}$ be a topos with a topology j satisfying Axioms 1-4. Then the inclusion $\mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$ is left exact, and finite limits in $\mathrm{D}_j\mathcal{E}$ are computed as in $\mathcal{E}$.*

Proof. It is useful to name the inclusion functor and the coreflector, say:

$$\begin{tikzcd} \mathrm{D}_j\mathcal{E} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{E}, \arrow[l, shift left, "R"] \end{tikzcd}$$

where

$$L \dashv R \qquad \text{and} \qquad R \circ L \cong id.$$

Recall that the associated discrete functor R is a known to have a right adjoint, since by Proposition 3.2, $\mathrm{D}_j\mathcal{E} \simeq \mathrm{Sh}_j\mathcal{E}$ and under this equivalence R is identified with the associated sheaf functor, which has a right adjoint.

The proof now proceeds by a series of lemmas.

Lemma 3.4. *The functor $LR: \mathcal{E} \to \mathcal{E}$ preserves finite products, monomorphisms, and all colimits.*

Proof. $LR: \mathcal{E} \to \mathcal{E}$ clearly preserves all colimits since both L and R are left adjoints. To show that it preserves the terminal object 1, it clearly suffices to show that 1 is discrete. By Axiom 2, we can present 1 as a subquotient of a discrete object D,

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d, two heads] & D \\ 1. & \end{tikzcd}$$

Since $S \twoheadrightarrow 1$ is epic, it follows that the unique morphism from D to 1 is also epic. Hence 1 is a quotient of a discrete object, and thus discrete by Lemma 2.7.

Binary products are preserved by Axiom 4.

It remains to show that LR preserves monos. Thus let $m: M \rightarrowtail N$ be a monomorphism in $\mathcal{E}$. For clarity, let us denote the composite functor LR by d. We write $\epsilon: d \Rightarrow id$ for the counit of the adjunction $L \dashv R$. Consider the following diagram

$$
\begin{tikzcd}
dM \arrow[dr, dotted, "\overset{\circ}{u}"] \arrow[drr, dashed, "u"] \arrow[drrr, bend left, "dm"] \arrow[ddrr, bend right, "\epsilon_M"'] & & & \\
& (\overset{\circ}{m^* dN}) \arrow[ul, dotted, bend left, "v"] \arrow[dr, tail, "a"'] & & \\
& & m^* dN \arrow[r, tail, "c"] \arrow[d, "b"'] \arrow[dr, phantom, "\lrcorner", very near start] & dN \arrow[d, "\epsilon_N"] \\
& & M \arrow[r, tail, "m"'] & N,
\end{tikzcd}
$$

where the inner square is a pullback. The outer (elongated) square commutes by definition of dm. Hence there exists a unique morphism $u\colon dM \to m^* dN$ such that

$$bu = \epsilon_M \qquad \text{and} \qquad cu = dm.$$

Since $(\overset{\circ}{m^* dN})$ is an open subobject of a discrete object dN, $(\overset{\circ}{m^* dN})$ is discrete by Axiom 3. Hence by couniversality of ϵ_M, there exists a unique morphism $v\colon (\overset{\circ}{m^* dN}) \to dM$ such that

$$\epsilon_M v = ba.$$

One now shows without difficulty that:

$$v\overset{\circ}{u} = 1 \qquad \text{and} \qquad \overset{\circ}{u}v = 1,$$

that is, that dM is isomorphic to $(\overset{\circ}{m^* dN})$, from which it follows that dm is monic, as required. □

Lemma 3.5. *Let $\mathcal{E}$ and $\mathcal{F}$ be toposes and suppose the functor $F\colon \mathcal{E} \to \mathcal{F}$ preserves finite products, monomorphisms, and pushouts. Then F is left exact.*

Proof. Folklore, but see [3, 2.61] for a related argument. □

Corollary 3.6. *The functor $LR\colon \mathcal{E} \to \mathcal{E}$ is left exact.*

Returning to the proof of Proposition 3.3, we now show that $L\colon \mathrm{D}_j\mathcal{E} \to \mathcal{E}$ is left exact:

L preserves finite products because the terminal object 1 in $\mathcal{E}$ is discrete and also terminal in $\mathrm{D}_j\mathcal{E}$ and the product (formed in $\mathcal{E}$) of two discrete objects X and Y is again discrete by Axiom 4.

To show that L preserves equalizers, we first show that it preserves monos. Let

$$X \overset{m}{\rightarrowtail} Y$$

be a mono in $\mathrm{D}_j\mathcal{E}$. Apply L and form the image factorization of Lm in $\mathcal{E}$ to get

$$
\begin{tikzcd}
LX \arrow[rr, tail, "Lm"] \arrow[dr, two heads] & & LY \\
& I. \arrow[ur, tail] &
\end{tikzcd}
$$

Now apply R to get

$$
\begin{tikzcd}
X \arrow[rr, tail, "m"] \arrow[d, "\cong"'] & & Y \arrow[d, "\cong"] \\
RLX \arrow[rr, tail, "RLm"] \arrow[dr, two heads] & & RLY \\
& RI. \arrow[ur, tail] &
\end{tikzcd}
$$

Note that R preserves epis as a left adjoint and monos as a right adjoint. Hence the indicated morphism $RLX \twoheadrightarrow RI$ is epic, but it also monic (since RLm is monic), so iso. Thus $RLX \cong RI$.

Now apply L again to get

$$
\begin{tikzcd}
LX \arrow[rr, "Lm"] \arrow[d, "\cong"'] & & LY \arrow[d, "\cong"] \\
LRLX \arrow[rr, tail, "LRLm"] \arrow[dr, two heads, "\cong"'] & & LRLY \\
& LRI. \arrow[ur, tail] &
\end{tikzcd}
$$

Since LR is left exact by Corollary 3.6, LRI is the image factorization of Lm, so $LRI \rightarrowtail LRLY$ is monic, whence Lm is so. Thus L preserves monos, as claimed.

To show that L preserves equalizers, let

$$
\begin{tikzcd}
X \arrow[r, tail, "m"] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z
\end{tikzcd}
$$

be an equalizer in $\mathrm{D}_j\mathcal{E}$. Apply L and form the equalizer E of Lf and Lg in $\mathcal{E}$ as indicated in:

$$
\begin{tikzcd}
LX \arrow[r, tail, "Lm"] & LY \arrow[r, shift left, "Lf"] \arrow[r, shift right, "Lg"'] & LZ \\
E \arrow[ur, tail, "n"'] & &
\end{tikzcd}
$$

Apply the functor R to get the following:

$$
\begin{tikzcd}
X \arrow[r, tail, "m"] \arrow[d, "\cong"] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\cong"] & Z \arrow[d, "\cong"] \\
RLX \arrow[r, tail, "RLm"] & RLY \arrow[r, shift left, "RLf"] \arrow[r, shift right, "RLg"'] & RLZ \\
RE \arrow[ur, "Rn"'] \arrow[uu, dashed, bend left, "u"] & &
\end{tikzcd}
$$

Since $m\colon X \rightarrowtail Y$ is an equalizer, there exists a unique arrow $u\colon RE \to X$ such that $Rn = m \circ u$. Finally, apply L one more time to get

$$
\begin{tikzcd}
LX \arrow[r, tail, "Lm"] \arrow[d, "\cong"] \arrow[dd, dashed, bend right=60, "v"'] & LY \arrow[r, shift left, "Lf"] \arrow[r, shift right, "Lg"'] \arrow[d, "\cong"] & LZ \arrow[d, "\cong"] \\
LRLX \arrow[r, tail, "LRLm"] & LRLY \arrow[r, shift left, "LRLf"] \arrow[r, shift right, "LRLg"'] & LRLZ \\
LRE \arrow[ur, "LRn"'] \arrow[uu, bend left, "Lu"] & &
\end{tikzcd}
$$

Since LR is left exact by Corollary 3.6, LRn is the equalizer of Lf and Lg, so there is a unique arrow $v\colon LX \to LRE$, as shown in the diagram. Now Lm and LRn are monic, by the fact just shown that L preserves monos, it is now easy to show that $v \circ Lu = 1$ and $Lu \circ v = 1$, whence $LX \cong LRE$ and thus $Lm\colon LX \to LY$ is an equalizer, as required. This completes the proof of Proposition 3.3. $\square$

We leave it to the reader to show that Axioms 1–4 are *sound*, in the sense that they are satisfied by every local map. (Hint: the least dense subobject U_X of $X \in \mathcal{E}$ is the image of the counit of $L \dashv \mathbf{a}$.)

Remark 3.7. The **axioms for bounded local maps** are as for localic local maps, except that Axiom 2 is replaced by the following two Axioms 2a and 2b.

Axiom 2a There is an object $G \in \mathcal{E}$ such that, for all $X \in \mathcal{E}$, there exists a discrete object D and a diagram

$$
\begin{tikzcd}
S \arrow[r, tail] \arrow[d, two heads] & D \times G \\
X &
\end{tikzcd}
$$

in $\mathcal{E}$, presenting X as a subquotient of $D \times G$.

Axiom 2b Given G as in $2a$, there is a discrete object G' and a diagram

$$
G' \twoheadrightarrow \mathring{G} \rightarrowtail G
$$

in $\mathcal{E}$.

The axioms for bounded local maps are also sound and complete, but we leave this to the reader.

4 Logic of Local Maps

We now show how the logic of the discrete objects $\mathrm{D}_j\mathcal{E}$ relates to the logic of $\mathcal{E}$.

We define $\mathrm{OpenSub}_j(\mathcal{E})$ to be the full subcategory of $\mathrm{Sub}(\mathcal{E})$ on the open subobjects, where $\mathrm{Sub}(\mathcal{E})$ is the total category of the subobject fibration over $\mathcal{E}$. The proof of the following proposition is a straightforward calculation.

Proposition 4.1. *The codomain functor* $\mathrm{cod}\colon \mathrm{OpenSub}_j(\mathcal{E}) \to \mathcal{E}$ *is a fibration with reindexing of* $X \rightarrowtail J$ *along* $u\colon I \to J$ *given by* $\overset{\circ}{u^*(X)}$*, the interior of the pullback of* X *along* u.

We let $\mathrm{ClSub}_j(\mathcal{E}) \to \mathcal{E}$ denote the fibration of closed subobjects over $\mathcal{E}$. We then have:

Proposition 4.2. *The interior operation and the closure operation establish a fibred equivalence, as in*

$$\begin{tikzcd}
\mathrm{OpenSub}_j(\mathcal{E}) \arrow[rr, bend left, "\overline{(-)}"] \arrow[dr] & \simeq & \mathrm{ClSub}_j(\mathcal{E}) \arrow[ll, bend left, "\circ"] \arrow[dl] \\
& \mathcal{E}. &
\end{tikzcd}$$

Proof. Easy using the already noted fact that $\mathring{\overline{X}} = \mathring{X}$ and $\overline{\mathring{X}} = \overline{X}$. $\square$

Proposition 4.3. *The fibration* $\begin{array}{c}\scriptstyle\mathrm{OpenSub}_j(\mathcal{E})\\ \downarrow\\ \scriptstyle\mathcal{E}\end{array}$ *of open subobjects is a higher-order fibration [5] with extensional entailment, in which the following hold (we label the connectives* etc. *in* $\begin{array}{c}\scriptstyle\mathrm{OpenSub}_j(\mathcal{E})\\ \downarrow\\ \scriptstyle\mathcal{E}\end{array}$ *with a subscript* $\circ$*):*

- $\bot_\circ$, $\vee_\circ$, $\exists_\circ$, $\mathrm{Eq}_\circ$ *are as for ordinary subobjects.*
- $\top_\circ = \mathring{\top}$, $X \wedge_\circ Y = \overset{\circ}{(X \wedge Y)}$, $X \supset_\circ Y = \overset{\circ}{(X \supset \overline{Y})}$, $(\forall_\circ)_f X = \overset{\circ}{(\forall_f \overline{X})}$, *and thus* $\neg_\circ(X) = \overset{\circ}{(X \supset \overline{\bot})}$.
- **true**: $1 \rightarrowtail \Omega$ *is a split generic object.*

Hence interior $\overset{\circ}{(-)}$ *defines a fibred functor* $\mathrm{Sub}(\mathcal{E}) \to \mathrm{OpenSub}_j(\mathcal{E})$ *over* $\mathcal{E}$ *which preserves all this structure, except the generic object.*

Proof. The first-order structure is defined categorically and thus preserved along equivalences. Therefore, the first-order structure is obtained from the well-known description of the logical operations of the closed subobject fibration (explicitly stated, *e.g.*, in [5]). For example, for $X, Y \in \mathrm{OpenSub}_j(\mathcal{E})$ over I we have that $X \vee_\circ Y = \overset{\circ}{\overline{X} \vee_j \overline{Y}}$, where $\vee_j$ is the disjunction in the closed subobject fibration, so $X \vee_\circ Y = \overset{\circ}{\overline{\overline{X} \vee \overline{Y}}} = (\overline{X} \vee \overline{Y})^{\circ} = \overset{\circ}{\overline{X}} \vee \overset{\circ}{\overline{Y}} = X \vee Y$ (where we used that interior preserves $\vee$ as a left adjoint). It is easy to verify that **true**: $1 \rightarrowtail \Omega$ is a split generic object. $\square$

Proposition 4.4. *There is a change-of-base situation*

$$\begin{tikzcd}
\mathrm{Sub}(\mathrm{D}_j\mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{OpenSub}_j(\mathcal{E}) \arrow[d] \\
\mathrm{D}_j\mathcal{E} \arrow[r, hook] & \mathcal{E}.
\end{tikzcd}$$

Proof. Let $X \rightarrowtail J$ be an open subobject of a discrete object J; then X itself is discrete by Axiom 3. Moreover, since the discrete objects are closed under finite limits in $\mathcal{E}$, the pullback $u^*(X)$ of X along a map $u\colon I \to J$ between discrete objects is discrete and hence also open. Thus the reindexing of X along u in $\begin{array}{c}\mathrm{OpenSub}_j(\mathcal{E})\\ \downarrow \\ \mathcal{E}\end{array}$, namely $\overset{\circ}{u^*(X)}$, is equal (as a subobject of I) to the reindexing of X in $\mathrm{Sub}(\mathrm{D}_j\mathcal{E})$, namely $u^*(X)$. $\square$

Combining the above proposition with Proposition 4.4 we have the following picture, complementing Lawvere's "adjoint cylinder" picture of local maps [9] (where the discrete objects come in to $\mathcal{E}$ on the left, the sheaves come in to $\mathcal{E}$ on the right, and the category of discrete objects is equivalent to the category of sheaves).

$$\begin{tikzcd}
\mathrm{Sub}(\mathrm{D}_j\mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{OpenSub}_j(\mathcal{E}) \arrow[rr, bend left, "\overline{}"] \arrow[rr, phantom, "\simeq"] \arrow[dr] & & \mathrm{ClSub}_j(\mathcal{E}) \arrow[ll, bend left, "\circ"] \arrow[dl] & \mathrm{Sub}(\mathrm{Sh}_j\mathcal{E}) \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathrm{D}_j\mathcal{E} \arrow[rr, hook] & & \mathcal{E} & & \mathrm{Sh}_j\mathcal{E} \arrow[ll, hook']
\end{tikzcd}$$

Combining Propositions 4.4 and 4.3, we of course derive a translation of the internal logic of $\mathrm{D}_j\mathcal{E}$ into the logic of $\mathcal{E}$. Since we are restricting attention to the discrete objects in the base, we can make some simplifications compared to what we get directly from Proposition 4.3:

Proposition 4.5. *The internal logical operations of* $\mathrm{D}_j\mathcal{E}$ *are given as follows (we label the connectives* etc. *with a subscript d):*

- *the geometric operations ($\top_d$, $\wedge_d$, $\bot_d$, $\vee_d$, $\exists_d$) are, of course, as for ordinary subobjects in $\mathcal{E}$*

- $X \supset_d Y = (X \overset{\circ}{\supset} Y)$ *and* $(\forall_d)_f X = (\overset{\circ}{\forall_f} X)$.

Proof. The first item is obvious since the inclusion of discrete objects is the inverse image of a geometric morphism. To show $X \supset_d Y = (X \overset{\circ}{\supset} Y)$ note that $X \supset_d Y = X \supset \overline{Y}$, by Propositions 4.4 and 4.3. Now let I be a discrete object of $\mathcal{E}$ and let $X, Y \in \mathrm{Sub}_{\mathcal{E}}(I)$ be subobjects of I. Suppose that X is open. Then $(X \overset{\circ}{\supset} Y) = \overset{\circ}{(X \supset \overset{\circ}{Y})}$ using Axiom 3 and the fact that discrete objects are closed under finite limits in $\mathcal{E}$. The case of $\forall$ is similar. □

Observe the following easy corollary of Proposition 4.5.

Corollary 4.6. *Let $u\colon I \to J$ be a morphism of discrete objects in $\mathcal{E}$ and let $X \in \mathrm{Sub}_{\mathcal{E}}(I)$ be a subobject of I. Then* $\overset{\circ}{(\forall_u \overset{\circ}{X})} = (\overset{\circ}{\forall_u} X)$.

4.1 Preservation of Valid Stable Formulas

We now show that a wider class of sentences than the geometric sentences is preserved by the inclusion of the discrete objects.

Let $\Gamma \vdash \varphi\colon \mathsf{Prop}$ be a formula (in context) of first-order logic over a first-order many-sorted language. Suppose that the basic types in the context Γ of the language are interpreted in $\mathcal{E}$ by discrete objects and that the atomic predicates are interpreted by open subobjects of discrete objects in $\mathcal{E}$, corresponding to subobjects in $\mathrm{D}_j\mathcal{E}$. We then write $[\![\varphi]\!]$ for the interpretation of φ in $\mathcal{E}$. Likewise, we write $[\![\varphi]\!]_d$ for the interpretation of φ in $\mathrm{D}_j\mathcal{E}$, *i.e.*, in the subobject fibration over $\mathrm{D}_j\mathcal{E}$. For notational simplicity we allow ourselves to consider $[\![\varphi]\!]_d$ as a subobject in $\mathcal{E}$, thus omitting the inclusion functor from discrete objects into $\mathcal{E}$. Finally, we say that φ **is valid in** $\mathcal{E}$, written in short as $\mathcal{E} \models \varphi$, iff $\top \leq [\![\varphi]\!]$ in $\mathrm{Sub}_{\mathcal{E}}([\![\Gamma]\!])$, where $[\![\Gamma]\!]$ is the interpretation of Γ. Likewise, we say that φ **is valid in** $\mathrm{D}_j\mathcal{E}$, written $\mathrm{D}_j\mathcal{E} \models \varphi$, if $\top_d \leq [\![\varphi]\!]_d$ in $\mathrm{Sub}_{\mathrm{D}_j\mathcal{E}}([\![\Gamma]\!]_d)$.

Definition 4.7. Let φ be a formula of first-order logic over a first-order many-sorted language. We say that φ is **stable** if, for all subformulas $(\psi \supset \vartheta)$ of φ, the formula ψ is geometric.

Lemma 4.8. *Let φ be a stable formula. Then* $\overset{\circ}{[\![\varphi]\!]} = [\![\varphi]\!]_d$.

Proof. The proof is by structural induction on φ. Note that $[\![\varphi]\!]_d$ is discrete, and thus open, so $\overset{\circ}{[\![\varphi]\!]_d} = [\![\varphi]\!]_d$. For φ atomic we clearly have $[\![\varphi]\!] = [\![\varphi]\!]_d$ and thus also $\overset{\circ}{[\![\varphi]\!]} = [\![\varphi]\!]_d$. Given the result for atomic formulas, for φ a geometric formula, we clearly also find that $[\![\varphi]\!] = [\![\varphi]\!]_d$, and thus also $\overset{\circ}{[\![\varphi]\!]} = [\![\varphi]\!]_d$. It remains to consider implication and universal quantification.

Suppose that $\varphi = (\psi \supset \vartheta)$. Then we have that

$$
\begin{aligned}
[\![\psi \supset \vartheta]\!]_d &= ([\![\psi]\!]_d \overset{\circ}{\supset} [\![\vartheta]\!]_d) && \text{see definition of } \supset_d \text{, Prop. 4.5} \\
&= \overset{\circ}{(\overset{\circ}{[\![\psi]\!]} \supset \overset{\circ}{[\![\vartheta]\!]})} && \text{by induction hypothesis} \\
&= \overset{\circ}{(\overset{\circ}{[\![\psi]\!]} \supset [\![\vartheta]\!])} && \text{by Prop. 4.5} \\
&= ([\![\psi]\!] \overset{\circ}{\supset} [\![\vartheta]\!]) && \text{since } \psi \text{ is geometric by stability of } \varphi,
\end{aligned}
$$

as required.

Finally, suppose that $\varphi = (\forall x\colon X.\ \psi)$. Then we have that

$$
\begin{aligned}
[\![\forall x\colon X.\ \psi]\!]_d &= (\overset{\circ}{\forall x\colon X}.\ [\![\psi]\!]_d) && \text{see definition of } \forall_d \text{, Prop. 4.5} \\
&= \overset{\circ}{(\forall x\colon X.\ \overset{\circ}{[\![\psi]\!]})} && \text{by induction} \\
&= (\overset{\circ}{\forall x\colon X}.\ [\![\psi]\!]) && \text{by Corollary 4.6} \\
&= [\![\overset{\circ}{\forall x\colon X}.\ \psi]\!],
\end{aligned}
$$

as required. □

Theorem 4.9. *If φ is stable, then $\mathcal{E} \models \varphi$ iff $\mathrm{D}_j\mathcal{E} \models \varphi$.*

Proof. Let $I = [\![\Gamma]\!] = [\![\Gamma]\!]_d$ be the discrete object interpreting Γ, the context of free variables of φ. Then, writing $\leq_d$ for the ordering in $\mathrm{Sub}_{\mathrm{D}_j\mathcal{E}}(I)$ and $\leq$ in $\mathrm{Sub}_{\mathcal{E}}(I)$, we have that

$$
\begin{aligned}
& && \mathrm{D}_j\mathcal{E} \models \varphi \\
&\iff && \top_d \leq_d [\![\varphi]\!]_d \\
&\iff && \top \leq [\![\varphi]\!]_d && \text{since } \top_d = \top \\
&\iff && \top \leq \overset{\circ}{[\![\varphi]\!]} && \text{by Lemma 4.8} \\
&\iff && \top \leq [\![\varphi]\!] && \text{since } I \text{ is discrete and thus open} \\
&\iff && \mathcal{E} \models \varphi.
\end{aligned}
$$

□

4.2 A Modal Logic for Local Maps

We now consider interior as a logical operator. Interior is not a logical operation in the subobject fibration over $\mathcal{E}$ because it does not commute with substitution, see Remark 2.2. (See also Lawvere's discussion of co-Heyting operations in presheaf toposes [11], where a similar phenomenon arises.) However, when we restrict attention to discrete objects, interior *does* commute with substitution:

Proposition 4.10. *Let $u\colon I \to J$ be a morphism between discrete objects I and J in $\mathcal{E}$ and suppose $X \rightarrowtail J$ is a subobject of J. Then $\mathring{(u^*X)} = u^*(\mathring{X})$ as subobjects of I.*

Proof. First note that $\mathring{X}$ is discrete by Axiom 3 and thus also $u^*(\mathring{X})$ is discrete and hence open. Thus $u^*(\mathring{X}) = \mathring{(u^*(\mathring{X}))} \leq \mathring{(u^*X)}$. The other direction always holds (regardless of I and J being discrete): $\mathring{(u^*X)} \leq u^*(\mathring{X})$ iff $u^*X \leq \overline{u^*(\mathring{X})} = u^*\overline{X}$. ☐

The following definition makes precise the idea of considering the logic of $\mathcal{E}$ restricted to discrete objects.

Definition 4.11. We define the fibration $\begin{smallmatrix}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{smallmatrix}$ of $\mathcal{E}$-predicates over $\mathrm{D}_j\mathcal{E}$ by change-of-base along $\mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$ as in

$$\begin{tikzcd} \mathrm{Pred} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Sub}(\mathcal{E}) \arrow[d] \\ \mathrm{D}_j\mathcal{E} \arrow[r, hook] & \mathcal{E}. \end{tikzcd}$$

Thus in the internal logic of $\begin{smallmatrix}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{smallmatrix}$, types and terms are interpreted by objects and morphisms of $\mathrm{D}_j\mathcal{E}$ and predicates over a type σ, interpreted by a discrete object I_σ, are interpreted as subobjects of I_σ in $\mathcal{E}$. In other words, we consider all the predicates of $\mathcal{E}$, but only on types and terms from $\mathrm{D}_j\mathcal{E}$.

The pulled-back fibration $\begin{smallmatrix}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{smallmatrix}$ is clearly a first-order fibration. By Proposition 4.10, the interior operation is a logical operation in $\begin{smallmatrix}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{smallmatrix}$. So is, of course, the closure operation. We can now give axioms for the interior and closure operations to obtain what we will refer to as a *modal logic for local maps*. In the syntactic calculus we denote interior by $\sharp$ and closure by $\flat$. The choice of this notation comes from our realizability model $\mathrm{RT}(A, A_\sharp)$ discussed in [1].

The calculus is an extension of standard intuitionistic first-order logic. We write logical entailment as $\Gamma \mid \varphi \vdash \psi$, where Γ is a context of the form $x_1\colon \sigma_1, \ldots, x_n\colon \sigma_n$ giving types to variables, and where φ and ψ formulas with free variables in Γ. There are two additional logical operations: if φ is a formula, also $\sharp\varphi$ and $\flat\varphi$ are formulas. Substitution of terms for variables in these new formulas is defined in the obvious way. There are the usual rules of many-sorted first-order intuitionistic logic plus the following axioms and rules:

$$\frac{}{\Gamma \mid \sharp\varphi \vdash \varphi}\,(3) \qquad \frac{}{\Gamma \mid \sharp\varphi \vdash \sharp\sharp\varphi}\,(4)$$

$$\frac{}{\Gamma \mid \top \vdash \sharp(\top)}\ (5) \qquad \frac{}{\Gamma \mid \sharp\varphi \wedge \sharp\psi \vdash \sharp(\varphi \wedge \psi)}\ (6)$$

$$\frac{\Gamma \mid \sharp\varphi \vdash \psi}{\Gamma \mid \varphi \vdash \flat\psi}\ (7) \qquad \frac{}{x : \sigma, y : \sigma \mid x =_\sigma y \vdash \sharp(x =_\sigma y)}\ (8)$$

Intuitively, Axiom (3) says that $\sharp$ is a deflationary operation, Axiom (4) then says that $\sharp$ is idempotent, Axioms (5) and (6) say that $\sharp$ is left exact, Rule (6) says that $\sharp$ is left adjoint to $\flat$, and Axiom (8) expresses that all the types are discrete and hence equality is $\sharp$.

From the above axioms and rules one can easily prove the necessitation rule:

$$\frac{\top \vdash \varphi}{\top \vdash \sharp\varphi}$$

and that $\sharp$ distributes over implication:

$$\frac{}{\sharp(\varphi \supset \psi) \vdash \sharp\varphi \supset \sharp\psi}$$

Thus $\sharp$ has the formal properties of the box operator in the modal logic S4, which is why we refer to the first-order logic axiomatized here as a *modal logic for local maps.*

References

[1] S. Awodey, L. Birkedal, and D.S. Scott. Local realizability toposes and a modal logic for computability. In L. Birkedal, J. van Oosten, G. Rosolini, and D.S. Scott, editors, *Tutorial Workshop on Realizability Semantics, FLoC'99, Trento, Italy, 1999*, volume 23 of *Electronic Notes in Theoretical Computer Science.* Elsevier, 1999.

[2] L. Birkedal. *Developing Theories of Types and Computability.* PhD thesis, School of Computer Science, Carnegie Mellon University, 1999.

[3] P.J. Freyd. Aspects of topoi. *Bull. Austral. Math. Soc.*, 7:1–76, 1972. Corrigendum in same volume, pp. 467–480.

[4] P.J. Freyd and G.M. Kelly. Categories of continuous functors, I. *Journal of Pure and Applied Algebra*, 2:169–191, 1972.

[5] B. Jacobs. *Categorical Logic and Type Theory*, volume 141 of *Studies in Logic and the Foundations of Mathematics.* Elsevier Science Publishers B.V., 1999.

[6] P.T. Johnstone. *Topos Theory.* Number 10 in L.M.S. Monographs. Academic Press, 1977.

[7] P.T. Johnstone and I. Moerdijk. Local maps of toposes. *Proc. London Math. Soc.*, 3(58):281–305, 1989.

[8] G.M. Kelly and F.W. Lawvere. On the complete lattice of essential localizations. *Bull. Soc. Math. Belg. Ser. A*, XLI(2):289–319, 1989.

[9] F.W. Lawvere. Categories of spaces may not be generalized spaces as exemplified by directed graphs. *Revista Colombiana de Matemáticas*, XX:179–186, 1986.

[10] F.W. Lawvere. Toposes generated by codiscrete objects in combinatorial topology and functional analysis. Notes for Colloquium lectures given at North Ryde, New South Wales, Australia on April 18, 1989 and at Madison, USA, on December 1, 1989, 1989.

[11] F.W. Lawvere. Intrinsic co-heyting boundaries and the leibniz rule in certain toposes. In A. Carboni, M.C. Pedicchio, and G. Rosolini, editors, *Category Theory. Proceedings of the International Conference held in Como, Italy, July 22–28, 1990*, volume 1488 of *Lecture Notes in Mathematics*, pages 279–281. Springer-Verlag, 1991.