Highlights

When No Price Is Right

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- Non-standard expected utility representation of non-Archimedean preferences.
- Unified treatment of Savage-style acts, horse lotteries, and de Finetti coherence.

When No Price Is Right

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ABSTRACT

In this paper, we show how to represent a non-Archimedean preference over a set of random quantities by a nonstandard utility function. Non-Archimedean preferences arise when some random quantities have no fair price. Two common situations give rise to non-Archimedean preferences: random quantities whose values are required to be greater than every real number, and strict preferences between random quantities that are deemed closer in value than every positive real number. We also show how to extend a non-Archimedean preference to a larger set of random quantities. The random quantities that we consider include real-valued random variables, horse lotteries in the sense of Anscombe and Aumann (1963), and acts in the theory of Savage (1954). In addition, we weaken the state-independent utility assumptions made by the existing theories and give conditions under which the utility that represents preference is the expected value of a state-dependent utility with respect to a probability over states.

1. Introduction

1.1. Motivation

The primary goal of this paper is to extend three well-known theories of decision making (described in Section 3 below) to allow for non-Archimedean (unbounded and/or discontinuous) preferences as defined in Definition 1.

Definition 1. Let $\precsim$ be a binary relation on a set $\mathcal{X}$.

1. We call $\precsim$ a *preorder* if it is both reflexive and transitive.
2. A preorder $\precsim$ on a set $\mathcal{X}$ is called *total* if, for all $X, Y \in \mathcal{X}$, $(X \precsim Y) \vee (Y \precsim X)$.
3. If $\precsim$ is a preorder on $\mathcal{X}$ and $X, Y \in \mathcal{X}$, we write
   - $X \sim Y$ if $(X \precsim Y) \wedge (Y \precsim X)$, and
   - $X \prec Y$ if $(X \precsim Y) \wedge [\neg(Y \precsim X)]$.
4. If a preorder $\precsim$ expresses an agent's preferences amongst elements of a convex set $\mathcal{X}$, we call the preferences *Archimedean* if, for all $X, Y, Z \in \mathcal{X}$,

$$(\alpha X + [1-\alpha]Y \prec Z \text{ for all real } 0 < \alpha \leq 1) \text{ implies } \neg(Z \prec Y). \tag{1}$$

5. If an agent's preferences are not Archimedean, we call them *non-Archimedean*.

Archimedean preferences have a continuity at $\alpha = 0$ for mixtures of the form in the first clause of (1). Also, if X is worth infinitely less than Z which in turn is just a little bit less valuable than Y, then (1) can fail.

All three Archimedean theories that we extend also assume that preferences are state-independent, are expressed via total preorders, and satisfy a linearity assumption. We relax the state-independence and total-preorder assumptions and drop altogether the assumption that preferences are Archimedean. Allowing utility to be state-dependent is particularly important in financial applications where different states of the world can entail different exchange rates between

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currencies as in Schervish, Seidenfeld and Kadane (1990) and/or different relative prices for commodities. We maintain a linearity assumption in order to achieve an expected-utility representation.

The use of lexicographies to represent discontinuous preference is widespread, e.g., Blume, Brandenburger and Dekel (1991b) use lexicographies in game theory to allow conditioning on events that would otherwise have zero probability and Rekola (2003); Gelso and Peterson (2005) use lexicographies to help model ecological preferences where certain options are essentially infinitely more valuable than others. See also Hausner (1954); Halpern (2010); Petri and Voorneveld (2016) for some theoretical considerations. The following is adapted from Definition 3.1 of Blume, Brandenburger and Dekel (1991a) and Definition 2.2 of Halpern (2010).

Definition 2. Let Ω be a set with a field Σ of subsets. A well-ordered set $\mathcal{L} = \{P_a\}_{a\in\aleph}$ of finitely-additive probabilities on (Ω, Σ) is called a *lexicographic probability system*. Let $\mathcal{X}$ be a set of standard-valued functions defined on Ω such that $P_a(X)$ is a finitely-additive expectation of X with respect to P_a for each $a \in \aleph$. The *lexicographic preference (derived from $\mathcal{L}$)* is defined as follows. For each $X, Y \in \mathcal{X}$,

- $X \sim Y$ if and only if $P_a(X) = P_a(Y)$ for all $a \in \aleph$.
- $X \prec Y$ if and only if there exists $a \in \aleph$ such that $P_a(X) \neq P_a(Y)$, and for the first such a, $P_a(X) < P_a(Y)$.

Examples 3.3 and 4.8 of Halpern (2010) show that there are cases of non-Archimedean preferences that cannot be modeled via lexicographic probability systems while showing that all lexicographic preferences can be represented by a nonstandard utility. See also Rizza (2015). In this way, a nonstandard representation is a strict generalization of lexicographic preferences. In this paper we use nonstandard-valued functions to represent non-Archimedean preferences.

1.2. Standard versus Nonstandard Numbers

For the remainder of the paper, we refer to the familiar real numbers in the set $\mathbb{R}$ as *standard* numbers to distinguish them from the nonstandard numbers that we describe in Appendix A and use liberally throughout the paper. We call a function *numerical* if it takes either standard or nonstandard values. In all cases, the calculations that are part of an agent's expressions of preference involve only standard numbers. We use nonstandard numbers to represent an agent's preferences after the fact and to infer a probability and utility to express that representation. Since we use multiple number systems, we need to be careful about what we mean by "linear" in various settings.

Definition 3. A space $\mathcal{W}$ of functions is a *standard-linear space* if $\alpha Y + \beta Z \in \mathcal{W}$ for all standard α, β and all $Y, Z \in \mathcal{W}$. A nonstandard-valued function U on a standard-linear space $\mathcal{W}$ is called a *standard-linear function* if $U(\alpha Y + \beta Z) = \alpha U(Y) + \beta U(Z)$, for all $Y, Z \in \mathcal{W}$ and all standard α, β. The *standard-linear span* of a set is the smallest standard-linear space containing the set.

Notice that $U(0) = 0$ for every standard-linear function U. Definition 3 restricts the coefficients in linear combinations to be standard even though the values of U might be nonstandard. Readers desiring a more thorough understanding of nonstandards than we present in Appendix A could read one of the many treatments such as Nelson (1987); Robinson (1996).

Other treatments of probability and/or decision theory that make use of nonstandard numbers include Pedersen (2012); Duanmu and Roy (2017); Benci, Horsten and Wenmackers (2013); Wenmackers (2019). Section 3.2 of Pedersen (2012) has extensive references along with some details of some of the attempts to make use of nonstandard-valued probabilities. The same author, in Pedersen (2014), investigates representations of non-Archimedean coherent preference over unconditional real-valued gambles. Our representation incorporates coherent conditional preferences in Section 4. The approach of Benci et al. (2013); Wenmackers (2019) is primarily to define probabilities that take infinitesimal values. For a probability P on a set Ω to be a "non-Archimedean probability," in their terminology, they impose a condition that requires all singletons $\{\omega\} \in \Omega$ to have probabilities that are standard multiples of a common infinitesimal ϵ. That is, there is an infinitesimal ϵ such that for every $\omega \in \Omega$, there is a standard $a_\omega > 0$ such that $P(\{\omega\}) = a_\omega \epsilon$. This assumption places severe restrictions on the forms of non-Archimedean preferences that can be expressed.

For those familiar with nonstandard models of the reals, all of our analysis is external rather than internal.[1] The main reason for an external analysis is that the nonstandards are non-Archimedean from an external perspective, but

[1]The distinction between internal and external analyses of nonstandards depends on some concepts of abstract set theory, such as what counts as a set. Nonstandard models of the reals, such as the ones in Appendix A, use standard objects, such as sequences and equivalence relations, to

are Archimedean from an internal perspective. An internal perspective is used by Duanmu and Roy (2017), who start with the familiar decision theory setup (loss functions, bayes rules, minimax rules, admissible rules, etc.) and obtain new results by allowing probabilities to take nonstandard values.

Narens (1974) develops a non-Archimedean theory of measurement. The theory leads to measurements whose values lie in nonstandard models of the reals. Narens' measurement systems have a number of features in common with probability and preference, so it is not surprising that nonstandard numbers are useful for representing non-Archimedean preference structures. See Halpern (2010) for a more thorough comparison of the uses of lexicographic preferences and nonstandard numbers in representing preferences.

1.3. Some Notation

Throughout this paper, Ω denotes a state space, $\mathcal{X}$ denotes a set of random quantities, which are functions from Ω to a space $\mathcal{O}$ of outcomes. Subsets of Ω are called *events*. When $\mathcal{X}$ is a set of random variables, the space $\mathcal{O}$ will be the standard numbers $\mathbb{R}$. For other cases, both $\mathcal{X}$ and $\mathcal{O}$ will be more complicated sets that are constructed later. We will make much use of the following concepts:

Definition 4. Let $\precsim$ be a binary relation on a set $\mathcal{X}$.

1. Let U be a numerical function defined on $\mathcal{X}$. We say that U *represents* $\precsim$ if, for all $X, Y \in \mathcal{X}$,

$$X \precsim Y \text{ if and only if } U(X) \leq U(Y). \tag{2}$$

2. If $(X \precsim Y) \wedge [\neg(Y \precsim X)]$ we write $X \prec Y$.

The following results follow easily from Definition 4.

Proposition 1. *Let U be a numerical function defined on a set $\mathcal{X}$.*

- *U represents a unique preorder $\precsim$ on $\mathcal{X}$, defined via (2) and $\precsim$ is total.*
- *U represents a preorder $\precsim$ if and only if $aU + b$ represents $\precsim$ for all positive a and all b.*

1.4. Expressed Preference

In our approach, preference amongst random quantities is expressed by willingness to trade.

Definition 5. Let X and Y be elements of a standard-linear space $\mathcal{X}$. If an agent is willing to trade X to receive Y, we write $X \precsim Y$. If both $X \precsim Y$ and $Y \precsim X$, we say that the agent is *indifferent* between X and Y, which we express by $X \sim Y$. If $(X \precsim Y) \wedge [\neg(Y \precsim X)]$ we write $X \prec Y$.

We deliberately give no name to the relation $\prec$ for reasons that will become apparent in Example 1 below. The first assumption that we make merely avoids the two extremes in which the agent either is willing to make no trades or is willing to make all trades.

Assumption 1. For all $X \in \mathcal{X}$, $X \precsim X$, and there exist $X, Y \in \mathcal{X}$ such that $X \prec Y$.

Our next assumption expresses the idea that willingness to trade depends only on the agent's net change in fortune, which we state formally as follows.

Assumption 2. Suppose that $X, X', Y, Y' \in \mathcal{X}$ and $Y - X = Y' - X'$. The agent is willing to give X to get Y if and only if the agent is willing to give X' to get Y'.

Our next assumption is the trading analog to de Finetti's assumption that an agent is willing to accept all finite sums of fair gambles.

construct new objects which play the roles of nonstandard numbers. These constructed objects are numbers when looked at internally, i.e., as objects that satisfy the Peano postulates and to which the Zermelo-Fraenkel axioms of set theory can be applied. When the constructed objects are looked at externally, i.e., as functions of standard objects, what counts as a set is defined in terms of the sets of standard objects from which they are built. Some of the most useful of these external sets include the standard numbers, the standard natural numbers and the infinitesimal numbers. These are sets when viewed externally, i.e. from the point of view of the standard objects from which they are defined. However, they do not satisfy the definition of "set" according to the Zermelo-Frankel axioms applied to the nonstandard numbers when viewed internally.

Assumption 3. Suppose that $X_j \precsim Y_j$ for $j = 1, 2$ and α_1, α_2 are positive standard numbers. Then

$$\alpha_1 X_1 + \alpha_2 X_2 \precsim \alpha_1 Y_1 + \alpha_2 Y_2.$$

Proposition 2 states some straightforward properties of the first three assumptions.

Proposition 2. *Suppose that $\precsim$ satisfies Assumptions 1–3. Then $\precsim$ is a preorder, $\prec$ is a strict partial order, and $\sim$ is an equivalence relation.*

A general preorder might not be total, and hence may leave some elements of $\mathcal{X}$ uncompared, i.e., neither $X \precsim Y$ nor $Y \precsim X$.

Example 1 (Consensus). Let $\aleph$ be a set, and let $\{\precsim_\alpha\}_{\alpha\in\aleph}$ be a collection, indexed by $\aleph$, of total preorders on a standard-linear space $\mathcal{X}$. Our agent might think of $\aleph$ as indexing a set of experts whose opinions the agent wants to adopt to the extent that they agree. Define the binary relation $\precsim$ on $\mathcal{X}$ by $X \precsim Y$ if, for all $\alpha \in \aleph$, $X \precsim_\alpha Y$. If each $\precsim_\alpha$ satisfies Assumptions 1–3, then so does $\precsim$, which will also be a preorder, but not necessarily total. In general, each instance of $X \precsim Y$ partitions $\aleph$ into two sets $\aleph_{X\sim Y}$ and $\aleph_{X\prec Y}$ as follows: $\alpha \in \aleph_{X\sim Y}$ if $X \sim_\alpha Y$, and $\alpha \in \aleph_{X\prec Y}$ if $X \prec_\alpha Y$. If either of the two sets is empty, there is unanimity about how the experts would trade X and Y. For example, if $\aleph_{X\sim Y} = \emptyset$, the agent is willing to trade X to get Y and will refuse to trade Y to get X. If both are nonempty, the agent is willing to trade X to get Y but has expressed neither willingness nor refusal to trade Y to get X. For example the agent might want to look more closely at which experts lie in each of the sets $\aleph_{X\sim Y}$ and $\aleph_{X\prec Y}$ before deciding whether to trade Y to get X. ⋄

As other authors have done, e.g. Giarlotta and Creco (2013); Giarlotta (2019); Nishimura and Ok (2020) we find it useful to allow an agent to distinguish preferences like the two cases that appear at the end of Example 1.

Definition 6. Let $\precsim$ satisfy Assumptions 1–3. For each case of $X \prec Y$, the agent can express whether this is an *unambiguous* one-way preference, which we denote $X \ll Y$ or an *ambiguous* one-way preference, which we denote $X \lhd Y$.

In order for an "unambiguous" one-way preference to mean what it sounds like, we impose the following assumption.

Assumption 4. The relations $\ll$ and $\lhd$ satisfy the following:

- $X \prec Y$ if and only if $(X \ll Y) \vee (X \lhd Y)$,
- $(X \ll Y) \wedge (Y \precsim Z)$ implies $X \ll Z$, and
- $(X \precsim Y) \wedge (Y \ll Z)$ implies $X \ll Z$.

If the second bullet in Assumption 4 were violated, the agent would be willing to trade Y to get Z and would be willing to contemplate trading Z to get X, which would violate the understanding of $X \ll Y$ as unambiguous willingness to trade one-way. A similar violation arises if the third bullet is violated. The first claim in Proposition 3 is a direct consequence of Theorem 3.4 of Giarlotta and Creco (2013), and the second claim is straightforward.

Proposition 3. *If $\precsim$ satisfies Assumptions 1–4, then $\precsim$ is a consensus as in Example 1. In Example 1, choosing $\ll$ to be $\prec$ satisfies Assumption 4, as does choosing $\ll$ to be empty.*

We are now ready to formalize our model for trading.

Definition 7. Let Ω be a set, and for each $\omega \in \Omega$ let $\mathcal{O}_\omega$ be a standard-linear space. Let $\mathcal{O}_\Omega = \prod_{\omega\in\Omega} \mathcal{O}_\omega$ and let $\mathcal{X} \subseteq \mathcal{O}_\Omega$ be a standard-linear space of functions with domain Ω. Let $\precsim$ and $\ll$ be binary relations on $\mathcal{X}$. If $\precsim$ and $\ll$ satisfy Assumptions 1–4, we call $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ a *trading system*. If $\precsim$ is a total preorder and $\ll$ is $\prec$, then $\mathcal{T}$ is a *total trading system*. The sum of finitely many terms of the form $\alpha(Y - X)$, where $X \sim Y$ and α is standard is called a *fair trade*. The sum of finitely many terms of the form $\alpha(Y - X)$, where $X \precsim Y$ and $\alpha > 0$ is standard is called an *acceptable trade*. Denote the set of acceptable trades as $\mathcal{V}_\mathcal{T}$.

Proposition 4 states some straightforward properties of trading systems.

Proposition 4. *Suppose that $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ is a trading system. The set $\mathcal{V}_\mathcal{T}$ of all acceptable trades is a convex cone, and it is the set of all trades V such that $0 \precsim V$. The set of all fair trades is a standard-linear space, and it is the equivalence class (under $\sim$) that contains the trade 0. Finally, $V \in \mathcal{V}_\mathcal{T}$ if and only if for all $X \in \mathcal{X}$, $X \precsim X + V$.*

1.5. Dominance and Coherence (Part One)

Suppose that a (possibly nonstandard-valued) function U on $\mathcal{X}$ represents a total trading system $(\mathcal{X}, \precsim, \prec)$. There is a necessary condition for $U(X)$ to be expressed as the expected value of the utility of $X(\omega)$ with respect to a probability over Ω. Loosely speaking, the condition is the following:

Let $X, Y \in \mathcal{X}$. If for all ω, $Y(\omega)$ is at least as valuable as $X(\omega)$ when state ω occurs, then $U(X) \leq U(Y)$.

In the theory of de Finetti (1974), where $\mathcal{X}$ is a linear space of standard-valued random variables, we can be more precise about the above condition. For each standard number x and each $\omega \in \Omega$ and each random variable X such that $X(\omega) = x$, x is assumed to be the utility value to the agent, when the state ω occurs, of receiving the random variable X. The condition then becomes "$X(\omega) \leq Y(\omega)$ for all ω implies $U(X) \leq U(Y)$."

In more general theories, where each $X(\omega)$ may be some non-numerical object $x \in \mathcal{O}$ (the codomain of X) and the utility of each object in $\mathcal{O}$ might vary with ω, the utility to the agent of receiving $X(\omega) = x$ could depend on both ω and x. Later (Definition 16 in Section 4.1) we define what we mean by $X(\omega) \leq Y(\omega)$ and $X(\omega) < Y(\omega)$ when $\mathcal{O} \neq \mathbb{R}$. Regardless of what are the objects in $\mathcal{O}$, there are several ways in which $X \leq Y$ but $X \neq Y$.

Definition 8. Let $X, Y \in \mathcal{X}$.

- We say that Y *weakly dominates* X or X *is weakly dominated by* Y if $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$ and there is $\omega \in \Omega$ such that $X(\omega) < Y(\omega)$.
- We say that Y *strictly dominates* X or X *is strictly dominated by* Y if $X(\omega) < Y(\omega)$ for all $\omega \in \Omega$.
- We say that Y *uniformly dominates* X or X *is uniformly dominated by* Y if there exists a standard $\epsilon > 0$ such that $X(\omega) \leq Y(\omega) - \epsilon$ for all $\omega \in \Omega$.

It is trivial to see that weak dominance is an extension of strict dominance which, in turn, is an extension of uniform dominance. Many of our results do not depend on which version of dominance an agent chooses. For those results that depend on the form of dominance (primarily in Section 4), we are explicit about which form is needed. We use $X \prec_{\text{Dom}} Y$ to denote "Y dominates X" in whichever sense the agent chooses. In de Finetti (1974), dominance means uniform dominance. Our next assumption formalizes the idea that more is better.

Assumption 5. The agent chooses one of the senses of dominance. Suppose that $X, Y \in \mathcal{X}$. If $X \leq Y$, then $X \precsim Y$. If $X \prec_{\text{Dom}} Y$, then $X \ll Y$.

Definition 9. A trading system $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ is called *coherent* if it satisfies Assumption 5.

When $\mathcal{O} = \mathbb{R}$, note that dominance is defined on all of $\mathbb{R}^{\Omega}$, while Assumption 5 pertains only to elements of $\mathcal{X}$. Until we can state Definition 16, Assumption 5 makes sense only when $\mathcal{O} = \mathbb{R}$. In the meantime, we state some results with clauses such as "If $\mathcal{T}$ is coherent ..." Those results that are not preceded by such clauses apply more generally.

2. Representing and Extending a Trading System

2.1. Representations of Total Trading Systems

In this section, we show how to represent a total trading system by a (possibly nonstandard-valued) numerical function.

Definition 10. If $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ is a total trading system and U represents $\precsim$ (recall Definition 4) then we say that U *represents* $\mathcal{T}$.

The following result follows easily from Definition 10.

Proposition 5. *A numerical function U represents $\precsim$ if and only if*

$$X \prec Y \text{ if and only if } U(X) < U(Y). \tag{3}$$

Next, we introduce a class of numerical functions that represent total trading systems.

Definition 11. A standard-linear function U (recall Definition 3) is called *monotone* if $X \le Y$ implies $U(X) \le U(Y)$. A monotone standard-linear function U is said to *respect dominance* if $X \prec_{\text{Dom}} Y$ implies $U(X) < U(Y)$.

Lemma 1. *Let U be a standard-linear function defined on a standard-linear space $\mathcal{X}$ of functions defined on a state space Ω. Then U represents a total trading system $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$. Also $\mathcal{T}$ is coherent if and only if U is monotone and respects dominance.*

Proof. Assume that U is a standard-linear function on a standard-linear space $\mathcal{X}$. Define the total preorder $\precsim$ on $\mathcal{X}$ by (2). For the first claim, we need to verify Assumptions 1–4. Assumption 1 follows because a preorder is reflexive. For Assumption 2, suppose that $Y - X = Y' - X'$. Since U is standard-linear,

$$U(Y) - U(X) = U(Y - X) = U(Y' - X') = U(Y') - U(X').$$

For Assumption 3, note that U being standard-linear implies that

$$U(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 U(X_1) + \alpha_2 U(X_2),$$

for all standard α_1, α_2 and $X_1, X_2 \in \mathcal{X}$. For Assumption 4, note that $\ll$ is $\prec$.

For the second claim, we need to prove that Assumption 5 holds if and only if U is monotone and respects dominance. For the "if" direction, assume that U is monotone and respects dominance. Since U is monotone, $X \le Y$ implies $U(X) \le U(Y)$ and $X \precsim Y$. Since U respects dominance, $X \prec_{\text{Dom}} Y$ implies $U(X) < U(Y)$ and $X \prec Y$, so Assumption 5 holds. For the "only if" direction, assume that Assumption 5 holds. To see that U is monotone, assume that $X \le Y$. The first requirement of Assumption 5 is that $X \precsim Y$, which implies that $U(X) \le U(Y)$, and U is monotone. To see that U respects dominance, assume that $X \prec_{\text{Dom}} Y$. The second requirement of Assumption 5 is that $X \prec Y$, which implies that $U(X) < U(Y)$, and U respects dominance. □

2.2. Agreement, Representation and Extension

If $\precsim$ is a not a total preorder, then there can be no numerical function U such that (2) holds. The problem is the "if" direction of (2) rather than the "only if" direction. In other words, representation in the sense of (2) is not achievable for preorders that are not total. On the other hand, a weaker version of (2) is available.

Definition 12. Let $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ be a trading system. A numerical function U on $\mathcal{X}$ *agrees with* $\mathcal{T}$ if

$$\begin{aligned} X \precsim Y \quad &\text{implies} \quad U(X) \le U(Y), \text{ and} \\ X \ll Y \quad &\text{implies} \quad U(X) < U(Y). \end{aligned} \tag{4}$$

When it comes to extension of a trading system, there are two modes of extension that are important to our analysis. One mode corresponds to adding more comparisons (amongst elements of a single set $\mathcal{X}$) to the preorder, bringing it closer to being total. The other mode corresponds to expanding the domain of definition of the preorder (from one set $\mathcal{X}$ to a larger set $\mathcal{X}'$.) Along with the second mode of extension comes a corresponding concept of restricting the domain of definition.

Definition 13. Let $\mathcal{X}$ and $\mathcal{X}'$ be standard-linear spaces with $\mathcal{X} \subseteq \mathcal{X}'$. Let ρ be a binary relation on $\mathcal{X}$, and let ρ' be a binary relation on $\mathcal{X}'$.

- If $\mathcal{X} = \mathcal{X}'$ and $X \rho Y$ implies $X \rho' Y$, we say that ρ' is an *extension*$_1$ of ρ.
- If $(X, Y \in \mathcal{X}) \wedge (X \rho Y)$ implies $X \rho' Y$, we say that ρ' is an *extension*$_2$ of ρ.
- If ρ' is an extension$_2$ of ρ, we say that ρ is the *restriction* of ρ' to $\mathcal{X}$.
- Suppose that $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ and $\mathcal{T}' = (\mathcal{X}, \precsim', \ll')$ are trading systems. If $\precsim'$ and $\ll'$ are extensions$_1$ of $\precsim$ and $\ll$ respectively, we call $\mathcal{T}'$ an *extension*$_1$ of $\mathcal{T}$.
- Let $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ and $\mathcal{T}' = (\mathcal{X}', \precsim', \ll')$ be trading systems. If $\mathcal{X} \subseteq \mathcal{X}'$ and if $\precsim'$ and $\ll'$ are extensions$_2$ of $\precsim$ and $\ll$ respectively, we call $\mathcal{T}'$ an *extension*$_2$ of $\mathcal{T}$.

To be clear, each binary relation and each trading system is both an extension$_1$ and an extension$_2$ of itself. The following result about extension$_2$ is key in our theorems on representation. Its proof appears in Appedix C.1.

Lemma 2. *Let $\mathcal{X}$ and $\mathcal{X}'$ be standard-linear spaces of functions with domain Ω and such that $\mathcal{X} \subseteq \mathcal{X}'$. Let $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ be a trading system. There exists a trading system $\mathcal{T}' = (\mathcal{X}', \precsim', \ll')$ that is an extension$_2$ of $\mathcal{T}$. If it is not desired that $\mathcal{T}'$ be coherent, $\mathcal{T}'$ can be chosen such that $\mathcal{V}_{\mathcal{T}'} = \mathcal{V}_{\mathcal{T}}$. If $\mathcal{T}$ is coherent and it is desired that $\mathcal{T}'$ be coherent, assume that $\leq$ and $\prec_{\mathrm{Dom}}$ are defined on $\mathcal{X}'$ and are extensions$_2$ of $\leq$ and $\prec_{\mathrm{Dom}}$ on $\mathcal{X}$. Then a coherent $\mathcal{T}'$ can be chosen so that for every $V' \in \mathcal{V}_{\mathcal{T}'}$ there is $V \in \mathcal{V}_{\mathcal{T}}$ such that $V \leq V'$.*

In addition to the proof of Lemma 2, Appendix C contains the lengthier proofs of results to be stated later. Note that Lemma 2 above (as well as Lemma 3 and Theorem 2 below) have language about $\leq$ and $\prec_{\mathrm{Dom}}$ on a larger space being extensions$_2$ of $\leq$ and $\prec_{\mathrm{Dom}}$ on a smaller space. When $\mathcal{O} = \mathbb{R}$, this condition is met trivially. The language is included to allow us to use these same results in other cases after Section 4.1 where $\leq$ and $\prec_{\mathrm{Dom}}$ are defined in terms of each specific trading system.

2.3. Finding Agreeing Functions

Our main representation Theorem 1 states that a trading system $\mathcal{T}$ has a standard-linear function U that agrees with it and an extension$_1$ to a total trading system that is represented by U. Results from Giarlotta and Creco (2013); Giarlotta (2019); Nishimura and Ok (2020) give the extension$_1$ for a general preorder, but without the representing function and without attention to the properties of a trading system. The following result has both Theorems 1 and 2 as special cases, and its proof appears in Appendix C.3.

Lemma 3. *Assume the following structure:*

- *$\mathcal{Y}$ and $\mathcal{W}$ are standard-linear spaces of functions defined on Ω with $\mathcal{Y}$ a proper subset of $\mathcal{W}$.*
- *$\mathcal{T}_{\mathcal{Y}} = (\mathcal{Y}, \precsim_{\mathcal{Y}}, \prec_{\mathcal{Y}})$ is a total trading system that is represented by the standard-linear function $U : \mathcal{Y} \to {}^*\mathbb{R}$, where ${}^*\mathbb{R}$ is a nonstandard model of the reals.*
- *$\mathcal{T}_{\mathcal{W}} = (\mathcal{W}, \precsim_{\mathcal{W}}, \ll_{\mathcal{W}})$ is the extension$_2$ of $\mathcal{T}_{\mathcal{Y}}$ obtained from Lemma 2.*

Then U can be extended to a standard-linear function $U' : \mathcal{W} \to {}^*\mathbb{R}'$, where ${}^*\mathbb{R}'$ contains ${}^*\mathbb{R}$ and such that U' represents a total trading system $\mathcal{T}' = (\mathcal{W}, \precsim', \prec')$ that is an extension$_2$ of $\mathcal{T}_{\mathcal{Y}}$. Also, if $\mathcal{T}_{\mathcal{Y}}$ is coherent and $\leq$ and $\prec_{\mathrm{Dom}}$ are defined on $\mathcal{W}$ so as to be extensions$_2$ of $\leq$ and $\prec_{\mathrm{Dom}}$ on $\mathcal{Y}$, then $\mathcal{T}'$ can be chosen to be coherent.

Theorem 1. *Let $\mathcal{T}$ be a trading system. There exists a standard-linear function U that agrees with $\mathcal{T}$ and total trading system $\mathcal{T}'$ that is an extension$_1$ of $\mathcal{T}$ such that U represents $\mathcal{T}'$. If $\mathcal{T}$ is coherent, $\mathcal{T}'$ can be chosen to be coherent.*

Proof. Apply Lemma 3 with $\mathcal{Y} = \{0\}$ (the trivial standard-linear space containing only the additive identity in $\mathcal{X}$,) $0 \precsim_{\mathcal{Y}} 0$, $\mathcal{W} = \mathcal{X}$, $\mathcal{T}_{\mathcal{W}} = \mathcal{T}$, $U(0) = 0$, and ${}^*\mathbb{R} = \mathbb{R}$. Let $\mathcal{T}'$ be the $\mathcal{T}'$ that results from Lemma 3, and let U be the corresponding U'. These satisfy the conclusion of Theorem 1. □

Theorem 2 of Skala (1974) shows that every total preorder can be represented by a nonstandard-valued function, but the standard-linear nature that we need is not proven in that paper.

2.4. Extending$_2$ a Trading System

Let $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ be a (coherent) trading system. Theorem 1 says that there exists a standard-linear function U that agrees with $\mathcal{T}$ and such that U represents a (coherent) total trading system $\mathcal{T}'$ that is an extension$_1$ of $\mathcal{T}$. Extension$_2$ is also possible if $\mathcal{X}$ is a subspace of a larger standard-linear space, as stated in Theorem 2.

Theorem 2. *Let $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ be a total trading system on a standard-linear space $\mathcal{X}$ with standard-linear representing function U. Let $\mathcal{X}'$ be a standard-linear space of functions that includes $\mathcal{X}$ as a proper subset. Then there is a total trading system $\mathcal{T}' = (\mathcal{X}', \precsim', \prec')$ that is an extension$_2$ of $\mathcal{T}$ and a standard-linear function U' on $\mathcal{X}'$ that represents $\mathcal{T}'$ and extends U to $\mathcal{X}'$. If $\mathcal{T}$ is coherent and $\leq$ and $\prec_{\mathrm{Dom}}$ are defined on $\mathcal{X}'$ so that they are extensions$_2$ of $\leq$ and $\prec_{\mathrm{Dom}}$ on $\mathcal{X}$, then $\mathcal{T}'$ can be chosen to be coherent.*

Proof. Apply Lemma 3 with $\mathcal{T}_{\mathcal{Y}} = \mathcal{T}$, $\mathcal{W} = \mathcal{X}'$, and U being the U in the statement of Theorem 2. The $\mathcal{T}'$ and U' that result from Lemma 3 satisfy the conclusion of Theorem 2. □

There are many examples of standard-linear spaces with proper supersets. One can easily imagine an agent determining a set of preferences over a small set $\mathcal{X}$ of objects and then being offered additional options in a set $\mathcal{X}'$. Here is an example of a situation that might seem of a different nature, but which still fits the setup of Theorem 2.

Example 2 (Refining a State Space). Suppose that an agent has a trading system $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ where each element of $\mathcal{X}$ is a function from Ω to a space $\mathcal{O}$. At some point, the agent realizes that the elements of Ω are not atomic. That is, each element ω appears to be, itself, a subset of a different set Ω'. To be specific, for each $\omega \in \Omega$, there is a subset $C_\omega \subseteq \Omega'$ such that the distinct elements of $\mathcal{C} = \{C_\omega : \omega \in \Omega\}$ form a partition of Ω'. Let $\sim_\Omega$ be the equivalence relation on Ω' that corresponds to $\mathcal{C}$, i.e., for all $x, y \in \Omega'$, $x \sim_\Omega y$ means that there is $\omega \in \Omega$ such that $x, y \in C_\omega$. In this way, Ω is the quotient space $\Omega'/\sim_\Omega$. Each element $X \in \mathcal{X}$ corresponds to a function $T(X) : \Omega' \to \mathcal{O}$ defined by $T(X)(x) = X(\omega)$ where $x \in C_\omega$. Each such $T(X)$ is constant on each element of $\mathcal{C}$. If at least one C_ω has more than one element, then there are additional functions defined on Ω' that do not have the form $T(X)$ for $X \in \mathcal{X}$. Let $\mathcal{X}'$ be a standard-linear space of functions defined on Ω' that contains $\mathcal{Y} = \{T(X) : X \in \mathcal{X}\}$ as a subset. A trading system $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ maps to a corresponding trading system $\mathcal{T}' = (\mathcal{Y}, \precsim', \ll')$. By

- $T(X) \precsim' T(Y)$ if and only if $X \precsim Y$ and
- $T(X) \ll' T(Y)$ if and only if $X \ll Y$.

Even if $\mathcal{T}$ is total, the resulting $\mathcal{T}'$ is not total. Theorem 2 can be used to extend$_2$ $\mathcal{T}'$ to a total trading system $\mathcal{T}^*$ on $\mathcal{X}'$. If $\mathcal{T}$ is coherent and $\mathcal{O} = \mathbb{R}$, then $\leq$ and $\prec_{\text{Dom}}$ are defined on $\mathcal{X}'$ so as to be extensions$_2$ of $\leq$ and $\prec_{\text{Dom}}$ on $\mathcal{X}$. In fact, $T(X) \prec_{\text{Dom}} T(Y)$ in $\mathcal{X}'$ if and only if $X \prec_{\text{Dom}} Y$ in $\mathcal{X}$. ⋄

3. Three Decision Theories

In this section, we show how the structure of Section 1 and the results of Section 2 extend three well-known theories of decision making.

3.1. Previsions for Random Variables

The first Archimedean theory to which our results apply is the theory of previsions of de Finetti (1974), which begins with an arbitrary set of standard-valued random variables. For each X in that set, an agent chooses a standard value $P(X)$ (called the prevision of X) such that the agent is willing to trade away either X or $P(X)$ in order to receive the other one. Specifically, the change in fortune $\alpha[X - P(X)]$ is considered a fair gamble for all standard α. Although de Finetti's theory deals only in fair trades (indifference,) there is an implicit assumption that "more is better," which is built into his notion of coherence (corresponding to uniform dominance in Definitions 8 and 9.) In de Finetti (1974), the agent is willing to accept every finite sum of fair gambles. In particular, the agent is willing to accept

$$\alpha[X - P(X)] - \alpha[Y - P(Y)] = \alpha(X - Y) + \alpha[P(Y) - P(X)], \tag{5}$$

for all standard α. If $P(X) = P(Y)$, the right-hand side of (5) is $\alpha(X - Y)$, and the agent is implicitly willing to trade X to get Y or to trade Y to get X. If $P(X) \neq P(Y)$, there is an implicit strict preference in one direction, e.g., if $P(Y) > P(X)$ and $\alpha < 0$, the fair trade (5) is strictly smaller than $|\alpha|(Y - X)$, so the agent is willing to trade X to get Y, but not the other way. In addition, willingness to accept all finite sums of fair trades implies that a coherent prevision P on an arbitrary set $\mathcal{Y}$ of random variables extends uniquely to a coherent prevision on the linear span $\mathcal{X}$ of $\mathcal{Y}$. Define the total preorder $\precsim$ on $\mathcal{X}$ defined "$X \precsim Y$ if and only if $P(X) \leq P(Y)$." It follows that $(\mathcal{X}, \precsim, \prec)$ is a total trading system that is represented by the linear function P. Our theory extends that of de Finetti (1974) by dropping the requirement that every element of $\mathcal{X}$ be indifferent to some standard constant.

A simple example of a random variable that is not indifferent to a standard constant arises with an "almost-fair" coin. For an even-money bet (odds equal 1) the agent strictly prefers the bet that pays on heads over the bet that pays on tails. But, for every bet that is not at even money (i.e., odds are different from 1,) the agent strictly prefers the side of the bet that pays the larger amount. Theorem 3.1 of Fishburn implies that there is no standard-valued prevision that ranks these bets in the order of the stated preferences. See also Debreu (1954). But a nonstandard-valued function can represent such preferences. Random variables with infinite previsions are also cases in which fair prices are not available. See Seidenfeld, Schervish and Kadane (2009) for some surprising examples.

3.2. Horse Lotteries

The second theory to which our results apply is that of Anscombe and Aumann (1963); VonNeumann and Morgenstern (1947) for decisions about horse lotteries, which are functions from Ω to the set of simple lotteries over a set of prizes.

3.2.1. Horse Lotteries in General

Definition 14. For each $\omega \in \Omega$, let $\mathcal{P}_\omega$ be the set of prizes available in state ω. A *simple lottery* r is a probability on a finite subset $\mathcal{P}(r) \subseteq \mathcal{P}_\omega$. Let $\mathcal{R}_\omega$ be the convex set of simple lotteries available in state ω.[2] For ease of notation, let $\mathcal{P} = \bigcup_{\omega\in\Omega} \mathcal{P}_\omega$ and $\mathcal{R} = \bigcup_{\omega\in\Omega} \mathcal{R}_\omega$ be respectively the sets of all prizes available in at least one state and all lotteries available in at least one state. Let $\mathcal{R}_\Omega = \prod_{\omega\in\Omega} \mathcal{R}_\omega$, which is a subset of $\mathcal{R}^\Omega$. A *horse lottery* is a function $h \in \mathcal{R}_\Omega$, i.e, $h(\omega) \in \mathcal{R}_\omega$ for every $\omega \in \Omega$. Let $\mathcal{H}$ stand for the set of horse lotteries under consideration, which we assume to be a convex subset of $\mathcal{R}_\Omega$.[3]

In each application, the set $\mathcal{H}$ of horse lotteries can be different, but each such $\mathcal{H}$ must be a convex subset of $\mathcal{R}_\Omega$. For $h_1, h_2 \in \mathcal{R}_\Omega$ and $\alpha \in [0,1]$, the meaning of $h_3 = \alpha h_1 + (1-\alpha)h_2$ is that $h_3(\omega) = \alpha h_1(\omega) + (1-\alpha)h_2(\omega) \in \mathcal{R}_\omega$, because $\mathcal{R}_\omega$ is convex. A set $\mathcal{H}$ of horse lotteries is not a linear space. Next, we show how to create a linear space that is equivalent to $\mathcal{H}$ in an appropriate sense.

3.2.2. A Linear Space for Horse Lotteries

The set $\mathcal{H}$ of horse lotteries is a convex subset of $\mathcal{R}_\Omega$, but is not a linear space. Hausner (1954) (Sections 2–4) assumes that $\precsim'$ is a total preorder that satisfies the following axiom, which is part of the theory of Anscombe and Aumann (1963); VonNeumann and Morgenstern (1947):

Independence Axiom: Let $\precsim'$ be a preorder on a convex set $\mathcal{H}$ of horse lotteries. For all $h_1, h_2, g \in \mathcal{H}$ and standard $0 < \alpha < 1$, $h_1 \precsim' h_2$ if and only if $\alpha h_1 + (1-\alpha)g \precsim' \alpha h_2 + (1-\alpha)g$.

Hausner (1954) shows how to create a standard-linear space $\mathcal{K}_0$ with a preorder $\precsim$ that satisfies our Assumptions 2 and 3 in Section 1.4 above. This is done as follows. For each $\omega \in \Omega$, let $\mathcal{O}_\omega$ be the set of all simple signed measures[4] on $\mathcal{P}_\omega$ that give measure 0 to the whole set $\mathcal{P}_\omega$. Let $\mathcal{O} = \bigcup_{\omega\in\Omega} \mathcal{O}_\omega$, and let $\mathcal{O}_\Omega = \prod_{\omega\in\Omega} \mathcal{O}_\omega$. Then

$$\mathcal{K}_0 = \{\alpha(h_1 - h_2) : h_1, h_2 \in \mathcal{H} \text{ and } \alpha \in \mathbb{R}\} \subseteq \mathcal{O}_\Omega$$

is a standard-linear space. Define $\precsim$ on $\mathcal{K}_0$ as follows. For each $k_1, k_2 \in \mathcal{K}_0$, express $k_2 - k_1 = \alpha(h_2 - h_1)$ with $\alpha > 0$ and $h_1, h_2 \in \mathcal{H}$. Then say that $k_1 \precsim k_2$ if $h_1 \precsim' h_2$. Hausner (1954) (Section 4) shows that $\precsim$ is well defined and satisfies Assumptions 2 and 3. The theory of Anscombe and Aumann (1963); VonNeumann and Morgenstern (1947) satisfies Assumption 4 vacuously since $\precsim$ is a total preorder. Dominance and coherence are not issues that arise in the theory of Anscombe and Aumann (1963); VonNeumann and Morgenstern (1947) as horse lotteries are not numerically comparable without further assumptions.

The state-independence assumption of Anscombe and Aumann (1963); VonNeumann and Morgenstern (1947) implies that all $\mathcal{R}_\omega$ sets are the same. Our theory is general enough to include cases in which the $\mathcal{R}_\omega$ sets might all be the same or might be different. We also drop the Archimedean axiom and allow $\precsim'$ to not be total as do other authors such as Aumann (1962); Dubra, Maccheroni and Ok (2004); Baucells and Shapley (2008). Our weaker state-independence Assumption 7 is stated in Section 4.1.

3.2.3. Representing Horse Lotteries

For the remainder of this paper, when we refer to the *horse-lottery case*, we will assume that $\mathcal{X}$ is the standard-linear space $\mathcal{K}_0$ defined in Section 3.2.2. (The case in which $\mathcal{X}$ is a linear space of standard-valued random variables will be called the *random-variable case*.) In the horse-lottery case, it would be easier on the intuition if each representing function of a trading system had $\mathcal{H}$ as its domain rather than $\mathcal{K}_0$. This is easily arranged. Let $\mathcal{T} = (\mathcal{K}_0, \precsim, \prec)$ be a total trading system in a horse-lottery case with standard-linear representing function U. Let $\mathcal{H}$ be the set of horse lotteries that corresponds to $\mathcal{K}_0$ as in Section 3.2.2. For each $k \in \mathcal{K}_0$, we can write $k = \alpha(h_1 - h_2)$ with $\alpha > 0$ standard. Then

[2] If $r_1, r_2 \in \mathcal{R}$ but $\mathcal{P}(r_1) \neq \mathcal{P}(r_2)$, $\mathcal{P}(\alpha r_1 + [1-\alpha]r_2) = \mathcal{P}(r_1) \bigcup \mathcal{P}(r_2)$ when $\alpha \in (0,1)$.

[3] Each $h \in \mathcal{H}$ is also a function from Ω to $\mathcal{R}$, but $\mathcal{R}$ may not be a convex set.

[4] A signed measure μ on a set $\mathcal{Y}$ is *simple* if there is a finite subset $\{y_1, \ldots, y_n\} \subseteq \mathcal{Y}$ and numbers $\alpha_1, \ldots, \alpha_n$ such that, for every $B \subseteq \mathcal{Y}$, $\mu(B) = \sum_{j=1}^n \alpha_j I_B(y_j)$.

$0 \precsim k$ is equivalent to $h_2 \precsim' h_1$ for a total preorder $\precsim'$ on $\mathcal{H}$. Let $h_0 \in \mathcal{H}$ be arbitrary, and define

$$V(h) = U(h - h_0). \tag{6}$$

It follows that $V(h_0) = 0$ and

$$U(\alpha[h_1 - h_2]) = \alpha[V(h_1) - V(h_2)]. \tag{7}$$

Also, V represents $\precsim'$ and satisfies

$$V(\beta h_1 + [1-\beta]h_2) = \beta V(h_1) + (1-\beta)V(h_2),$$

for all $h_1, h_2 \in \mathcal{H}$ and all standard $\beta \in [0, 1]$.

3.3. Savage-style Acts

The third theory to which our results apply is that of Savage (1954). This theory makes some assumptions (including state-independence) about preferences amongst acts (functions) from a state space Ω to a set $\mathcal{P}$ of consequences (prizes) and then proves an expected-utility representation for those preferences. Lemma 4 below starts with those same assumptions and shows that there is a set of lotteries over the acts with an implied willingness to trade that satisfies the assumptions that appear in Section 1.4 of this paper. We then weaken the original assumptions of Savage (1954) and show how to use our results for the horse-lottery case to represent non-Archimedean and state-dependent preferences over the acts of Savage (1954). Whenever we refer to "the horse-lottery case" in this paper, we implicitly include the theory of Savage in that case.

At this point, we can show how a non-Archimedean version of the theory of Savage (1954) becomes a special case of trading systems in the horse-lottery case without any additional assumptions or choices by the agent. The proof of Lemma 4 is in Appendix C.4.

Lemma 4. *Let $\mathcal{F}$ be a set of functions from Ω to $\mathcal{P}$, and let $\precsim'$ be a total preorder on $\mathcal{F}$ that satisfies the seven postulates (P1–P7) of Savage Savage (1954). Let $\mathcal{H}$ be the set of finite mixtures of elements of $\mathcal{F}$. Then $\precsim'$ extends to a total preorder on $\mathcal{H}$, and the $\mathcal{K}_0$ and $\precsim$ constructed from $\mathcal{H}$ and $\precsim'$ in Section 3.2.2 form a total trading system $(\mathcal{K}_0, \precsim, \prec)$ that satisfies Assumptions 1–4.*

Despite the fact that Savage worked hard to avoid making the assumption that his set of acts contained the mixtures that we assume, his postulates are sufficient to show that his preorder extends to a total trading system that satisfies our assumptions without any further choices needed from the agent. For the purposes of this paper, instead of assuming a subset of P1–P7 or some weakened versions of them, assume only that there is a set $\mathcal{F}$ of Savage-style acts with a preorder (not necessarily total) $\precsim'$. Then embed $\mathcal{F}$ into the convex set $\mathcal{H}$ of Lemma 4 which is a special case of a set of horse lotteries. We then proceed with the same analysis and assumptions as in Sections 3.2.2 and 1.4. In particular, we make Assumptions 1–4. By so doing, we implicitly weaken some of Savage's postulates so as to allow non-Archimedean preferences, In addition, all of the extension and representation results in Section 2 above apply to the resulting trading system, as well as the results in Section 4 below. In the end, if the agent does not want to think about mixtures of Savage-style acts, we show (in Section 4.7) how to restrict the results of Section 4 to the original Savage-style acts.

4. Probability and Expected Utility

In this section, we explore the relationship between finitely-additive expectation and the representing function of a total trading system. Throughout the section, $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ denotes a total trading system on a standard-linear space $\mathcal{X}$ of functions from Ω to a standard-linear space $\mathcal{O}$. Let U be a standard-linear function that represents $\mathcal{T}$. Let Σ be a field of subsets of Ω. In order to construct a probability from a trading system, the indicators of elements of Σ must play a role in the elements of $\mathcal{X}$. We make the following assumption about a total trading system $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ throughout this section.

Assumption 6. For all $B \in \Sigma$ and all $X \in \mathcal{X}$, $XI_B \in \mathcal{X}$, where

$$(XI_B)(\omega) = \begin{cases} X(\omega) & \text{if } \omega \in B, \\ 0(\omega) & \text{if } \omega \in B^C. \end{cases}$$

If Assumption 6 is not met, one can apply Theorem 2 to find an extension$_2$ of $\mathcal{T}$ to $\mathcal{T}' = (\mathcal{X}', \precsim', \prec')$, where $\mathcal{X}'$ is the standard-linear span of $\mathcal{X} \bigcup \{XI_B : X \in \mathcal{X}, B \in \Sigma\}$. This extension needs to be done before attempting to infer a probability on (Ω, Σ) or attempting to interpret a representing function U as an expected utility. Once Assumption 6 is met, we can define conditional preference.

Definition 15. Let $\mathcal{W}$ be a standard-linear space of functions defined on a set $\mathcal{Z}$. Let Γ be a field of subsets of $\mathcal{Z}$. Suppose that $XI_B \in \mathcal{W}$ for each $B \in \Gamma$ and $X \in \mathcal{W}$. Let $\precsim$ be a total preorder on $\mathcal{W}$. For $X, Y \in \mathcal{W}$, if $0 \prec (Y - X)I_B$, we write $X \prec Y|B$ and we say that Y *is conditionally preferred to* X *given* B. If $0 \precsim (Y - X)I_B$, we write $X \precsim Y|B$. If both $X \precsim Y|B$ and $Y \precsim X|B$, we write $X \sim Y|B$. An event $B \in \Gamma$ is *null* if $X \sim Y|B$ for all $X, Y \in \mathcal{W}$. An event is *non-null* if it is not null. If $z \in \mathcal{Z}$ and $\{z\}$ is a null event, we call z a *null state*. A state z is *non-null* if $\{z\}$ is non-null.

4.1. Dominance and Coherence (Part Two)

In this section, we extend the concepts of dominance and coherence to certain horse-lottery cases. This extension is useful in Section 4.4 where we show how, in both the random-variable and horse-lottery cases, a standard-linear function that represents a coherent trading system can be interpreted as an expected value of a (possibly state-dependent) utility function defined on the codomain $\mathcal{O}$ of the elements of $\mathcal{X}$.

For the remainder of this section, we assume that $\mathcal{X}$ is the space $\mathcal{K}_0$ defined in Section 3.2.2 and $\mathcal{O}$ is a set of simple signed measures on subsets of the prize set $\mathcal{P}$ and that assign signed measure 0 to $\mathcal{P}$. As such, each element of $\mathcal{K}_0$ is a function from Ω to $\mathcal{O}$. Dominance for horse lotteries is defined in terms of conditional preference on a state-by-state basis. Suppose that there is a non-null event B that consists entirely of null states, i.e., every $\omega \in B$ is a null state. Then a state-by state comparison of two elements X and Y of $\mathcal{X}$ given the elements of B tells us nothing about how X and Y compare (or should compare) given B. The reason is that $X \sim Y|\{\omega\}$ for every X and Y and every null state ω. To circumvent this problem, we make an assumption that is a generalization (weaker assumption) of the state-independent utility assumptions made by Anscombe and Aumann (1963) (Assumption 1,) Savage (1954) (P3–P4,) and Blume et al. (1991a) (Axioms 5 and 5$'$.) Assumption 7 below allows varying degrees of state dependence for the utilities of prizes.

Assumption 7. There exists a partition $\mathcal{B} \subseteq \Sigma$ of Ω into non-null events such that, for each $B \in \mathcal{B}$,

- for all $\omega \in B$, $\mathcal{O}_\omega$ is the same set $\mathcal{O}(B)$,
- for each $x \in \mathcal{O}(B)$, there is $X_x \in \mathcal{X}$ such that $X_x(\omega) = x$ for all $\omega \in B$, and
- for each $x \in \mathcal{O}(B)$, $0 \prec X_x|B$ if and only if $0 \prec X_x|C$ for every non-null $C \subseteq B$.

For each ω, we will use B_ω to denote the element of $\mathcal{B}$ that contains ω.

The second bullet assumes that certain functions defined on $\mathcal{O}_\Omega$ are in $\mathcal{X}$. If these functions are not in $\mathcal{X}$ and the first bullet is satisfied, we can apply Theorem 2 to find an extension$_2$ of $\mathcal{T}$ to $\mathcal{T}' = (\mathcal{X}', \precsim', \prec')$ where $\mathcal{X}'$ is the standard-linear span of $\mathcal{X} \bigcup \{X_x I_B I_A : B \in \mathcal{B}, x \in \mathcal{O}(B), A \in \Sigma\}$. Such an extension$_2$ will continue to satisfy Assumption 6 and the first bullet of Assumption 7. We will assume that such an extension$_2$ has been done for the remainder of the paper. Note that there is no guarantee that such a $\mathcal{T}'$ satisfies the third bullet of Assumption 7. But we cannot even check whether the third bullet is satisfied until the second bullet is satisfied.

The state-independence assumptions made by Anscombe and Aumann (1963); Savage (1954); Blume et al. (1991a) correspond to the extreme case of Assumption 7 in which $\mathcal{B} = \{\Omega\}$, i.e., utility is independent of state for the whole state space. The opposite extreme case of Assumption 7 has $\mathcal{B} = \{\{\omega\} : \omega \in \Omega\}$, i.e., every state is non-null, in which case all three bullets of Assumption 7 are tautological because each B is a singleton. There are cases between these two extremes, such as Example 5 in Section 4.2. A more concrete example would be the following.

- Each state consists of a specification of exchange rates between a set of currencies and a specification of a set of meteorological conditions.
- The values of the prizes depend only on the exchange rates and not on the meteorological conditions.
- Each set in the partition $\mathcal{B}$ is the set of states with a fixed specification of exchange rates.

Definition 16 below defines $\leq$ and dominance in those horse-lottery cases that satisfy Assumption 7. It allows us to talk about coherent trading systems in the horse-lottery case with no modifications to Assumption 5 or the definition of coherence (Definition 9). As in the random-variable case, whichever (if any) sense of dominance is reflected in the agent's preferences, we express "Y dominates X" by $X \prec_{\text{Dom}} Y$.

As in Definition 8 for the random-variable case, Definition 16 defines dominance on a larger set of objects than just $\mathcal{X}$. The reason is that we may need elements of that larger set of objects in order to infer the existence of a probability on the field Σ. (See Lemma 6 in Section 4.2.)

Lemma 5. *Let $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ be a total trading system that satisfies Assumptions 6 and 7. Let $X \in \mathcal{O}_\Omega$, and let $\omega_0 \in \Omega$. There exists $G_{X,\omega_0} \in \mathcal{X}$ such that $G_{X,\omega_0}(\omega) = X(\omega_0)$ for all $\omega \in B_{\omega_0}$.*

Proof. Because $x = X(\omega_0) \in \mathcal{O}(B_{\omega_0})$, Assumption 7 says that there is $X_x \in \mathcal{X}$ such that $X_x(\omega) = x = X(\omega_0)$ for all $\omega \in B_{\omega_0}$. Rename X_x to be G_{X,ω_0}. □

Lemma 5 gives us what we need to define dominance on all of $\mathcal{O}_\Omega$.

Definition 16. Suppose that a trading system satisfies Assumptions 6 and 7. For each $\omega_0 \in \Omega$ and each $X \in \mathcal{O}_\Omega$, let $G_{X,\omega_0} \in \mathcal{X}$ be from Lemma 5.

- If $G_{X,\omega_0} \precsim G_{Y,\omega_0} | B_{\omega_0}$, we write $X(\omega_0) \leq Y(\omega_0)$.
- If $G_{X,\omega_0} \prec G_{Y,\omega_0} | B_{\omega_0}$, we write $X(\omega_0) < Y(\omega_0)$.
- If, for all $B \in \mathcal{B}$ and all $\omega_0 \in B$, $X(\omega_0) \leq Y(\omega_0)$, we write $X \leq Y$.
- If, for all ω_0, $X(\omega_0) < Y(\omega_0)$, we say that Y *strictly dominates* X.
- If $X \leq Y$ and there exists ω_0 such that $X(\omega_0) < Y(\omega_0)$, we say that Y *weakly dominates* X.

Now, the definition of "coherent" (Definition 9) applies in the horse-lottery case. We did not define "uniform dominance" because the $<$ symbol in Definition 16 is not a relation between numerical values, but rather a non-numerical preference relation. Once we define a state-dependent utility function (Definition 18 in Section 4.3) we can define uniform dominance in the horse-lottery case.

4.2. Numeraires and Probability

Definition 17. Assume the conditions stated in Definition 15. Let $Z \in \mathcal{W}$. If $0 \prec Z$ and $0 \precsim Z|B$ for all B, we call Z a *numeraire for* Γ. If, in addition, $0 \prec Z|B$ for every non-null B, we call Z a *strong numeraire for* Γ.

In the random-variable case with a coherent trading system, every non-negative function in $\mathcal{X}$ that is strictly preferred to 0 is a numeraire. Nevertheless, even positive constants might not be strong numeraires.

Example 3. Let $\Omega = \mathbb{Z}^+$, the positive integers, and let Σ be the finite/cofinite field, i.e., the collection of all finite subsets of Σ and their complements. Let $G : \Omega \to \mathbb{R}$ be $G(\omega) = \omega$. Let $\mathcal{X}$ be the standard-linear span of all standard-valued bounded functions and the functions $\{GI_B : B \in \Sigma\}$. Each $X \in \mathcal{X}$ can be written uniquely as

$$X = X_b + \alpha_X G I_E, \tag{8}$$

where X_b is bounded, α_X is standard, and $E = \{11, 12, \dots\}$.[5] Define

$$U(X) = \frac{1}{10} \sum_{\omega=1}^{10} X(\omega) + 2\alpha_X. \tag{9}$$

Also, $U : \mathcal{X} \to \mathbb{R}$ is standard-linear (and standard-valued.) If $X \prec_{\text{Dom}} Y$ (uniform or strict, but *not* weak dominance) then $U(X) < U(Y)$, so U represents a coherent (with uniform or strict dominance) total trading system $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ with $X \precsim Y$ meaning $U(X) \leq U(Y)$. Since $U(1) = 1$, $0 \prec 1$. Since $0 \leq 1$, $0 \precsim 1|B$ for all $B \in \Sigma$, and 1 is numeraire for Σ. Since $0 \prec G|E$, E is a non-null event. On the other hand, $0 \sim 1|E$, so 1 is not a strong numeraire for Σ. The non-null events are all cofinite sets and all elements of Σ that are supersets of the singletons $\{1\}, \dots, \{10\}$. For each $\omega = 1, \dots, 10$, $0 \prec G|\{\omega\}$, and for each cofinite set B, $0 \prec G|B$, so G is a strong numeraire for Σ. ⋄

[5] Note that $\alpha_X = \lim_{\omega\to\infty} X(\omega)/G(\omega)$, and $X_b = X - \alpha_X G I_E$.

In Section 4.5, we show how different numeraires relate to each other within the same trading system.

We take probability to be finitely additive in Lemma 7 below and elsewhere due to the difficulty in defining countable sums of nonstandard numbers. (See the results and examples in Appendix A.4.) In general, we need a numeraire in order to derive a probability on (Ω, Σ) from a trading system. Since 1 is a numeraire in the random-variable case, the following result is needed only in the horse-lottery case.

Lemma 6. *Let $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ be a coherent total trading system in the horse-lottery case that satisfies Assumptions 6 and 7. There is a coherent extension$_2$ $\mathcal{T}' = (\mathcal{X}', \precsim', \prec')$ of $\mathcal{T}$ such that*

- *there is a numeraire Z for Σ,*
- *Z is constant on each element of $\mathcal{B}$ (from Assumption 7,) and*
- *$0 \prec Z|B$ for every $B \in \mathcal{B}$.*

Proof. First, we show that, for each $B \in \mathcal{B}$, there is $x_B \in \mathcal{O}(B)$ such that $0 \prec X_{x_B}|B$, where X_x is defined in the second bullet of Assumption 7. Let $B \in \mathcal{B}$, which, by definition, is non-null. So there is $Y \in \mathcal{X}$ such that $0 \prec Y|B$. For each $\omega \in B$, let $y_\omega = Y(\omega)$. If $X_{y_\omega} \precsim 0|B$ for every $\omega \in B$, then $YI_B \leq 0$ and $Y \precsim 0|B$ by coherence. This contradicts $0 \prec Y|B$, so there must be $\omega_0 \in B$ such that $0 \prec X_{x_B}|B$ for $x_B = y_{\omega_0}$.

Define $Z : \Omega \to {}^*\mathbb{R}$ as follows: For all $B \in \mathcal{B}$ and all $\omega \in B$, let $Z(\omega) = x_B$. By construction $0 \prec Z|B$ for all $B \in \mathcal{B}$ and $0 \prec_{\text{Dom}} Z$ by either definition of dominance. If $Z \in \mathcal{X}$, let $\mathcal{T}' = \mathcal{T}$. If not, let $\mathcal{X}'$ be the standard-linear span of $\mathcal{X} \bigcup \{ZI_D : D \in \Sigma\}$. Use Theorem 2 to extend$_2$ $\mathcal{T}$ to a coherent $\mathcal{T}'$. In $\mathcal{T}'$, $0 \precsim' Z|D$ for every $D \in \Sigma$ by construction. Since $0 \prec' Z$, Z is a numeraire for Σ. □

For the remainder of the paper, assume that each trading system $\mathcal{T}$ in the horse-lottery case is an extension$_2$ from Lemma 6.

Lemma 7. *Let $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ be a coherent total trading system that satisfies Assumption 6 (and Assumption 7 in the horse-lottery case.) Let Z be a numeraire for Σ, and let U be a standard-linear function that represents $\mathcal{T}$. Define $P(B) = U(ZI_B)/U(Z)$ for each $B \in \Sigma$. Then P is a finitely-additive (possibly nonstandard-valued) probability on Σ.*

Proof. First, note that $U(Z) > 0$ and $P(\Omega) = U(ZI_\Omega)/U(Z) = 1$. For each $B \in \Sigma$, $0 \precsim Z|B$ and $P(B) = U(ZI_B)/U(Z) \geq 0$. Suppose that B_1 and B_2 are disjoint events. Then $ZI_{B_1} + ZI_{B_2} = ZI_{B_1 \cup B_2}$. Since U is standard-linear, we have $U(ZI_{B_1}) + U(ZI_{B_2}) = U(ZI_{B_1 \cup B_2})$ and $P(B_1) + P(B_2) = P(B_1 \cup B_2)$. □

Example 4 (Continuation of Example 3). The two numeraires, 1 and G for Σ lead to two different probabilities P_1 and P_G on (Ω, Σ) respectively. From (9), it is clear that $P_1(\{\omega\}) = 1/10$ for $\omega = 1, \ldots, 10$, and $P_1(B) = 0$ for each B that is a subset of E. In particular, $P_1(E) = 0$ despite E being non-null. For each ω, $U(GI_{\{\omega\}}) = \omega/10$ for $\omega \in E^C$ and 0 for $\omega \in E$. For each cofinite set $B \subseteq E$, $U(GI_B) = 2$, since $\alpha_G = 1$. Then $P_G(\{\omega\}) = \omega/75$ for $\omega \in E^C$ and 0 for $\omega \in E$. Also, $P_G(B) = 4/15$ for every cofinite subset B of E, including E itself. ⋄

Example 5. Let $\Omega = (0, 1)$, the unit interval with Σ the Borel σ-field. Let $\mathcal{P} = \{a, b\}$. Let $\mathcal{R}$ be the set of all lotteries over $\mathcal{P}$. Each element $r \in \mathcal{R}$ is characterized by $r(\{b\})$. Let $\mathcal{H}$ be that subset of $\mathcal{R}^\Omega$ for which $g_h(\omega) = h(\omega)(\{b\})$ is a Borel-measurable function of ω. In this example, for states $\omega < 1/2$, prize a is better than b, and the situation is the reverse for all $\omega > 1/2$. We include the null event $\{1/2\}$ with $\omega > 1/2$ for the rest of the example. Define

$$V(h) = \int_0^{1/2} [1 - g_h(\omega)]d\omega + \int_{1/2}^1 g_h(\omega)d\omega. \tag{10}$$

Each element of $\mathcal{K}_0$ has the form $\alpha(h_1 - h_2)$ for standard $\alpha \geq 0$ and $h_1, h_2 \in \mathcal{H}$. Define

$$U(\alpha[h_1 - h_2]) = \alpha[V(h_1) - V(h_2)].$$

The trading system represented by U satisfies Assumption 7 with $\mathcal{B} = \{(0, 1/2), [1/2, 1)\}$. There are many numeraires for Σ. Let $g_{h_0}(\omega) = I_{(0,1/2)}(\omega)$, so that $V(h_0) = 0$. Each h with $g_h(\omega) < 1$ for $\omega < 1/2$ and $g_h(\omega) > 0$ for $\omega \geq 1/2$ has $V(h) > 0$ and can be combined with h_0 to define a numeraire for Σ as follows. Define $Z_h = (h - h_0)/V(h)$. Since

$U(Z_h I_A) \geq 0$ for each Borel set A, $0 \precsim Z_h | A$. Also, $0 \prec Z_h$, so each such Z_h is a numeraire for Σ. As (10) would suggest, the probability derived from Z_h has the following density with respect to Lebesgue measure:

$$f_h(\omega) = \frac{1}{V(h)} \left([1 - g_h(\omega)] I_{(0,1/2)}(\omega) + g_h(\omega) I_{[1/2,1)}(\omega) \right).$$

Since $f_h > 0$, Z_h is a strong numeraire for Σ. ⋄

4.3. State-Dependent Utility

In the random-variable case, one can think about a numeraire as the units that correspond to numerical values of the random variables. For example, suppose that the random variables in a trading system $\mathcal{T}$ are in units of dollars. A European foreign-exchange trader might be more comfortable comparing units of euros rather than dollars. If $G(\omega)$ is the exchange rate (in dollars per euro) in state ω, $X(\omega)/G(\omega)$ is the utility of X in state ω to the European trader measured in the preferred currency units of euros. We need something analogous to X/G for the horse-lottery case.

Definition 18. Let Σ be a field of subsets of Ω, and let $\mathcal{T}$ be a coherent total trading system that satisfies Assumption 6 (and Assumption 7 in the horse-lottery case.) Let $U : \mathcal{X} \to {}^*\mathbb{R}$ be a standard-linear function that represents $\mathcal{T}$. In the horse-lottery case, let Z be a numeraire of the sort obtained through Lemma 6. In the random-variable case, let Z be a strictly positive numeraire. Define, for each $X \in \mathcal{X}$, each $\omega \in \Omega$, and each $x \in \mathcal{O}_\omega$:

$$U^*_Z(\omega, x) = \begin{cases} \dfrac{x}{Z(\omega)} & \text{in the random-variable case,} \\ \dfrac{U(xI_{B_\omega})}{U(ZI_{B_\omega})} & \text{in the horse-lottery case.} \end{cases}$$

We call U^*_Z the *state-dependent utility function (relative to the numeraire Z.)* Let $X, Y \in \mathcal{X}$. If there exists a standard $\epsilon > 0$ such that $U^*_Z(\omega, X(\omega)) \leq U^*_Z(\omega, Y(\omega)) - \epsilon$ for all ω, we say that X *is uniformly dominated by Y relative to Z* or Y *uniformly dominates X relative to Z*.

As Definition 18 makes explicit, which elements of $\mathcal{X}$ uniformly dominate each other depends on which numeraire Z is used. Example 6 illustrates this fact.

Example 6 (Continuation of Example 5). The state-dependent utility that corresponds to the numeraire Z_h is, for each $X = \alpha(h_1 - h_2) \in \mathcal{K}_0$,

$$U^*_h(\omega, X(\omega)) = \frac{\alpha}{f_h(\omega)} \begin{cases} g_{h_2}(\omega) - g_{h_1}(\omega) & \text{for } \omega < 1/2, \\ g_{h_1}(\omega) - g_{h_2}(\omega) & \text{for } \omega \geq 1/2, \end{cases}$$

so that $U^*_h(\omega, Z_h(\omega)) = 1$ for all ω, and Z_h uniformly dominates 0 relative to Z_h. The same could be said for each numeraire constructed in the same fashion. However, what "uniformly dominates" 0 depends on which numeraire is used to construct the probability. To be specific, suppose that $g_h(\omega) = 1/2$ for all ω so that f_h is constant. Let h' be another horse lottery with $g_{h'}(\omega) = 1 - \omega$ for all ω, so that $f_{h'}$ is $\wedge$-shaped and $U^*_{h'}(\omega, Z_{h'}(\omega)) = 1$ for all ω. Then h' uniformly dominate 0 relative to $Z_{h'}$, but

$$U^*_h(\omega, Z_{h'}(\omega)) = \begin{cases} \omega & \text{for } \omega < 1/2 \\ 1 - \omega & \text{for } \omega \geq 1/2, \end{cases}$$

so h' does not uniformly dominate 0 relative to Z_h. ⋄

To see that U^*_Z is well defined in the horse-lottery case, note that for each $B \in \mathcal{B}$ of Assumption 7, $U(ZI_B) > 0$ because $0 \prec Z|B$. In general U^*_Z is a nonstandard-valued function defined on the set $\bigcup_{\omega\in\Omega}(\{\omega\} \times \mathcal{O}_\omega)$. The interpretation of $U^*_Z(\omega, x)$ is the utility to the agent of X in state ω measured in units of numeraire Z when $X(\omega) = x$. In Section 4.4, Theorem 3 shows that $U(X)$ can be interpreted as an expected value of the state-dependent utility of X with respect to the probability P on (Ω, Σ) that corresponds to the chosen numeraire Z via Lemma 7.

4.4. Expected Utility

The following definition is a generalization of the concept of Daniell integral to the finitely-additive nonstandard-valued case. See Schervish, Seidenfeld and Kadane (2014) for discussion of the finitely-additive standard-valued case. We give additional motivation for this definition in Appendix B.

Definition 19. Let $^*\mathbb{R}$ be a nonstandard model of the reals, and let $\mathcal{Z}$ be a set. Let $\mathcal{W}$ be a standard-linear space that is a subset of $(^*\mathbb{R})^{\mathcal{Z}}$, and that contains all standard constants. Let $W : \mathcal{W} \to {}^*\mathbb{R}$ be a standard-linear function that satisfies

- $W(1) = 1$, and
- for $w_1, w_2 \in \mathcal{W}$, $w_1 \leq w_2$ implies $W(w_1) \leq W(w_2)$ (i.e., W is *monotone*.)

Then W *acts as an expected value on* $\mathcal{W}$. Suppose, in addition, that Γ is a field of subsets of $\mathcal{Z}$ and for every $B \in \Gamma$ and $w \in \mathcal{W}$, $wI_B \in \mathcal{W}$. Define $P(B) = W(I_B)$ for $B \in \Gamma$ so that P is a finitely-additive probability on $(\mathcal{Z}, \Gamma)$. Then W *acts as an expected value on* $\mathcal{W}$ *with respect to* P. For each $w \in \mathcal{W}$, we also say that $W(w)$ *is an expected value of* w *with respect to* P. We also use the notation $P(w)$ to denote $W(w)$.

It should be apparent that Definition 19 agrees with the familiar countably-additive definition of expected value for simple probabilities when $^*\mathbb{R} = \mathbb{R}$. Furthermore, all finite standard-valued countably-additive expected values satisfy Definition 19, but they also have an additional continuity property that does not carry over to the nonstandard-valued case.

Theorem 3. *Assume the conditions from and terms defined in Definition 18. Let* $U_X(\omega) = U^*_Z(\omega, X(\omega))$ *for each* $X \in \mathcal{X}$ *and* $\omega \in \Omega$, *and let* P *be the probability on* (Ω, Σ) *from Lemma 7 using* Z *as the numeraire for* Σ. *Let* $\mathcal{W}_Z = \{U_X : X \in \mathcal{X}\}$. *Then*

- $\mathcal{W}_Z$ *is a standard-linear space of functions from* Ω *to* $^*\mathbb{R}$,
- $W_Z(U_X) = U(X)/U(Z)$ *defines an expected value of* U_X *with respect to* P, *and*
- *if dominance means weak dominance, then every non-empty element of* Σ *is non-null and has positive probability.*

Proof. First, we show that $U_X \leq U_Y$ implies $X \leq Y$. This is immediate in the random-variable case. In the horse-lottery case, if $U_X \leq U_Y$, then for each $B \in \mathcal{B}$ and each $\omega_0 \in B$, $G_{X,\omega_0} \precsim G_{Y,\omega_0} | B$ in the notation of Lemma 5 and Definition 16. It follows that $X \leq Y$.

Next, we show that W_Z is well defined. Let $X, Y \in \mathcal{X}$ with $U_X = U_Y$. Then $U_X \leq U_Y$ and $U_Y \leq U_X$. We just proved that $X \leq Y$ and $Y \leq X$. By coherence of $\mathcal{T}$, $U(X) \leq U(Y)$ and $U(Y) \leq U(X)$, so $U(X) = U(Y)$ and W_Z is well defined.

Next, we prove the three bullets in the theorem. Since U is standard-linear, so is $U^*_Z(\omega, \cdot)$ for each ω. Since $U_{\alpha X + \beta Y}(\omega) = \alpha U_X(\omega) + \beta U_Y(\omega)$ for all $X, Y \in \mathcal{X}$, $\omega \in \Omega$, and standard α, β, we see that $\mathcal{W}_Z$ is a standard-linear space and that W_Z is standard-linear. Since $U_Z(\omega) = 1$ for all ω, we see that $\mathcal{W}_Z$ contains all constants and $W_Z(1) = U(Z)/U(Z) = 1$. Since $U_{XI_B}(\omega) = U_X(\omega)I_B(\omega)$ for all $X \in \mathcal{X}$, $B \in \Sigma$, and $\omega \in \Omega$, we see that $\mathcal{W}_Z$ contains wI_B for all $w \in \mathcal{W}_Z$ and $B \in \Sigma$. Thus, the first bullet and part of the second bullet are proven.

For the rest of the second bullet we must show that (i) $W_Z(I_B) = P(B)$ for all $B \in \Sigma$, and (ii) W_Z is monotone. For (i), note that for each $B \in \Sigma$, $U_{ZI_B} = I_B$, and $W_Z(I_B) = U(ZI_B) = P(B)$ (including $B = \Omega$.) For (ii) $U_X \leq U_Y$ implies $X \leq Y$ by what we proved earlier. Then, $W_Z(U_X) = U(X)/U(Z) \leq U(Y)/U(Z) = W_Z(U_Y)$, where the inequality follows from coherence of $\mathcal{T}$.

For the third bullet, assume that dominance means weak dominance and that $E \in \Sigma$ is non-empty. The construction of Z implies that $0 \prec_{\text{Dom}} ZI_E$, so $0 \prec Z|E$, and E is non-null. Also, $0 = U(0)/U(Z) < U(ZI_E)/U(Z) = P(E)$. □

Corollary 1. *Assume the conditions of Theorem 3 in the random-variable case with numeraire* $Z = 1$. *Then* $\mathcal{W}_Z = \mathcal{X}$, $U_X = X$ *for all* $X \in \mathcal{X}$, *and* U *acts as an expected value on* $\mathcal{X}$ *with respect to* P.

4.5. Changes of Numeraire

Assume the conditions stated in Definition 18. Let Z_1, Z_2 be two numeraires of the sort described there with corresponding probabilities P_{Z_1}, P_{Z_2}. Then

$$\mathcal{W}_{Z_j} = \{U^*_{Z_j}(\cdot, X(\cdot)) : X \in \mathcal{X}\},$$

for $j = 1, 2$. And $W_{Z_j} : \mathcal{W}_{Z_j} \to {}^*\mathbb{R}$, defined by

$$W_{Z_j}(U^*_{Z_j}(\cdot, X(\cdot))) = U(X)/U(Z_j),$$

acts as an expected value on $\mathcal{W}_{Z_j}$ with respect to P_{Z_j}. Note that

$$\begin{aligned} U^*_{Z_2}(\cdot, Z_1(\cdot)) &= \frac{1}{U^*_{Z_1}(\cdot, Z_2(\cdot))}, \text{ and} \\ \mathcal{W}_{Z_2} &= \left\{ wU^*_{Z_2}(\cdot, Z_1(\cdot)) : w \in \mathcal{W}_{Z_1} \right\}. \end{aligned}$$

For each $w \in \mathcal{W}_{Z_2}$,

$$W_{Z_2}(w) = W_{Z_1}\left[wU^*_{Z_2}(\cdot, Z_1(\cdot))\right]\frac{U(Z_1)}{U(Z_2)}.$$

In other words, $U^*_{Z_2}(\cdot, Z_1(\cdot))U(Z_1)/U(Z_2)$ has the defining feature of a Radon-Nikodym derivative of W_{Z_2} with respect to W_{Z_1} (when restricted to indicators of events,) without referring to absolute continuity.

Example 7 (Continuation of Example 4). In the notation of Theorem 3, the two numeraires, 1 and G, along with their probabilities P_1 and P_G, correspond to linear spaces $\mathcal{W}_1 = \mathcal{X}$ and $\mathcal{W}_G = \{X/G : X \in \mathcal{X}\}$ with the standard-linear functions $W_1(X) = U(X)$ (for $X \in \mathcal{W}_1$) and $W_G(Y) = U(YG)/U(G)$ (for $Y \in \mathcal{W}_G$.) So, $U^*_1(\omega, x) = x$ while $U^*_G(\omega, x) = x/\omega$. It then appears as if $dP_G/dP_1(\omega) = 1/[\omega U(G)]$ acts as a Radon-Nikodym derivative despite the fact that no existing definition of *absolutely continuous* has P_G absolutely continuous with respect to P_1. Nevertheless, we can still express $P_G(B) = P_1(BG)/P_1(G)$ for each event $B \in \Sigma$, including $B = E = \{11, 12, \ldots\}$. ◇

4.6. Another Layer of Expected Utility

As in Section 3.2.3, in the horse-lottery case, it would be easier on the intuition if state-dependent utility could be expressed as a function of lotteries in state ω rather than elements of $\mathcal{O}_\omega$. Let $\mathcal{W}$ and U^*_Z be as in Theorem 3. Let $h_0 \in \mathcal{H}$ be arbitrary, and define $V^*_Z : \bigcup_{\omega\in\Omega}\left(\{\omega\} \times \mathcal{R}_\omega\right) \to {}^*\mathbb{R}$ by

$$V^*_Z(\omega, r) = U^*_Z(\omega, r - h_0(\omega)). \tag{11}$$

This has the effect of shifting the utility in state ω so that $h_0(\omega)$ has value 0. It follows that, for $r_1, r_2 \in \mathcal{R}_\omega$,

$$U^*_Z(\omega, \alpha[r_1 - r_2]) = \alpha[V^*_Z(\omega, r_1) - V^*_Z(\omega, r_2)], \tag{12}$$

which gives us equivalent ways to express state-dependent utility in both $\mathcal{H}$ and $\mathcal{K}_0$. Equivalent ways to express marginal utilities were given in (6) and (7) in Section 3.2.3. Finally, for each $\omega \in \Omega$ and $p \in \mathcal{P}_\omega$, let r^p stand for the simple lottery that assigns probability 1 to the prize p, and define

$$V^*_{0,Z}(\omega, p) = V^*_Z(\omega, r^p). \tag{13}$$

A useful consequence of the above notation is the following simple corollary of Theorem 3.

Corollary 2. *Assume the conditions of Theorem 3. Then $V(h)$ is an expected value of $V^*_Z(\cdot, h(\cdot))$ with respect to P.*

Our final goal, in this section, is to show that $V^*_Z(\omega, r)$ can be interpreted as an expected value of $V^*_{0,Z}(\omega, \cdot)$ with respect to the simple probability r.

Lemma 8. *For each $\omega \in \Omega$ and $r \in \mathcal{R}_\omega$, $V^*_Z(\omega, r)$ is an expected value of $V^*_{0,Z}(\omega, \cdot)$ with respect to r.*

Proof. Let $\omega \in \Omega$, and let $r \in \mathcal{R}_\omega$ be a simple lottery with $\mathcal{P}(r) = \{p_1, \ldots, p_n\}$ and $r(\{p_j\}) = \alpha_j$ for $j = 1, \ldots, n$. For $j = 1, \ldots, n$, let $x_j = r^{p_j} - h_0(\omega) \in \mathcal{O}_\omega$, where r^{p_j} stands for the simple lottery that assigns prize p_j with probability 1. Then $r - h_0(\omega) = \sum_{j=1}^n \alpha_j x_j$. Hence

$$\begin{aligned} V_Z^*(\omega, r) &= U^*(\omega, r - h_0(\omega)) \\ &= \sum_{j=1}^n \alpha_j U_Z^*(\omega, x_j) \\ &= \sum_{j=1}^n \alpha_j [V_Z^*(\omega, r^{p_j}) - V_Z^*(\omega, h_0(\omega))] \\ &= \sum_{j=1}^n \alpha_j V_{0,Z}^*(\omega, p_j), \end{aligned}$$

where the first equality follows from (11), the second follows from standard-linearity of $U_Z^*(\omega, \cdot)$, the third follows from (12), and the last follows from the definition of $V_{0,Z}^*$ and the fact that $V_Z^*(\omega, h_0(\omega)) = 0$. Let $Z = \mathcal{P}(r)$, $\Gamma = 2^{\mathcal{P}(r)}$ and $\mathcal{W}$ equal to the standard-linear span of $\{V_{0,Z}^*(\omega, \cdot)I_A : A \in \Gamma\} \bigcup \{I_A : A \in \Gamma\}$ in Definition 19. The two bullets in the definition are clearly satisfied. □

Corollary 2 and Lemma 8 combine to say that $V(h)$ is an iterated expected value of $V_{0,Z}^*(\omega, p)$ where, for each $\omega \in \Omega$, the inner expected value is with respect to the probability $h(\omega)$ on $\mathcal{P}(h(\omega))$, and the outer expected value is with respect the the probability P on Ω. To be precise, $Q(\omega) = h(\omega)[V_{0,Z}^*(\omega, \cdot)]$ is the inner expected value, and $V(h) = P[Q(\cdot)]$ is the outer expected value.

4.7. The Original Savage-style Acts

In this section, we assume that a coherent total trading system $\mathcal{T} = (\mathcal{K}_0, \precsim, \prec)$ that satisfies Assumptions 1–7 was generated from a set $\mathcal{F}$ of Savage-style acts by creating the set $\mathcal{H}$ of horse lotteries that are simple mixtures of elements of $\mathcal{F}$. The probability P that results from Lemma 7 is a function from Σ to ${}^*\mathbb{R}$, and as such can be associated with the set $\mathcal{F}$, regardless of how its existence was proved. Similarly, the function V of (6), when restricted to the elements of $\mathcal{F}$, represents the preorder $\precsim'$ on $\mathcal{F}$ that is the restriction of $\precsim$. Also, the state-dependent utility of act $f \in \mathcal{F}$ in state ω is $V_{0,Z}^*(\omega, f(\omega))$ from (13). Hence the expected-utility interpretation of Corollary 2, when restricted to elements of $\mathcal{F}$ gives an interpretation of V as an expected value of the state-dependent utility $V_{0,Z}^*(\omega, f(\omega))$ with respect to P.

4.8. Existence of Coherent Trading Systems in the Horse-Lottery Case

In the random-variable case, all forms of dominance are defined independently of an agent's willingness to trade. Furthermore, a respect for a chosen form of dominance can be enforced while the agent is stating preferences. The same is not true for the horse-lottery case, where dominance is defined based on an existing total trading system. If that trading system does not respect dominance, the agent either has to start over or can try to modify the stated preferences.

There is a way for the agent to enforce respect for a chosen form of dominance, but it places restrictions on the order in which preferences can be stated. To mimic the random-variable case, for each $X, Y \in \mathcal{X}$, the agent needs to be able to determine, prior to saying which trades between X and Y are acceptable, whether $X \prec_{\text{Dom}} Y$ or $Y \prec_{\text{Dom}} X$ or neither. To do this,one needs a partition $\mathcal{B}$ of Ω into non-null events that satisfy the first two bullets of Assumption 7. The agent also needs to be willing to require preferences to satisfy the third bullet.

If the agent wishes to respect weak dominance, then all states will be non-null, and $\mathcal{B}$ can be taken to be $\{\{\omega\} : \omega \in \Omega\}$. The agent would first determine whether $X \prec Y|\{\omega\}$ or $Y \prec X|\{\omega\}$ or neither. Such comparisons reduce to checking, for each $x, y \in \mathcal{O}_\omega$, how $xI_{\{\omega\}}$ and $yI_{\{\omega\}}$ compare. Once all such determinations are made, weak dominance is defined on all of $\mathcal{O}_\Omega$, and all instances of $X \prec_{\text{Dom}} Y$ can be labelled as $X \ll Y$ before the rest of the trading system is determined.

If the agent wishes only to respect strict dominance, the agent needs to determine, for each $B \in \mathcal{B}$ and each $(x, y) \in \mathcal{O}(B)^2$, whether $x \prec y|B$. If so, the agent then declares that $x \prec y|C$ for every non-null subset C of B. As above, strict dominance is now defined on all of $\mathcal{O}_\Omega$, and all instances of $X \prec_{\text{Dom}} Y$ can be labelled as $X \ll Y$ before the rest of the trading system is determined.

4.9. Conditional Trading

Conditional preference (see Definition 15) can be interpreted as a willingness to trade given that some event occurs. Throughout this section, assume the conditions of Definition 18 and Theorem 3. In particular:

- $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ is a coherent total trading system with standard-linear representing function U.
- Σ is a field of events for which Assumption 6 holds.
- A numeraire Z for Σ exists with corresponding probability P as in Lemma 7.
- There exists a state-dependent utility $U_Z : \mathcal{O}_\Omega \to {}^*\mathbb{R}$ such that $U(X) = U(Z)P[U_Z(\cdot, X(\cdot))]$.

The following result is straightforward from standard-linearity of U.

Proposition 6. *Let $E_1, \dots, E_n$ be a finite partition of Ω. If $X \precsim Y|E_j$ for $j = 1, \dots, n$, then $X \precsim Y$. If, in addition, $X \prec Y|E_j$ for at least one j, then $X \prec Y$.*

In the spirit of conditioning on an event, the following result shows how to restrict a trading system on a state space Ω to a smaller state space consisting of an element E of Σ with $P(E) > 0$. The restriction $X|_E$ of a function to a subset E of its domain is defined to be the function

$$X|_E(\omega) = X(\omega) \text{ for all } \omega \in E,$$

which maps E into the codomain of X. The corresponding restriction of a field Σ is $\Sigma_E = \{A \cap E : A \in \Sigma\}$.

Lemma 9. *Let E be a non-null element of Σ, and define*

- *$\mathcal{X}_E = \{X|_E : X \in \mathcal{X}\}$, and*
- *$\precsim_E$ to mean $X|_E \precsim_E Y|_E$ if and only if $X \precsim Y|E$.*

Then, $\mathcal{T}_E = (\mathcal{X}_E, \precsim_E, \prec_E)$ is a total trading system with representing function $U_E(X|_E) = U(XI_E)$, and which satisfies Assumption 6. If $P(E) > 0$, then (a) $Z|_E$ is a numeraire for Σ_E with associated probability $P_E(B) = P(B \cap E)/P(E)$, and (b) there is an expected state-dependent utility representation for U_E with respect to P_E. Finally, if dominance in $\mathcal{T}$ means weak dominance, then $\mathcal{T}_E$ is coherent with weak dominance.

Proof. It is straightforward that $\precsim_E$ is a well-defined total preorder on $\mathcal{X}_E$ as is the fact that Assumptions 1–4 hold in $\mathcal{T}_E$. Since U is standard-linear and the operation of restriction to E commutes with linear combinations, it follows that U_E is also standard-linear. Since $X \precsim Y|E$ means $XI_E \precsim YI_E$, it is clear that U_E represents $\mathcal{T}_E$. For Assumption 6, the appropriate field is Σ_E, and the assumption holds in $\mathcal{T}_E$.

Next, assume that $P(E) > 0$ so that $0 \precsim Z|E$ and $Z|_E$ is a numeraire for Σ_E. Since $U(ZI_E) = P(E)$ and $U(ZI_{B\cap E}) = P(B \cap E)$, the probability associated with $Z|_E$ is P_E as stated. Also, for each $X \in \mathcal{X}$,

$$\begin{aligned} U_E(X|_E) &= U(XI_E) = P[U_Z^*(\cdot, X(\cdot))I_E] \\ &= P_E[U_Z^*(\cdot, X(\cdot))]P(E) \\ &= P_E[U'_{Z,E}(\cdot, X|_E)], \end{aligned}$$

where $U'_{Z,E}(\omega, x) = U_Z^*(\omega, x)P(E)$ for $\omega =\in E$ and $x \in \mathcal{O}_\omega$.

Finally, assume that dominance in $\mathcal{T}$ means weak dominance. If $X|_E \prec_{\text{Dom}} Y|_E$ in $\mathcal{T}_E$, then $X|_E \leq Y|_E$, and there is $\omega \in E$ such that $X|_E(\omega) < Y|_E(\omega)$ so it follows that $XI_E \prec_{\text{Dom}} YI_E$ and $XI_E \prec YI_E$ in $\mathcal{T}$. Hence $X|_E \prec_E Y|_E$, and Assumption 5 holds in $\mathcal{T}_E$. □

As a corollary, we have a version of the law of total probability/expectation for conditional trading systems. Note that, if E is a non-empty null event then $P(E) = 0$ and $U_E(X|_E) = 0$ for all $X \in \mathcal{X}$.

Corollary 3. *Let n be a standard finite integer, and let $E_1, \dots, E_n$ be a partition of Ω into non-empty events. Then, for each $X \in \mathcal{X}$,*

$$U(X) = \sum_{j=1}^{n} U_{E_j}(X|_{E_j})P(E_j).$$

The following example illustrates why coherence of $\mathcal{T}_E$ in Lemma 9 requires $P(E) > 0$ for uniform and strict dominance.

Example 8 (Continuation of Examples 3 and 4). Recall that the trading system $\mathcal{T}$ is coherent using either uniform or strict dominance, and that the set $E = \{11, 12, \dots\}$ is non-null. The two probabilities P_1 and P_G computed in Example 4 differ most notably by the fact that $P_1(E) = 0$ while $P_G(E) > 0$. Suppose that we try to restrict the trading system to the set E as is done in Lemma 9. With $Z = 1$ as numeraire, $P_1(E) = 0$, so Lemma 9 doesn't apply. In particular, $0 \sim 1|E$ which violates all forms of dominance. With $Z = G$ as numeraire, $P_G(E) > 0$, but $U^*_G(\omega, x) = x/\omega$. In order for X to uniformly dominate 0 there must be a standard $\epsilon > 0$ such that $X(\omega)/\omega > \epsilon$ for all ω. A necessary condition for this is $\alpha_X > 0$, so only some unbounded functions uniformly dominate 0, and all $X \in \mathcal{X}$ with $\alpha_X > 0$ satisfy $0 \prec X|E$. So $\mathcal{T}'$ is coherent using uniform dominance with numeraire G. Finally, recall that $\mathcal{T}'$ is not coherent using strict dominance because $0 \sim 1|E$, despite the fact that $\mathcal{T}$ is coherent using strict dominance. ⋄

5. Discussion

The major contributions of this paper are

- a systematic representation of coherent preferences amongst random variables or horse lotteries regardless of how strong is the form of dominance that one wishes to respect,
- the use of nonstandard models of the reals to represent non-Archimedean preferences,
- an extension theorem from one standard-linear space of random variables to a larger space,
- an expected utility interpretation for the nonstandard representation in special cases (including those of existing theories,) and
- a derivation of conditional preferences.

Some, but not all, of the examples of non-Archimedean preferences arise from the use of weak dominance in the definition of coherence. For example, it is impossible to have an Archimedean preference structure that respects weak dominance on the set of random variables defined on an uncountable state space.

Weak dominance is the weakest of the three dominance concepts in Definition 8. Weak dominance is the same as the form of dominance used to define inadmissibility in statistical decision theory. The strongest of the three dominance concepts is the one used in de Finetti's theory, namely uniform dominance. Strict dominance is intermediate to the other two. Since dominance is used to prevent calling a trading system coherent, the stronger the dominance condition, the weaker the sense of coherence, i.e., the easier it is to call a trading system coherent. Since some of our results use the weakest form of dominance, those results use the strongest form of coherence.

There is room for future work. Assumption 7, in the presence of the other assumptions, is sufficient to prove the existence of an expected-utility representation of preference, but it is not necessary. If we define the finitely-additive signed measure $\mu_X(B) = U(XI_B)$ for $B \in \Sigma$, then $U^*_Z(\cdot, X(\cdot))$ behaves like a Radon-Nikodym derivative of μ_X with respect to P in Theorem 3. The missing necessary condition would be equivalent to a Radon-Nikodym theorem for nonstandard-valued finitely-additive signed measures. A standard-valued finitely-additive Radon-Nikodym theorem for bounded measures was proved by Maynard (1979), but it is heavily dependent on standard real numbers. Here is an example of a total trading system with an expected-utility representation that fails Assumption 7.

Example 9. Let $\Omega = (0, 1)$ with $\mathcal{P} = \{a, b, c\}$ and Σ being the field generated by the intervals (the unions of finitely many disjoint intervals, including singletons.) Each $r \in \mathcal{R}$, the set of all lotteries, is a simple probability $(r(\{a\}), r(\{b\}), r(\{c\}))$. Let $V^*(\omega, r) = \omega r(\{b\}) + r(\{c\})$. This corresponds to a having utility 0 in every state, b having utility ω in state ω, and c having utility 1 in every state. Let $\mathcal{H}$ be the set of functions $h : \Omega \to \mathcal{R}$ such that $V^*(\omega, h(\omega))$ is a Borel-measurable function of ω. We will work in the space $\mathcal{H}$ rather than $\mathcal{K}_0$ where possible. Define

$$V(h) = \int_0^1 V^*(\omega, h(\omega))d\omega.$$

Let $h \precsim' g$ mean $V(h) \le V(g)$. Let $Z = h_c - h_a$, where $h_c(\omega) = (0, 0, 1)$ for all ω and $h_a(\omega) = (1, 0, 0)$ for all ω. Then $V^*(\omega, h_a) = 0$ and $V^*(\omega, h_c) = 1$ for all ω. This makes $Z \in \mathcal{K}_0$ a numeraire, and V^* is actually V^*_Z in the notation of

Section 4.6. The probability corresponding to Z is Lebesgue measure on (Ω, Σ). Every non-null event B contains an interval C. Let α be the midpoint of C, and partition $C = C_1 \bigcup C_2$ by splitting at the midpoint, which makes C_1 and C_2 non-null. Notice that $(0, 1, 0) \prec' (1-\alpha, 0, \alpha)|C_1$ and $(1-\alpha, 0, \alpha) \prec' (0, 1, 0)|C_2$. Hence no partition of the kind required by Assumption 7 exists. ⋄

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A. Overview of Nonstandard Models

This appendix is intended to give only examples and an intuitive overview of the concept of nonstandard models of the reals. Those needing a more thorough understanding should read one of the many treatments such as Robinson (1996).

A nonstandard model of the reals is an embedding of the real numbers $\mathbb{R}$ into a superset $^*\mathbb{R}$ that preserves many of the familiar properties of the reals (e.g., being a linearly ordered algebraic field) while introducing others that are convenient for certain analyses (e.g., "infinite" numbers that obey the usual rules of arithmetic.) (For convenience, we take $\mathbb{R}$ to be a nonstandard model of the reals, despite its being standard.)

Definition 20. A linearly-ordered algebraic field that contains the standard reals $\mathbb{R}$ as a subfield is called a *nonstandard model of the reals*.

In this paper, each nonstandard model $\mathcal{F}$ of the reals (other than $\mathbb{R}$) will be non-Archimedean in one of the many equivalent senses, such as the following: There exist $x, y \in \mathcal{F}$ with $x < y$ such that $nx < y$ for every standard integer n. For those with more knowledge of nonstandard models, our analysis is entirely external. The formal meaning of "external" is not important here, but it includes the ability to refer to subsets of the standard reals (the familiar $\mathbb{R}$) as subsets of $^*\mathbb{R}$. The cost of an external analysis includes, among other things, the inability to carry theorems and proofs back and forth between the standard and nonstandard models. The external approach also requires us to distinguish between standard and nonstandard notions of finite, infinite, and countable. The main thing that we gain from the external approach is the non-Archimedean nature of $^*\mathbb{R}$ as opposed to $\mathbb{R}$. One manifestation of a non-Archimedean property is the non-existence of suprema and/or infima for certain bounded external subsets of $^*\mathbb{R}$.

Next, we describe a popular class of nonstandard models of the reals known as *ultraproducts*. They rely on the concept of *ultrafilter*.

A.1. Ultrafilters

Definition 21. Let $\mathcal{Z}$ be a set. A nonempty subset $\mathcal{U}$ of $2^{\mathcal{Z}}$ is called an *ultrafilter* on $\mathcal{Z}$ if it has the following properties:

- $(A \in \mathcal{U}) \wedge (A \subseteq B)$ implies $B \in \mathcal{U}$.
- $(A \in \mathcal{U}) \wedge (B \in \mathcal{U})$ implies $A \cap B \in \mathcal{U}$.
- For each $A \subseteq \mathcal{Z}$ either $A \in \mathcal{U}$ or $A^C \in \mathcal{U}$ but not both.

The simplest example of an ultrafilter is to let $z_0 \in \mathcal{Z}$ and define

$$\mathcal{U} = \{A \subseteq \mathcal{Z} : z_0 \in A\}.$$

Such an ultrafilter is called *principal*. All other ultrafilters are called *non-principal*.

Ultrafilters on $\mathcal{Z}$ are equivalent to $0-1$-valued probabilities on $\mathcal{Z}$. It is straightforward to show that a (possibly finitely-additive) probability P defined on $(\mathcal{Z}, 2^{\mathcal{Z}})$ takes only the values 0 and 1 if and only if $\{A \subseteq \mathcal{Z} : P(A) = 1\}$ is an ultrafilter. Principal ultrafilters correspond to countably-additive probabilities, while non-principal ultrafilters correspond to merely finitely-additive probabilities. The existence of ultrafilters for general sets $\mathcal{Z}$ depends on the axiom of choice. Theorem 7.1 of Comfort and Negrepontis (1974) gives a simple condition that insures that a subset of $2^{\mathcal{Z}}$ can be extended to an ultrafilter. That theorem relies on the following concept.

Definition 22. A nonempty collection $\mathcal{F}$ of subsets of a set $\mathcal{Z}$ has the *finite intersection property* if the intersection of every nonempty finite collection of elements of $\mathcal{F}$ is nonempty.

Theorem 7.1 of Comfort and Negrepontis (1974) then says that if $\mathcal{F}$ has the finite intersection property, then there is an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$.

Example 10. Let $\mathcal{Z} = \mathbb{Z}$, the positive integers. Let C be the collection of all subsets of the form $\{n, n+1, n+2, \ldots\}$. There are many ultrafilters that contain C. All such ultrafilters are non-principal, and they all correspond to merely finitely-additive probabilities. The reason is that, if $C \subseteq \mathcal{U}$ and $n_0 \in \mathbb{Z}$, then $\{n_0+1, n_0+2, \ldots\} \in \mathcal{U}$, so $P(\{n_0\}) = 0$. ⋄

A.2. Ultraproducts

In this section we introduce a popular method of constructing and extending nonstandard models of the reals. The construction and extension of nonstandard models are viturally identical. In what follows, the initial construction uses $\mathcal{F} = \mathbb{R}$, the standard reals. Later extensions (if needed) use $\mathcal{F}$ equal to an existing nonstandard model.

Let $\mathcal{Z}$ be an infinite set, let $\mathcal{U}$ be an ultrafilter on $\mathcal{Z}$, and let $\mathcal{F}$ be a nonstandard model of the reals (possibly $\mathbb{R}$ itself.) The elements of $\mathcal{F}^{\mathcal{Z}}$ are functions from $\mathcal{Z}$ to $\mathcal{F}$. Define a binary relation of $\mathcal{F}^{\mathcal{Z}}$ by

$$f \sim_{\mathcal{U}} g \text{ if and only if } \{z \in \mathcal{Z} : f(z) = g(z)\} \in \mathcal{U}.$$

It is easy to see that $\sim_{\mathcal{U}}$ is an equivalence relation because the intersection of finitely many elements of $\mathcal{U}$ is in $\mathcal{U}$. Call ${}^*\mathcal{F}_{\mathcal{U}} = \mathcal{F}^{\mathcal{Z}} / \sim_{\mathcal{U}}$ (the set of equivalence classes corresponding to $\sim_{\mathcal{U}}$) the *ultraproduct* corresponding to $\mathcal{F}$, $\mathcal{Z}$ and $\mathcal{U}$. The embedding of $\mathcal{F}$ into ${}^*\mathcal{F}_{\mathcal{U}}$ (which we call the *natural embedding*) is $x \mapsto [d_x]_{\mathcal{U}}$, where (for $x \in \mathcal{F}$) $d_x \in \mathcal{F}^{\mathcal{Z}}$ is the constant function $d_x(z) = x$ for all $z \in \mathcal{Z}$. We take the liberty of using the symbol x to stand for $[d_x]_{\mathcal{U}}$ when $x \in \mathcal{F}$. If $\mathcal{U}$ is a principal ultrafilter with $\{z_0\} \in \mathcal{U}$, then ${}^*\mathbb{R}_{\mathcal{U}}$ is essentially $\mathcal{F}$ because $g \in [d_{g(z_0)}]_{\mathcal{U}}$ for every $g \in \mathcal{F}^{\mathcal{Z}}$. The classic ultraproducts are those that start with $\mathcal{F} = \mathbb{R}$, $\mathcal{Z} = \mathbb{Z}$ (the positive integers,) and $\mathcal{U}$ a non-principal ultrafilter on $\mathcal{Z}$.

The following examples of ultraproducts are non-Archimedean extensions of the standard real numbers, and we will use one of them the first time that we need to construct nonstandard numbers.

Example 11. Let $\mathcal{Z} = \mathbb{Z}$, let $\mathcal{U}$ be a non-principal ultrafilter of subsets of $\mathcal{Z}$ that contains $\{n, n+1, \ldots\}$ for every integer n, and let $\mathcal{F} = \mathbb{R}$. Let ${}^*\mathbb{R}_{\mathcal{U}}$ denote the ultraproduct nonstandard model ${}^*\mathcal{F}_{\mathcal{U}}$ in the above construction. Let $f(n) = 1/n$ for all n. It is easy to check that $[f]_{\mathcal{U}} < x$ for every strictly positive standard real x. Just note that each set of the form $\{n : f(n) < x\}$ equals $\{m_x, m_x+1, m_x+2, \ldots\}$ where m_x is the first integer m such that $m > 1/x$. At the other extreme, let $g(n) = n$ for all n. Then $[-g]_{\mathcal{U}} < x < [g]_{\mathcal{U}}$ for every standard x. Finally, let x be an arbitrary finite standard. For every standard $y < x$ and every standard $z > x$, we have $y < [f-x]_{\mathcal{U}} < x < [f+x]_U < z$. In words, ${}^*\mathbb{R}_{\mathcal{U}}$ includes numbers that are squeezed in between x and every standard less than (or greater than) x. In this way ${}^*\mathbb{R}_{\mathcal{U}}$ is non-Archimedean. ⋄

Definition 23. A nonstandard z such that $|z| < y$ for every positive standard y is called *infinitesimal*. A nonstandard z such that $|z| > y$ for every positive standard y is called *externally infinite*. A nonstandard that is not externally infinite is called *externally finite*.

The infinitesimals and standard reals are externally finite, as are hybrid nonstandards such as $1 + x$, where x is infinitesimal.

Lemma 10. *The ultraproducts constructed at the start of this section are nonstandard models of the reals.*

Proof. The extension of $\leq$ to ${}^*\mathcal{F}_{\mathcal{U}}$ is $[f]_{\mathcal{U}} \leq [g]_{\mathcal{U}}$ if $\{z : f(z) \leq g(z)\} \in \mathcal{U}$. This $\leq$ is a linear order on ${}^*\mathcal{F}_{\mathcal{U}}$. The extension of each arithmetic operation $\circ \in \{+, -, \times, /\}$ to ${}^*\mathcal{F}_{\mathcal{U}}$ is $[f]_{\mathcal{U}} \circ [g]_{\mathcal{U}} = [f \circ g]_{\mathcal{U}}$.[6] All of these extensions are well-defined because $\mathcal{U}$ is closed under finite intersections of its elements. The additive identity (zero element) in ${}^*\mathcal{F}_{\mathcal{U}}$ is $[0]_{\mathcal{U}}$ where 0 is the additive identity in $\mathcal{F}$. Similarly, the multiplicative identity (1 element) in ${}^*\mathcal{F}_{\mathcal{U}}$ is $[1]_{\mathcal{U}}$. The additive and multiplicative inverses of an element $[f]_{\mathcal{U}}$ are respectively $[-f]_{\mathcal{U}}$ and $[1/f]_{\mathcal{U}}$, the latter applying only when f is not the zero element of $\mathcal{F}$. ☐

It is straigtforward that each ultraproduct extension of a non-Archimedean nonstandard model of the reals is also non-Archimedean.

Lemma 11. *Each externally finite nonstandard x has a nearest standard $\mathfrak{R}(x)$.*

Proof. If x is itself standard, no other standard is closer to x, so $\mathfrak{R}(x) = x$. For the remainder of the proof, assume that x is not itself standard. Let $L = \{z \in \mathbb{R} : z < x\}$ and $U = \{z \in \mathbb{R} : x < z\}$. Then both L and U are nonempty, $L \cap U = \emptyset$, and $L \cup U = \mathbb{R}$. Also, each element of U is an upper bound for L and each element of L is a lower bound for U. Since both L and U are sets of standards, $\sup L = \inf U$, and we call the common value x', which is standard. We now show that $\mathfrak{R}(x) = x'$ is the nearest standard to x. Let $z < x'$ so that $z < (z + x')/2 < x'$. Since $(z + x')/2$ is closer to x than z, no standard less than x' is closer to x. Similarly, let $y > x'$ so that $(y + x')/2$ is closer to x than y, so no $y < x'$ is closer to x. ☐

We call $\mathfrak{R}(x)$ the *standard part of* x. Note that x and z have the same standard part if and only if $x - z$ is infinitesimal. For convenience, we say that the standard part of an externally infinite x is infinite and express the fact as $\mathfrak{R}(x) = \infty$ or $\mathfrak{R}(x) = -\infty$ as appropriate. Note that Lemma 11 doesn't depend on how the nonstandard model is constructed.

The externally finite nonstandards have upper and lower bounds that are externally infinite, but there is neither a least upper bound nor a greatest lower bound. An example of a bounded external set of nonstandards that has no supremum or infimum is the following.

Example 12. Let x_0 be a standard real, and let $A = \{x : \mathfrak{R}(x) = x_0\}$. It is clear that, for every $x \in A$, $x + z \in A$ for every positive infinitesimal z, so no element of A is an upper bound for A. Similarly, no element of A is a lower bound for A. Hence, every upper bound y for A has standard part $\mathfrak{R}(y) > x_0$. For every such y, $(x_0 + \mathfrak{R}(y))/2$ is an upper bound for A that is smaller than y. Similarly, for every lower bound z for A, $(x_0 + \mathfrak{R}(z))/2$ is a larger lower bound for A. Hence, A has neither a greatest lower bound nor a least upper bound. ◇

In the theorems of Section 2, we assign numerical values to objects in a trading system sequentially. There are two different situations when the number system $\mathcal{F}$ we are using does not have a value that is appropriate for the next object. One situation arises when the next object to be assigned a value is strictly preferred (dispreferred) to every element of a set B and each number in $\mathcal{F}$ is already less (greater) than or equal to the value assigned to an element of B. In this case, we need to expand $\mathcal{F}$ to include values that are larger (smaller) than everything already in $\mathcal{F}$. The second situation arises when the next object needs to be assigned a value strictly between two non-empty sets B_1 and B_2 that already partition $\mathcal{F}$. The following lemma shows how to extend a number system $\mathcal{F}$ in each of those situations.

Lemma 12. *Let $\mathcal{F}$ be a nonstandard model of the reals.*

1. *There exists a nonstandard model ${}^*\mathcal{F}$ of the reals such that (i) $\mathcal{F}$ is naturally embedded in ${}^*\mathcal{F}$ and (ii) there exist $z_-, z_+ \in {}^*\mathcal{F}$ such that $z_- < x < z_+$ for all $x \in \mathcal{F}$.*
2. *Let $B_1 \subseteq \mathcal{F}$ and $B_2 = B_1^C$. Suppose that, for all $b_1 \in B_1$ and all $b_2 \in B_2$, $b_1 < b_2$. There exists a nonstandard model ${}^*\mathcal{F}$ of the reals such that (i) $\mathcal{F}$ is naturally embedded in ${}^*\mathcal{F}$ and (ii) there exists $z \in {}^*\mathcal{F}$ such that $b_1 < z < b_2$ for all $b_1 \in B_1$ and $b_2 \in B_2$.*

Proof. If $\mathcal{F} = \mathbb{R}$, the nonstandard model of Example 11 satisfies both claims 1 and 2. If $\mathcal{F}$ is already nonstandard, start with claim 1. Let $\mathcal{Z} = \mathcal{F}$, and let $\mathcal{C}$ be the collection of all subsets of $\mathcal{Z}$ of the form $\{z \in \mathcal{Z} : z > z_0\}$ for some $z_0 \in \mathcal{Z}$. The collection $\mathcal{C}$ has the finite intersection property because every finite subset of $\mathcal{Z}$ has a maximum element. Let $\mathcal{U}$ be an ultrafilter of subsets of $\mathcal{Z}$ that contains $\mathcal{C}$, so that $\mathcal{U}$ is non-principal. Let ${}^*\mathcal{F}$ be the corresponding ultraproduct nonstandard model. Then $\mathcal{F}$ is naturally embedded in ${}^*\mathcal{F}$. Let $f(z) = z$ define a function in $\mathcal{F}^{\mathcal{Z}}$. The equivalence class

[6] $[f/g]_{\mathcal{U}}$ is defined if and only if $\{z : g(z) = 0\} \notin \mathcal{U}$, which is equivalent to $[g]_{\mathcal{U}} \neq [0]_{\mathcal{U}}$.

$[f]_{\mathcal{U}}$ is greater than every element of $\mathcal{F}$ by construction and $[-f]_{\mathcal{U}}$ is smaller than every element of $\mathcal{F}$. For claim 2, if B_1 has a least upper bound b, then b is also a greatest lower bound for B_2. In this case $x = b + [1/f]_{\mathcal{U}}$ satisfies claim 2 if $b \in B_1$, and $x = b - [1/f]_{\mathcal{U}}$ satisfies claim 2 if $b \in B_2$.

The only case remaining is claim 2 when B_1 has no least upper bound. In this case, let $\mathcal{Z} = B_2$ with C being the collection of sets of the form $\{z \in \mathcal{Z} : z < z_0\}$ for some $z_0 \in \mathcal{Z}$. Then C has the finite intersection property. Let $\mathcal{U}$ be a (non-principal) ultrafilter that contains C, and let $f(z) = z$ for all $z \in \mathcal{Z}$. The equivalence class $[f]_{\mathcal{U}} > b_1$ for all $b_1 \in B_1$ because every $f(z) > b_1$. Also, $[f]_{\mathcal{U}} < b_2$ for all $b_2 \in B_2$ because $\{z : f(z) < b_2\} \in \mathcal{U}$. □

Some of our results rely on the possibility of applying Lemma 12 infinitely many times, in a well-ordered manner. To be specific, let Γ be an ordinal. Let $\mathcal{F}_0 = \mathbb{R}$. For each successor $\gamma \le \Gamma$, let $\mathcal{F}_\gamma$ be the result of applying Lemma 12 to $\mathcal{F}_{\gamma-1}$. For each limit $\gamma \le \Gamma$ (if any), let $\mathcal{F}_{\gamma-} = \bigcup_{\delta<\gamma} \mathcal{F}_\delta$. It is straightforward to show that $\mathcal{F}_{\gamma-}$ is a nonstandard model of the reals when γ is a limit ordinal.

A.3. Some Notes About Infinity

The use of the symbol ∞ to stand for "larger than every standard number" has a long history, and rarely causes trouble when discussing standard reals. Certain conventions allow some arithmetic with ∞. For example,

- for all finite x, $\infty + x = \infty$, and
- for all finite, non-zero x, $x\infty$ equals $\pm\infty$, with the sign matching that of x.

However, there is no place for standard infinity in a nonstandard model of the reals. Externally infinite, but internally finite, nonstandards replace standard infinity, and they require no special conventions to allow internally finitary arithmetic. Whenever we need to represent something in a nonstandard model that is larger than every number in the model, we appeal to Lemma 12 which essentially iterates the ultraproduct construction to produce a larger nonstandard model that contains internally finite numbers to represent what we need.

A.4. Countable Additivity

Nonstandard models of the reals are designed to have all of the finitary properties of the reals along with non-Archimedean structure. One must be careful not to expect everything that one knows about infinite sets of standard reals to apply to nonstandards. We already saw examples of bounded sets with no suprema or infima. (See Example 12.) A related property is that externally countable sums do not behave the same in nonstandard arithmetic as they do in standard arithmetic. See Section 4.3 of Halpern (2010) for more discussion of this point. For example, when the sequence of finite partial sums of a standard positive countable sequence is bounded, one can define the sum of the entire sequence to be the supremum of the partial sums. The same is not always possible for nonstandards.

Example 13. Let $\epsilon > 0$ be infinitesimal, and let $x_n = \epsilon/2^n$ for each standard positive integer n. One might think that that $\sum_{n=1}^{\infty} x_n = \epsilon$. However, we show next that, despite the fact that the sequence of finite partial sums is bounded, there is no least upper bound. Clearly, each finite partial sum $y_m = \sum_{n=1}^{m} x_n < \epsilon$ for every standard positive integer m, so that ϵ is an upper bound on the finite partial sums. Let w be an arbitrary finite upper bound on the finite partial sums. Next, we show that there is a smaller upper bound than w.

Let $z = \epsilon^2$ so that $z < x_n$ for every standard positive integer n. It follows that, for each standard positive integer m,

$$\begin{aligned} y_m + x_{m+1} &< w, \\ y_m + z &< w, \\ y_m &< w - z. \end{aligned}$$

Hence $w - z$ is a smaller upper bound on the sequence of partial sums. ⋄

The remainder of this section is devoted to exploring the extent to which one can make sense of the sum of externally countably many elements of a nonstandard model of the reals. First, we note that the sequence of finite partial sums $\left\{y_m = \sum_{k=1}^{m} x_k\right\}_{m=1}^{\infty}$ of an externally countable sequence $\{x_k\}_{k=1}^{\infty}$ in a nonstandard model of the reals converges to an element x of the same model if and only if $\{|y_m - x|\}_{m=1}^{\infty}$ converges to 0. So, we state our results in terms of externally countable sequences of strictly positive numbers converging to 0. To say that a strictly positive externally countable sequence $\{w_k\}_{k=1}^{\infty}$ converges to 0 means that, for every $\epsilon > 0$ in the model, there is a standard integer N_ϵ such that for all $k > N_\epsilon$, $w_k < \epsilon$. To show that such a sequence fails to converge, we need to find an $\epsilon > 0$ in the model such that $w_k > \epsilon$ for infinitely many k. The first step is the following.

Lemma 13. *Let $^*\mathbb{R}$ be the nonstandard model in Example 11. No externally countable sequence of strictly positive elements of $^*\mathbb{R}$ converges to 0.*

Proof. Let $\{t_k\}_{k=1}^{\infty}$ be an externally countable sequence of strictly positive elements of $^*\mathbb{R}$. Represent each $t_k = [(y_{k,1}, y_{k,2}, \ldots)]_{\mathcal{U}}$, where $\mathcal{U}$ is the appropriate ultrafilter. Since each $t_k > 0$, we can assume without loss of generality that $y_{k,n} > 0$ for all k, n. Construct another strictly positive nonstandard $u = [(v_1, v_2, \ldots)]_{\mathcal{U}}$ as follows. Let $0 < v_1 < y_{1,1}$, and for each $n > 1$, let $0 < v_n < \min\{y_{1,n}, \cdots, y_{n,n}\}$. For each k,

$$\{n : 0 < v_n < y_{k,n}\} \supseteq \{k, k+1, \ldots\} \in \mathcal{U}.$$

Hence, $0 < u < t_k$ for all k. □

The next step is to extend the conclusion of Lemma 13 to further applications of Lemma 12.

Lemma 14. *Let $\mathcal{F}$ be a nonstandard model of the reals such that no externally countable sequence of strictly positive elements of $\mathcal{F}$ converges to 0. Apply Lemma 12 to create $^*\mathcal{F}$. Then no externally countable sequence of strictly positive elements of $^*\mathcal{F}$ converges to 0.*

Proof. Let $\mathcal{Z}$ and $\mathcal{U}$ be the set and ultrafilter used in the extension of $\mathcal{F}$ to $^*\mathcal{F}$. Let $\{[x_k]_{\mathcal{U}}\}_{k=1}^{\infty}$ be an externally countable sequence of strictly positive elements of $^*\mathcal{F}$. For each $z \in \mathcal{Z}$, $\{x_k(z)\}_{k=1}^{\infty}$ is an externally countable sequence of strictly positive elements of $\mathcal{F}$, hence there exists $\epsilon(z) > 0$ in $\mathcal{F}$ such that $x_k(z) > \epsilon(z)$ for all k. Then $[x_k]_{\mathcal{U}} > [\epsilon]_{\mathcal{U}} > 0$ for all k, and $\{[x_k]_{\mathcal{U}}\}_{k=1}^{\infty}$ does not converge to 0 in $^*\mathcal{F}$. □

Recall that convergence of an externally countable sum to an element x of a nonstandard model of the reals requires convergence to 0 of the difference between x and the finite partial sums. Since convergence to 0 of a countable sequence in nonstandard models of the reals is so difficult to ensure, we assume only that probabilities are finitely additive. Furthermore, the non-Archimedean nature of the preferences that we model leads to utilities being nonstandard-valued. Integrating a nonstandard-valued function with respect to countably-additive probability runs into the same problems we just exhibited for nonstandard-valued probabilities. The limits required to define the integral of a non-simple nonstandard-valued function do not exist in general. The next section discusses a finitely-additive definition of expected value that is suitable for use with nonstandard-valued functions and/or probabilities.

B. Expected Values for Nonstandard-Valued Functions

In the theory of countably-additive probability and expected value for standard-valued functions, each probability P on a measurable space $(\mathcal{Z}, \Gamma)$ has a unique extension from indicators of elements of Γ to an expected value for all bounded measurable functions and for all non-negative measurable functions (where ∞ may be the resulting expected value.) Finally, there is a unique extension to all functions whose positive and negative parts don't both have ∞ as their expected valued. The first step is the trivial extension to the simple functions, those that assume only finitely many values. This trivial extension applies equally well in the nonstandard-valued and/or the finitely-additive cases, namely

$$P\left(\sum_{j=1}^{n} a_j I_{A_j}\right) = \sum_{j=1}^{n} a_j P(A_j).$$

The extension of expected value to a bounded measurable function f is done by a sequence of uniform approximations of f by a sequence of pairs of simple functions $\{(f_{n,<}, f_{n,>}\}_{n=1}^{\infty}$ where $f_{n,<}(z) \le f_{n,>}(z)$ and $f_{n,>}(z) - 1/n \le f(z) \le f_{n,<}(z) + 1/n$ for all $z \in \mathcal{Z}$ and all $n \in \mathbb{Z}^+$. The sequences of expected values $\{P(f_{n,>})\}_{n=1}^{\infty}$ and $\{P(f_{n,>})\}_{n=1}^{\infty}$ both converge to the same number, and that number is $P(f)$. This feature applies in the finitely-additive standard-valued case, but not so much to nonstandard-valued cases. In particular, we can get uniform approximations of bounded functions by simple functions to within each positive standard value, which allows us to pin down the standard part of $P(f)$, but we cannot uniquely determine the value of $P(f)$ from these simple functions alone. To get a uniform approximation to within an infinitesimal amount requires a "simple" function with an externally infinite nonstandard integer number of terms. The sum of countably many nonstandard values is generally not possible to define, and countably-additive probabilities are not generally additive over an externally infinite nonstandard number of values.

The standard-valued countably-additive extension to a non-negative measurable function f is done by

$$P(f) = \sup_{\text{simple } g \le f} P(g), \tag{14}$$

which is still possible in the standard-valued finitely-additive case. In the nonstandard-valued cases, there are many sets of finite numbers, even bounded sets of finite numbers, for which no supremum (least upper bound) exits, as noted in Appendix A. So, one cannot use (14) to define the expected value of a non-negative nonstandard-valued function. Instead, $P(f)$ must be at least as large as $P(g)$ for every simple $g \le f$. This makes expected values of unbounded standard-valued functions and general nonstandard-valued functions non-unique extensions of the underlying probabilities, be they countably-additive or merely finitely-additive. Example 8 in Section 4.9 is a case of a countably-additive probability with a finitely-additive extension to unbounded functions.

The implications of the non-uniqueness of extensions are handled as follows. In the finitely-additive standard-valued case, it is coherent, in the sense of de Finetti (1974) and for a single unbounded non-negative function f, to assign a value to $P(f)$ that equals the right-hand side of (14) plus c for $c > 0$. However, doing so has implications for the expected values of other unbounded non-negative functions. Our extension Theorems 1 and 2 are set up to take into account all of those implications if and when they arise. The reader should also note that, (14) often forces $P(f) = \infty$ because there can be simple functions $g \le f$ with arbitrarily large $P(g)$. In our nonstandard approach, we would assign a nonstandard externally infinite expected value as $P(f)$. In fact, the very idea of what counts as "bounded" or "unbounded" changes when nonstandard externally infinite numbers are being used.

For the above reasons, we use Definition 19 to define expected values of nonstandard-valued functions with respect to a finitely-additive (or even a countably-additive) probability P. Essentially, an expected-value functional W with respect to P is a standard-linear mapping $W : \mathcal{W} \to {}^*\mathbb{R}$, where $\mathcal{W}$ is a standard-linear space of (possibly nonstandard-valued) functions that includes the indicators of the sets on which P is defined as well as other functions for which one desires expected values. The functional W needs to have two additional properties: (i) $g \le f$ implies $W(g) \le W(f)$ (monotonicity) and (ii) $W(1) = 1$ (normalized.) Each finitely-additive probability P has multiple extensions to each $\mathcal{W}$ that includes non-simple (even unbounded) functions. Each extension involves the space of functions whose expected values need to be computed as well as the specific expected values assigned to the functions. For each finitely-additive probability P and each standard-linear space $\mathcal{W}$ of functions whose expected values we want, there is a convex set $\mathcal{E}_{P,\mathcal{W}}$ of possible extensions of P to $\mathcal{W}$.

Our assumptions refer to an agent's willingness to engage in various trades amongst elements of a set $\mathcal{X}$. If the agent's willingness to trade satisfies our assumptions, then there is a (possibly nonstandard-valued) function U on $\mathcal{X}$ that represents the trades that the agent is willing to make. We then show that $U(X)$ can be interpreted as an expected value of a state-dependent utility of the value of X in state ω with respect to a probability over (Ω, Σ). Unless we impose more restrictions on which trades an agent should be willing to make, i.e., make more restrictive assumptions, the inferred expected-value functionals could be arbitrary elements of $\mathcal{E}_{P,\mathcal{W}}$.

C. Lengthy Proofs

This appendix contains the lengthier proofs of the results in the main paper.

C.1. Proof of Lemma 2

If $\mathcal{X}' = \mathcal{X}$, then $\mathcal{T}' = \mathcal{T}$ satisfies the conclusions of the lemma. For the remainder of the proof, assume that $\mathcal{X}$ is a proper subset of $\mathcal{X}'$.

We start with the case in which it is not required that $\mathcal{T}'$ be coherent. Define $\precsim'$ on $\mathcal{X}'$ as follows: For $X, Y \in \mathcal{X}'$, $X \precsim' Y$ if $Y - X \in \mathcal{V}_{\mathcal{T}}$. Then $\precsim'$ is an extension$_2$ of $\precsim$. If $\ll$ on $\mathcal{X}$ is nonempty, define $\ll'$ on $\mathcal{X}'$ as follows: For $X, Y \in \mathcal{X}'$, $X \ll' Y$ if $0 \ll Y - X$. Since $0 \in \mathcal{V}_{\mathcal{T}}$, Assumption 1 holds. For Assumption 2, suppose that $X, Y, X', Y' \in \mathcal{X}'$ with $Y - X = Y' - X'$. Then $X \precsim' Y$ if and only if $Y' - X' = Y - X \in \mathcal{V}_{\mathcal{T}}$ if and only if $X' \precsim' Y'$. For Assumption 3, suppose that $X_j \precsim' Y_j$ for and $\alpha_j > 0$ is standard for $j = 1, 2$. Then

$$\alpha_1 Y_1 + \alpha_2 Y_2 - (\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 (Y_1 - X_1) + \alpha_2 (Y_2 - X_2) \in \mathcal{V}_{\mathcal{T}},$$

hence $\alpha_1 X_1 + \alpha_2 X_2 \precsim' \alpha_1 Y_1 + \alpha_2 Y_2$. For Assumption 4 on $\mathcal{X}'$, only the final two bullets need to be proven. To that end, $(X \ll' Y) \wedge (Y \precsim' Z)$ implies $[0 \ll (Y - X)] \wedge [0 \precsim (Z - Y)]$, and $(X \precsim' Y) \wedge (Y \ll' Z)$ implies $[0 \precsim (Y - X)] \wedge [0 \ll (Z - Y)]$. Each of the last two implies $0 \ll (Z - X)$, hence $X \ll' Z$.

If $\mathcal{T}$ is coherent and it is desired that $\mathcal{T}'$ be coherent, define $\precsim'$ on $\mathcal{X}'$ as follows: For $X, Y \in \mathcal{X}'$, $X \precsim' Y$ if there is $V \in \mathcal{V}_{\mathcal{T}}$ such that $V \le Y - X$. Then $\precsim'$ extends$_2$ $\precsim$ and $\mathcal{V}_{\mathcal{T}'}$ satisfies the final claim of the lemma. Define $\ll'$ on $\mathcal{X}'$ as follows: For $X, Y \in \mathcal{X}'$, $X \ll' Y$ if there is $V \in \mathcal{V}_{\mathcal{T}}$ such that either $V \prec_{\text{Dom}} Y - X$ or $0 \ll V \le Y - X$. This makes $\mathcal{T}'$ satisfy Assumption 5. Since $0 \in \mathcal{V}_{\mathcal{T}}$ and $0 \le X - X$, Assumption 1 holds. For Assumption 2, suppose that $X, Y, X', Y' \in \mathcal{X}'$ with $Y - X = Y' - X'$. Then $X \precsim' Y$ if and only if there is $V \in \mathcal{V}_{\mathcal{T}}$ such that $V \le Y - X = Y' - X'$ if and only if $X' \precsim' Y'$. For Assumption 3, suppose that $X_j \precsim' Y_j$ and $\alpha_j > 0$ is standard for $j = 1, 2$. For $j = 1, 2$, let $V_j \in \mathcal{V}_{\mathcal{T}}$ be such that $V_j \le Y_j - X_j$. Then $V = \alpha_1 V_1 + \alpha_2 V_2 \in \mathcal{V}_{\mathcal{T}}$, and

$$V \le \alpha_1 Y_1 + \alpha_2 Y_2 - (\alpha_1 X_1 + \alpha_2 X_2).$$

hence $\alpha_1 X_1 + \alpha_2 X_2 \precsim' \alpha_1 Y_1 + \alpha_2 Y_2$. For Assumption 4 on $\mathcal{X}'$, the first bullet is immediate from the definition of $\triangleleft'$. For the last two bullets,

$$\begin{array}{lll} (X \ll' Y) \wedge (Y \precsim' Z) & \text{implies} & X \ll' Z, \text{ and} \\ (X \precsim' Y) \wedge (Y \ll' Z) & \text{implies} & X \ll' Z, \end{array}$$

there are several things that could lead to the left-hand clauses:

(i) there is $V_1 \in \mathcal{V}_{\mathcal{T}}$ such that $V_1 \prec_{\text{Dom}} Y - X$,

(ii) there is $V_2 \in \mathcal{V}_{\mathcal{T}}$ such that $0 \ll V_2 \le Y - X$,

(iii) there is $V_3 \in \mathcal{V}_{\mathcal{T}}$ such that $V_3 \le Z - Y$,

(iv) there is $V_4 \in \mathcal{V}_{\mathcal{T}}$ such that $V_4 \le Y - X$,

(v) there is $V_5 \in \mathcal{V}_{\mathcal{T}}$ such that $V_5 \prec_{\text{Dom}} Z - Y$,

(vi) there is $V_6 \in \mathcal{V}_{\mathcal{T}}$ such that $0 \ll V_6 \le Z - Y$.

Similarly, there are two ways to achieve the right-hand clause(s):

(vii) there is $V_7 \in \mathcal{V}_{\mathcal{T}}$ such that $V_7 \prec_{\text{Dom}} Z - X$,

(viii) there is $V_8 \in \mathcal{V}_{\mathcal{T}}$ such that $0 \ll V_8 \le Z - X$.

We need to prove two implications based on the above possibilities:

1 [{(i) or (ii)} and (iii)] implies [(vii) or (viii)], and
2 [(iv) and {(v) or (vi)}] implies [(vii) or (viii)].

For 1, [(i) and (iii)] implies $V_1 + V_3 \prec_{\text{Dom}} Z - Y + Y - X = Z - X$, which implies (vii) with $V_7 = V_1 + V_3$. Alternatively, [(ii) and (iii)] implies $0 \ll V_2 + V_3 \le Y - X + Z - Y = Z - X$, which implies (vii) with $V_8 = V_2 + V_3$. For 2, (iv) and (v) implies $V_4 + V_5 \prec_{\text{Dom}} Y - X + Z - Y = Z - X$, which implies (vii) with $V_7 = V_4 + V_5$. Alternatively (iv) and (vi) implies $0 \ll V_4 + V_6 \le Z - Y + Y - X = Z - X$, which implies (viii) with $V_8 = V_4 + V_6$.

C.2. Lemma 15 and Its Proof

The proofs of Theorems 1 and 2 are transfinite inductions. Lemma 15 is a template for the successor ordinal steps in the transfinite inductions. Lemma 3 (whose proof is in Appendix Appendix C.3) is the remainder of the transfinite induction, including the limit ordinal steps.

The proof of Lemma 15 uses an argument that resembles the proof of de Finetti's fundamental theorem of prevision. The main step is constructing bounds for the possible values of the agreeing function (prevision in de Finetti's case, U in Theorem 1) at a new object Z given previously chosen values of the agreeing function. In de Finetti's theorem, one uses existing previsions of random variables X for which either $X \le Z$ or $Z \le X$. In Theorem 1, we replace prevision by an agreeing function U, and we replace $X \le Z$ by a combination of $X \precsim Z$, $X \ll Z$, and/or $X \prec_{\text{Dom}} Z$. Additional steps are needed to deal with strict preferences of a non-Archimedean nature and with sets of nonstandards that don't have suprema and/or infima.

Lemma 15. *Assume the following structure:*

- *$\mathcal{Y}$ and $\mathcal{W}$ are linear spaces of functions from Ω to $\mathcal{O}$ with $\mathcal{Y}$ a proper subset of $\mathcal{W}$.*
- *$\mathcal{T}_Y = (\mathcal{Y}, \precsim_{\mathcal{Y}}, \prec_{\mathcal{Y}})$ is a total trading system that is represented by the standard-linear function $U : \mathcal{Y} \to {}^*\mathbb{R}$, where ${}^*\mathbb{R}$ is a nonstandard model of the reals.*
- *$\mathcal{T}_{\mathcal{W}} = (\mathcal{W}, \precsim_{\mathcal{W}}, \ll_{\mathcal{W}})$ is the extension$_2$ of $\mathcal{T}_Y$ obtained from Lemma 2.*

Let $Z \in \mathcal{W}$. Let $\mathcal{Z}$ be the standard-linear span of $\mathcal{Y} \bigcup \{Z\}$. Then U can be extended to a standard-linear function $U' : \mathcal{Z} \to {}^*\mathbb{R}'$, where ${}^*\mathbb{R}'$ contains ${}^*\mathbb{R}$ and such that U' represents a total trading system $\mathcal{T}' = (\mathcal{Z}, \precsim', \prec')$ that is an extension$_2$ of $\mathcal{T}_Y$. Also, if $\mathcal{T}_{\mathcal{Y}}$ is coherent, then $\mathcal{T}'$ can be chosen to be coherent.

Proof. If coherence is an issue, note that dominance has the same type (uniform, strict, or weak) in both $\mathcal{Y}$ and $\mathcal{W}$, so we will use the same notation $X \prec_{\text{Dom}} Y$ to mean that Y dominates X regardless of whether X, Y are both in $\mathcal{Y}$, both in $\mathcal{W}$ or one in each. We have

$$\mathcal{Z} \setminus \mathcal{Y} = \{\alpha Z + X : X \in \mathcal{Y}, \alpha \in \mathbb{R} \setminus \{0\}\}. \tag{15}$$

It is straightforward to show that the representation of elements of $\mathcal{Z} \setminus \mathcal{Y}$ in (15) is unique. Define $U'(X) = U(X)$ for $X \in \mathcal{Y}$.

Start with the case in which there is $Y \in \mathcal{Y}$ such that $Y \sim_{\mathcal{W}} Z$. In this case, set $U'(Z) = U(Y)$, ${}^*\mathbb{R}' = {}^*\mathbb{R}$, and

$$U'(\alpha Z + X) = \alpha U'(Z) + U(X), \tag{16}$$

for all other elements of $\mathcal{Z}$. Set $\precsim'$ to be the total preorder on $\mathcal{Z}$ that U' represents. The only thing that remains to show, in this case, is that $\mathcal{T}' = (\mathcal{Z}, \precsim', \prec')$ is coherent if $\mathcal{T}_{\mathcal{W}}$ is coherent. Suppose that $\alpha Z + W \prec_{\text{Dom}} \alpha' Z + Y$, for α, α' standard and $W, Y \in \mathcal{Y}$. If $\alpha = \alpha'$, then $W \prec_{\text{Dom}} Y$ and $W \ll_{\mathcal{Y}} Y$, so $W \prec_{\mathcal{Y}}, W \prec' Y$, and $\alpha Z + W \prec' \alpha Z + Y$. If $\alpha > \alpha'$, then $Z \prec_{\text{Dom}} (Y - W)/(\alpha - \alpha')$ and $Z \prec_{\mathcal{W}} (Y - W)/(\alpha - \alpha')$. Let $X \sim_{\mathcal{W}} Z$. Then $X \prec_{\mathcal{W}} (Y - W)/(\alpha - \alpha')$, $X \prec_{\mathcal{Y}} (Y - W)/(\alpha - \alpha')$, $X \prec' (Y - W)/(\alpha - \alpha')$, and $Z \prec' (Y - W)/(\alpha - \alpha')$. Hence $\alpha Z + W \prec' \alpha Z + Y$. A similar argument works of $\alpha < \alpha'$.

For the remainder of the proof, assume that for all $X \in \mathcal{Y}$, $\neg(X \sim Z)$. We start by choosing a value for $U'(Z)$. After that, we make U' standard-linear by defining it through (16). Then, we show that setting $\ll'$ to $\prec'$ satisfies Assumption 4. Finally, we prove that the trading system $\mathcal{T}'$ that U' represents (recall Lemma 1) is coherent if $\mathcal{T}_{\mathcal{Y}}$ is coherent. Since U' extends U, $\mathcal{T}'$ extends $\mathcal{T}_Y$.

When we attempt to choose a value for $U'(Z)$, we need to attend to instances of $\ll_{\mathcal{W}}$, if any.

$$\begin{aligned}
\mathcal{L}_1 &= \{U(X) : X \in \mathcal{Y}, X \precsim_{\mathcal{W}} Z\}, \\
\mathcal{U}_1 &= \{U(X) : X \in \mathcal{Y}, Z \precsim_{\mathcal{W}} X\}, \\
\mathcal{L}_2 &= \{U(X) : X \in \mathcal{Y}, X \ll_{\mathcal{W}} Z\}, \\
\mathcal{U}_2 &= \{U(X) : X \in \mathcal{Y}, Z \ll_{\mathcal{W}} X\}.
\end{aligned}$$

The definition of $\ll_{\mathcal{W}}$ and the fact that U represents $\mathcal{T}_Y$ guarantee that, for $j = 1, 2$, $\ell < u$ for all $\ell \in \mathcal{L}_j$ and $u \in \mathcal{U}_j$. Also, $\mathcal{U}_2 \subseteq \mathcal{U}_1$ and $\mathcal{L}_2 \subseteq \mathcal{L}_1$. (If $\ll_{\mathcal{W}}$ is empty, then $\mathcal{L}_2 = \mathcal{U}_2 = \emptyset$.)

There are several cases (and subcases) to handle:

(a) Both $\mathcal{L}_1$ and $\mathcal{U}_1$ are nonempty, and

(a)(i) there is $x \in {}^*\mathbb{R}$ such that $\ell \le x \le u$ for all $\ell \in \mathcal{L}_1$ and $u \in \mathcal{U}_1$, and at least one such x satisfies $x \notin \mathcal{L}_2 \bigcup \mathcal{U}_2$, or

(a)(ii) there is no x as described in case (a)(i).

(b) $\mathcal{L}_1$ is empty, $\mathcal{U}_1$ is nonempty, and

(b)(i) there is $x \in {}^*\mathbb{R}$ such that $x \le u$ for all $u \in \mathcal{U}_1$, and at least one such x satisfies $x \notin \mathcal{U}_2$, or

(b)(ii) there is no x as described in case (b)(i).

(c) $\mathcal{U}_1$ is empty, $\mathcal{L}_1$ is nonempty, and

(c)(i) there is $x \in {}^*\mathbb{R}$ such that $\ell \le x$ for all $\ell \in \mathcal{L}_1$, and at least one such x satisfies $x \notin \mathcal{L}_2$, or

(c)(ii) there is no x as described in case (c)(i).

(d) Both $\mathcal{L}_1$ and $\mathcal{U}_1$ are empty.

In cases (a)(i), (b)(i), and (c)(i) set $U'(Z) = x$ and ${}^*\mathbb{R}' = {}^*\mathbb{R}$.
In case (a)(ii), apply claim 2 of Lemma 12 (in Appendix A.2) to find an extension ${}^*\mathbb{R}'$ of ${}^*\mathbb{R}$ and $x \in {}^*\mathbb{R}'$ such that $\ell < x < u$ for all $\ell \in \mathcal{L}_1$ and all $u \in \mathcal{U}_1$, and set $U'(Z) = x$.
In case (b)(ii), apply claim 1 of Lemma 12 to find an extension ${}^*\mathbb{R}'$ of ${}^*\mathbb{R}$ and $x \in {}^*\mathbb{R}'$ such that $x < u$ for all $u \in \mathcal{U}_1$, and set $U'(Z) = x$.
In case (c)(ii), apply claim 1 of Lemma 12 to find an extension ${}^*\mathbb{R}'$ of ${}^*\mathbb{R}$ and $x \in {}^*\mathbb{R}'$ such that $x > \ell$ for all $\ell \in \mathcal{L}_1$, and set $U'(Z) = x$.
In case (d), we have two choices. One choice is to let $U'(Z) \in {}^*\mathbb{R}$ and set ${}^*\mathbb{R}' = {}^*\mathbb{R}$. The other choice is to apply either claim of Lemma 12 to find an extension ${}^*\mathbb{R}'$ of ${}^*\mathbb{R}$ and let $x \in {}^*\mathbb{R}'$.
By construction, we have that U' extends U from $\mathcal{Y}$ to $\mathcal{Z}$.

We define $\mathcal{T}' = (\mathcal{Z}, \precsim', \prec')$ by saying that

$$X \precsim' Y \text{ if and only if } U'(X) \le U'(Y).$$

It follows that $\precsim'$ extends$_2$ $\precsim_{\mathcal{Y}}$ from $\mathcal{Y}$ to $\mathcal{Z}$. If $\ll_{\mathcal{W}}$ is empty, the proof is over.

For the remainder of the proof, assume that $\ll_{\mathcal{W}}$ is not empty. We must show that, if $X, Y \in \mathcal{Z}$ and $X \ll_{\mathcal{W}} Y$, then $X \prec' Y$. This involves comparing $U'(X)$ to $U'(Y)$ for various $X, Y \in \mathcal{Z}$. Represent such X and Y as in (15) by

$$\begin{aligned} X &= q(X)Z + X', \\ Y &= q(Y)Z + Y', \end{aligned}$$

with $X', Y' \in \mathcal{Y}$ and $q(X), q(Y) \in \mathbb{R}$. Then

$$Y - X = [q(Y) - q(X)]Z + Y' - X', \tag{17}$$

$$U'(Y) - U'(X) = [q(Y) - q(X)]x + U(Y' - X'), \tag{18}$$

where $x = U'(Z)$. If $q(X) = q(Y)$, then $Y - X = Y' - X'$, $X' \prec_{\mathcal{Y}} Y'$, and $U'(Y) - U'(X) = U(Y' - X') > 0$, so that $X \prec' Y$. If $q(X) < q(Y)$, then

$$\frac{Y' - X'}{q(X) - q(Y)} \ll_{\mathcal{W}} Z,$$

and $x = U'(Z) > [U(X' - Y')]/[q(X) - q(Y)]$ by construction. It follows from (18) that $U'(X) < U'(Y)$, as needed. If $q(X) > q(Y)$, a similar argument shows that $U'(X) < U'(Y)$, so $\mathcal{T}'$ preserves instances of $X \ll_{\mathcal{W}} Y$.

Finally, assume that $\mathcal{T}_Y$ is coherent. It follows from the previous paragraph, that U' respects dominance. We complete the proof by showing that U' is monotone. Suppose that $X, Y \in \mathcal{Z}$ with $X \le Y$. If $q(X) = q(Y)$, then (17) yields $0 \le Y' - X' = Y - X$ and

$$U'(Y) - U'(X) = U(Y' - X') \ge 0.$$

If $q(X) > q(Y)$, then

$$\frac{X' - Y'}{q(Y) - q(X)} \le Z,$$

and $x = U'(Z) \ge [U(X' - Y')]/[q(Y) - q(X)]$ by construction. It follows from (18) that

$$U'(Y) - U'(X) \ge U(X' - Y') + U(Y' - X') = 0.$$

If $q(X) < q(Y)$, a similar argument shows that $U'(Y) \ge U'(X)$, so U' is monotone. $\square$

C.3. Proof of Lemma 3

The proof proceeds by transfinite induction on $\mathcal{W} \setminus \mathcal{Y}$. Let Λ be an ordinal, and let $\{X_\lambda\}_{0<\lambda<\Lambda}$ be a well-ordering of the elements of $\mathcal{W} \setminus \mathcal{Y}$. Let $\mathcal{X}_0 = \mathcal{Y}$, $\mathcal{T}_0 = \mathcal{T}_\mathcal{Y}$, ${}^*\mathbb{R}_0 = {}^*\mathbb{R}$, and $U_0 = U$. Then the following induction hypothesis holds for $\lambda = 0$:

Induction hypothesis: Let $\lambda < \Lambda$ be an ordinal. There is a total trading system $\mathcal{T}_\lambda = (\mathcal{X}_\lambda, \precsim_\lambda)$ such that

- $\mathcal{X}_\lambda$ contains $\{X_\gamma\}_{\gamma\le\lambda}$,
- $\precsim_\lambda$ is a total preorder and is an extension$_2$ of $\precsim_\mathcal{Y}$ to $\mathcal{X}_\lambda$,
- $\mathcal{T}_\lambda$ is represented by a standard-linear function $U_\lambda : \mathcal{X}_\lambda \to {}^*\mathbb{R}_\lambda$, where ${}^*\mathbb{R}_\lambda$ is a nonstandard model of the reals that contains ${}^*\mathbb{R}_\gamma$ for each $\gamma < \lambda$, and
- $\mathcal{T}_\lambda$ is coherent if $\mathcal{T}_\mathcal{Y}$ is coherent.

Next, we deal with an arbitrary successor ordinal γ. Assume that the induction hypothesis holds for $\lambda = \gamma - 1$. We must prove that the induction hypothesis holds for $\lambda = \gamma$. Apply Lemma 15 with $\mathcal{Y} = \mathcal{X}_{\gamma-1}$, $Z = X_\gamma$, $U = U_{\gamma-1}$, $\precsim_\mathcal{Y} = \precsim_{\gamma-1}$, and ${}^*\mathbb{R} = {}^*\mathbb{R}_{\gamma-1}$. Then $\mathcal{X}_\gamma$ is the $\mathcal{Z}$ in Lemma 15. Let U_γ and ${}^*\mathbb{R}_\gamma$ be respectively the U' and ${}^*\mathbb{R}'$ that result from Lemma 15. Then, the induction hypothesis holds for $\lambda = \gamma$.

Finally, we prove that the induction hypothesis holds for each limit ordinal λ. We start by creating objects to play the roles of $\mathcal{Y}$, U, $\precsim_\mathcal{Y}$, and ${}^*\mathbb{R}$ in the statement of Lemma 15. Define

$$\begin{aligned} {}^*\mathbb{R}_{<\lambda} &= \bigcup_{\gamma<\lambda} {}^*\mathbb{R}_\gamma, \\ \mathcal{X}_{<\lambda} &= \bigcup_{\gamma<\lambda} \mathcal{X}_\gamma. \end{aligned}$$

Clearly, ${}^*\mathbb{R}_{<\lambda}$ is a nonstandard model of the reals that contains ${}^*\mathbb{R}_\gamma$ for all $\gamma < \lambda$. For $X \in \mathcal{X}_{<\lambda}$, let $U_{<\lambda}(X) = U_\gamma(X)$, where γ is the first ordinal such that $X \in \mathcal{X}_\gamma$. Each such γ is strictly less than λ. This makes $U_{<\lambda} : \mathcal{X}_{<\lambda} \to {}^*\mathbb{R}_{<\lambda}$. Define $\precsim_{<\lambda}$ on $\mathcal{X}_{<\lambda}$ by $X \precsim_{<\lambda} Y$ if $X \precsim_\gamma Y$ for γ being the first ordinal such that both $X, Y \in \mathcal{X}_\gamma$. Then $\gamma < \lambda$ and $U_{<\lambda}$ represents $\precsim_{<\gamma}$ on $\mathcal{X}_{<\lambda}$. To see that $U_{<\lambda}$ is standard-linear, let $X^1, X^2 \in \mathcal{X}_{<\lambda}$. Let γ be the first ordinal for which both X^1 and X^2 are in $\mathcal{X}_\gamma$. Then $\gamma < \lambda$ and $U_{<\lambda}(X^j) = U_\gamma(X^j)$ for $j = 1, 2$. Since U_γ is standard-linear,

$$\begin{aligned} U_{<\lambda}(\alpha X^1 + \beta X^2) &= U_\gamma(\alpha X^1 + \beta X^2) \\ &= \alpha U_\gamma(X^1) + \beta U_\gamma(X^2) \\ &= \alpha U_{<\lambda}(X^1) + \beta U_{<\lambda}(X^2), \end{aligned}$$

so $U_{<\lambda}$ is standard-linear.

To complete the proof, apply Lemma 15 with $\mathcal{Y} = \mathcal{X}_{<\lambda}$, ${}^*\mathbb{R} = {}^*\mathbb{R}_{<\lambda}$, $Z = X_\lambda$, and $U = U_{<\lambda}$.

C.4. Proof of Lemma 4

Note that each function $f \in \mathcal{F}$ is a special case of a horse-lottery h for which each lottery $h(\omega)$ puts probability 1 on a single prize (consequence) $f(\omega)$. In this way, we can think of $\mathcal{F}$ as a subset of a set of horse lotteries. Savage (1954) proves that there is a probability P on Ω and a utility $U : \mathcal{P} \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$, $f \precsim' g$ if and only if $P[U(f(\cdot))] \le P[U(g(\cdot))]$. Let $\mathcal{H}$ be the set of finite mixtures of elements of $\mathcal{F}$, and let $\mathcal{P}'$ be the set of finite mixtures of elements of $\mathcal{P}$. Define U' on $\mathcal{P}'$ by $U'\left(\sum_{j=1}^n \alpha_j p_j\right) = \sum_{j=1}^n \alpha_j U(p_j)$.

Next, we show that U' is well defined. Suppose that

$$\sum_{j=1}^n \alpha_j p_j = \sum_{k=1}^m \alpha'_k p'_k, \tag{19}$$

with all α_j and all α'_k strictly positive. A necessary condition for (19) is that the set of distinct p_j be the same as the set of distinct p'_k. Another necessary condition is that, if $p_j = p'_k$, the sums of the α_j and/or α'_k corresponding to repeated values of p_j and/or p'_k must be equal. This makes U' well defined. Hence, $U'[h(\omega)]$ is well defined for every ω and every $h \in \mathcal{H}$. It follows that $V(h) = P(U'[h(\cdot)])$ is well defined, and can be used to represent a preorder $\precsim^*$ on $\mathcal{H}$ by "$h \precsim^* g$ if and only if $V(h) \leq V(g)$." It is straightforward that $V(\alpha h + [1-\alpha]g) = \alpha V(h) + (1-\alpha)V(g)$ for all $\alpha \in [0,1]$ and all $h, g \in \mathcal{H}$. There is a corresponding $\mathcal{K}_0 = \{\alpha(h-g) : h, g \in \mathcal{H}\}$, and $U^\dagger(\alpha[h-g]) = \alpha[V(h) - V(g)]$ is also well defined on $\mathcal{K}_0$.

Next, we show that $U^\dagger$ is standard-linear. Let $k_j = \alpha_j(h_j - g_j)$ for $j = 1, 2$. Then

$$\begin{aligned} \beta_1 k_1 + \beta_2 k_2 &= \beta_1\alpha_1(h_1 - g_1) + \beta_2\alpha_2(h_2 - g_2) \\ &= \beta_1\alpha_1 h_1 + \beta_2\alpha_2 h_2 - \beta_1\alpha_1 g_1 - \beta_2\alpha_2 g_2 \\ &= \gamma(J_1 - J_2), \end{aligned}$$

where γ, J_1, J_2 depend on the signs of $\alpha_1, \beta_1, \alpha_2, \beta_2$. For example, if both $\alpha_j > 0$ and both $\beta_j < 0$,

$$\begin{aligned} \gamma &= \beta_1\alpha_1 + \beta_2\alpha_2, \\ J_1 &= \frac{\beta_1\alpha_1}{\gamma} h_1 + \frac{\beta_2\alpha_2}{\gamma} h_2, \\ J_2 &= \frac{\beta_1\alpha_1}{\gamma} g_1 + \frac{\beta_2\alpha_2}{\gamma} g_2. \end{aligned}$$

Each of $J_1, J_2 \in \mathcal{K}_0$, so

$$\begin{aligned} U^\dagger(\beta_1 k_1 + \beta_2 k_2) &= \gamma[V(J_1) - V(J_2)] \\ &= \beta_1\alpha_1 V(h_1) + \beta_2\alpha_2 V(h_2) - \beta_1\alpha_1 V(g_1) - \beta_2\alpha_2 V(g_2) \\ &= \beta_1 U^\dagger(k_1) + \beta_2 U^\dagger(k_2). \end{aligned}$$

There are fifteen other combinations of signs that produce various formulae for γ, J_1, J_2, but all of them lead to the same conclusion. Lemma 1 says that $U^\dagger$ represents a total trading system $\mathcal{T}$ that satisfies Assumptions 1–4.

Finally, we show that $\mathcal{T} = (\mathcal{K}_0, \precsim, \prec)$ as described in Section 3.2.2. According to the discussion in Section 3.2.2, $\precsim$ corresponds to $\precsim^*$ as follows. Let $k_1, k_2 \in \mathcal{K}_0$ be expressed as $k_j = \alpha_j(h_j - g_j)$ with $\alpha_j > 0$ and $h_j, g_j \in \mathcal{H}$ for $j = 1, 2$. First, we need to express $k_2 - k_1 = \gamma(s_2 - s_1)$ with $\gamma > 0$ and $s_1, s_2 \in \mathcal{H}$. Then, we need to show that $k_1 \precsim k_2$ if and only if $s_1 \precsim^* s_2$. First, note that

$$\begin{aligned} k_2 - k_1 &= \alpha_1 h_1 + \alpha_2 g_2 - [\alpha_2 g_1 + \alpha_1 h_2] \\ &= \gamma(s_2 - s_1), \end{aligned}$$

where $\gamma = \alpha_1 + \alpha_2$, and

$$\begin{aligned} s_1 &= \beta h_1 + (1-\beta) g_2, \\ s_2 &= \beta h_2 + (1-\beta) g_1, \end{aligned}$$

where $\beta = \alpha_1/\gamma$. Next, note that $k_1 \precsim k_2$ if and only if $U^\dagger(k_1) \leq U^\dagger(k_2)$, which is true if and only if

$$\alpha_1[V(h_1) - V(g_1)] \leq \alpha_2[V(h_2) - V(g_2)],$$

which is true if and only if

$$V(\beta h_1 + [1-\beta] g_2) \leq V([1-\beta] h_2 + \beta g_1),$$

which is true if and only if $s_1 \precsim^* s_2$.

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