Some strategic aspects of forecasting with strictly proper scoring rules

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Based on joint work with M.J.Scherivsh and J.B.Kadane:

Exchange Rates, Phil. Sci. (2013) 80: 504-532

Outline

1. **Some familiar, strategic aspects of coherent$_1$ strategies in (de Finetti's) *Prevision Game*.**
2. **Avoiding such strategic aspects with coherent$_2$ strategies in (de Finetti's) *Forecasting Game*.**
3. **The role of the *numeraire* and under-determination of the canonical SEU representation.**
4. **Strategic aspects of coherent$_2$ strategies in the *Forecasting Game*, which are absent in the *Prevision Game*.** **See §6 of *Exchange Rates*.**
5. **Concluding thoughts: Strategic aspects in units of "accuracy"?**

Call an agent's choices *coherent* when they respect *simple dominance* relative to a (finite) partition.

$\Omega = \{\omega_1, \ldots, \omega_n\}$ is a finite partition of the sure event: a set of *states*.

Consider two acts A_1, A_2 defined by their outcomes relative to Ω.

	ω_1	ω_2	ω_3	…	ω_n
A_1	o_{11}	o_{12}	o_{13}	…	o_{1n}
A_2	o_{21}	o_{22}	o_{23}	…	o_{2n}

Suppose the agent can compare the desirability of different outcomes at least within each state. Suppose that in each state ω_j, outcome o_{2j} is (strictly) preferred to outcome o_{1j}, $j = 1, \ldots, n$.

Then A_2 *simply dominates* A_1 with respect to Ω.

- ***Coherence*: When A_2 simply dominates A_1 in some finite partition, then A_1 is inadmissible in any choice problem where A_2 is feasible.**

Background on de Finetti's two senses of *coherence*

De Finetti (1937, 1974) developed two games and two senses of *coherence* (*coherence*$_1$ and *coherence*$_2$), which he extended also to infinite partitions.

The games focus on assessing random variables:

Let $\Omega = \{\omega_1, \ldots, \omega_n, \ldots\}$ be a countable partition of the sure event: a finite or denumerably infinite set of *states*.

Let $\chi = \{X_i: \Omega \rightarrow \Re;\ i = 1, \ldots\}$ be a countable class of (bounded) real-valued random variables defined on Ω.

That is, $X_i(\omega_j) = r_{ij}$ and for each $X \in \chi$, $-\infty < inf_\Omega X(\omega) \leq sup_\Omega X(\omega) < \infty$.

Part 1: The Prevision Game.

In game 1, the *Prevision Game*, the random variables are commodities, identified with their associated numerical outcomes.

	ω_1	ω_2	ω_3	…	ω_n	…
X_1	r_{11}	r_{12}	r_{13}	…	r_{1n}	…
X_2	r_{21}	r_{22}	r_{23}	…	r_{2n}	…
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_i	r_{i1}	r_{i2}	r_{i3}	…	r_{in}	…
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

***Coherence$_1$*: de Finetti's (1937) the *Prevision Game* – pricing variables.**

In order to highlight issues of *strategic pricing*, game #1 is formulated as a 2-person, 0-sum, sequential game.

The players in the *Prevision Game*:

- **The *Bookie* (or *Merchant*) – for each random variable *X* in χ ,**

  the *Bookie* plays first and announces a *prevision* (a *fair price*), *P*(*X*), for buying/selling *X*.

- **The *Gambler* – (or *Customer*) plays second and makes finitely many (non-trivial) contracts with the *Bookie* at the *Bookie*'s announced prices.**

The *Bookie* first announces the price *P(X)* for buying/selling X.

The *Gambler* then fixes the term α_X that determines the direction of the sale and the quantity of *X* traded.

In state ω, the contract has an *outcome* to the *Bookie*
(and the opposite-valued outcome to the *Gambler*) of

$$\alpha_X[X(\omega) - P(X)] = O_\omega(X(\omega), P(X), \alpha_X).$$

When $\alpha_X > 0$, the *Bookie* buys α_X-many *X* from the *Gambler*.

When $\alpha_X < 0$, the *Bookie* sells α_X-many *X* to the *Gambler*.

The *Gambler* may choose finitely many non-zero ($\alpha_X \neq 0$) contracts.

The *Bookie*'s net *outcome* in state ω is the sum of the payoffs from the finitely many non-zero contracts: $\sum_{X \in \chi} O_\omega(X(\omega), P(X), \alpha_X) = O(\omega)$.

***Coherence*$_1$: The *Bookie*'s previsions $\{P(X): X \in \chi\}$ are *coherent*$_1$ provided that there is no strategy for the *Gambler* that results in a sure (uniform) net loss for the *Bookie*.**

$$\neg\, \exists(\{\alpha_{X1}, \ldots, \alpha_{Xk}\}, \varepsilon > 0),\ \forall \omega \in \Omega \quad \sum_{X \in \chi} O_{\omega}(X, P(X), \alpha_X) \leq -\varepsilon.$$

Otherwise, the *Bookie*'s previsions are *incoherent*$_1$.

The net outcome *O* is just another random variable.

The *Bookie*'s *coherent*$_1$ previsions do not allow the *Gambler* contracts where the *Bookie*'s net-payoff is uniformly dominated by *Abstaining*.

	ω_1	ω_2	ω_3	…	ω_n	…
O	$O(\omega_1)$	$O(\omega_2)$	$O(\omega_3)$	…	$O(\omega_n)$	…
Abstain	0	0	0	…	0	…

Theorem **(de Finetti, 1937):**

A set of previsions $\{P(X)\}$ is *coherent*$_1$.

if and only if

There exists a (finitely additive) probability P such that the previsions are the P-Expected values of the corresponding variables

$$E_P[X] = P(X).$$

***Corollary*: When the variables are 0-1 indicator functions for events,**

e.g., $A(\omega) = 1$ if $\omega \in A$ and $A(\omega) = 0$ if $\omega \notin A$,

then de Finetti's theorem asserts:

Coherent prices agree with the values of a (finitely additive)

probability distribution over these same events.

Otherwise, they are incoherent.

***Example* 1:**

Consider pricing the two events {*A*, A^c} – pricing their indicator functions.

***A Bookie*'s two previsions, {*P*(*A*)=.6; *P*(A^c)=.7}, are incoherent$_1$**

The *Bookie* has overpriced the two variables.

A *Book* is achieved against these previsions with the *Gambler*'s strategy

$\alpha_A = \alpha_{A^c} = 1$, requiring the *Bookie* to buy each variable at the announced price.

The net payoff to the *Bookie* is -0.3 regardless which state ω obtains.

<u>Two examples where the Bookie engages in *strategic pricing*</u>.

Common theme: the *Bookie* anticipates the *Gambler*'s fair-prices.

***<u>Example</u>* 2: Regarding event A, the *Bookie* has a *straightforward* fair price (a *credence*) $P_B(A) = p$, but models the *Gambler* as having a higher fair-price, $P_G(A) = q > p$.**

Knowing this, the Bookie offers a strategic "fair-price" $P(A) = (p+q)/2$.

The *Gambler* will find this price attractive and will buy *A* from the *Bookie* (i.e., *Gambler bets on* A: $\alpha_A < 0$) at the elevated price, $(p+q)/2 > p$.

So, the *Bookie* does better by *strategic* pricing – gets paid more and pays out less – compared with *straightforward* pricing.

- ***Strategic pricing* dominates *straightforward pricing*, given the *Bookie*'s model of the *Gambler*.**

Example **3: Betting against an "expert."**

The *Bookie* has to price the indicator *A* for event A, but believes that the *Gambler* already knows which of $\{A, A^c\}$ obtains.

If the *Bookie* announces a prevision $0 < P(A) < 1$, then the *Bookie* anticipates that the *Gambler* will choose α_A so that *Gambler* wins and *Bookie* loses: $\alpha_A < 0$ if A obtains, and $\alpha_A > 0$ if A^c obtains.
Then, though the *Bookie* loses for sure, she/he is not *incoherent*$_1$.

If p_A is the *Bookie*'s "*straightforward*" fair-price (her/his credence) the *Bookie* plays *strategically* and announces:

$$P(A) = 1 \text{ if } p_A > .5$$
$$P(A) = 0 \text{ if } p_A < .5$$
$$\text{either } P(A) = 1 \text{ or } P(A) = 0 \text{ if } p_A = .5.$$

Then *Bookie* assigns a subjective probability, $max\{ p_A , (1-p_A) \} \geq .5$ to breaking-even, rather than losing for sure.

- **Bold play is optimal in an unfavorable game!**

An historical observation: De Finetti – a radical operationalist – was concerned that issues relating to *strategic pricing* undermined his theory of Subjective probability. Because, then strategic "*fair-prices*" offered by the *Bookie* in the *Prevision Game* are not the *Bookie*'s subjective expectations for those same random variables.

- **What the *Bookie* announces depends upon who is the *Gambler*.**

We can appreciate the problem of strategic pricing even without endorsing de Finetti's radical operationalism:

- ***Strategic* play by the *Bookie* in the *Prevision Game* corrupts the elicitation of the *Bookie*'s subjective expectations.**

For instance, in *Example* 3, all one learns from the *Bookie*'s announced price, "$P(A)$ = 1," is that the *Bookie*'s credence, p_A, is at least .5.

Part 2: Starting in about 1960, de Finetti switched to a *Forecasting Game*, in order to mitigate problems for his theory of Subjective Probability, posed by strategic pricing in the *Prevision Game*.

Game #2: de Finetti's (1974) *Forecasting Game* (with Brier Score)
There is only the one player in the *Forecasting Game*, the *Forecaster*.

- **The *Forecaster* – for random variable X in χ announces a real-valued *forecast* $F(X)$, subject to a squared-error loss outcome.**

In state ω, the *Forecaster* is penalized $-[X(\omega) - F(X)]^2 = O_\omega(X, F(X))$.

The *Forecaster*'s net score in state ω from forecasting finitely variables $\{F(X_i): i = 1, \ldots, k\}$ is the sum of the k-many individual losses:

$$\sum_{i=1}^{k} O_\omega(X, F(X_i)) = \sum_{i=1}^{k} -[X_i(\omega) - F(X_i)]^2 = O(\omega).$$

***Coherence*$_2$: The *Forecaster*'s forecasts $\{F(X): X \in \chi\}$ are *coherent*$_2$ provided that there is no finite set of variables, $\{X_1, \ldots, X_k\}$ and set of rival forecasts $\{F'(X_1), \ldots, F'(X_k)\}$ that yields a uniform smaller net loss for the *Forecaster* in each state.**

$$\neg \exists(\{F'(X_1), \ldots, F'(X_k)\}, \varepsilon > 0), \forall \omega \in \Omega$$

$$\sum_{i=1}^{k} -[X_i(\omega) - F(X_i)]^2 \leq \sum_{i=1}^{k} -[X_i(\omega) - F'(X_i)]^2 - \varepsilon.$$

Otherwise, the *Forecaster*'s forecasts are *incoherent*$_2$.

The *Forecaster*'s *coherent*$_2$ previsions do not allow rival forecasts that uniformly dominate in Brier Score (i.e., squared-error).

	ω_1	ω_2	ω_3	…	ω_n	…
O	$O(\omega_1)$	$O(\omega_2)$	$O(\omega_3)$	…	$O(\omega_n)$…	
O'	$O'(\omega_1)$	$O'(\omega_2)$	$O'(\omega_2)$	…	$O'(\omega_n)$	…

Theorem **(de Finetti, 1974):**

A set of previsions {*P*(*X*)} is *coherent*$_1$.

if and only if

The same *forecasts* {*F*(*X*): *F*(*X*) = *P*(*X*)} are coherent$_2$.

if and only if

There exists a (finitely additive) probability P such that these quantities are the P-Expected values of the corresponding variables

$$E_P[X] = F(X) = P(X).$$

Example 1 (continued) – slides 16-18 may be skipped.

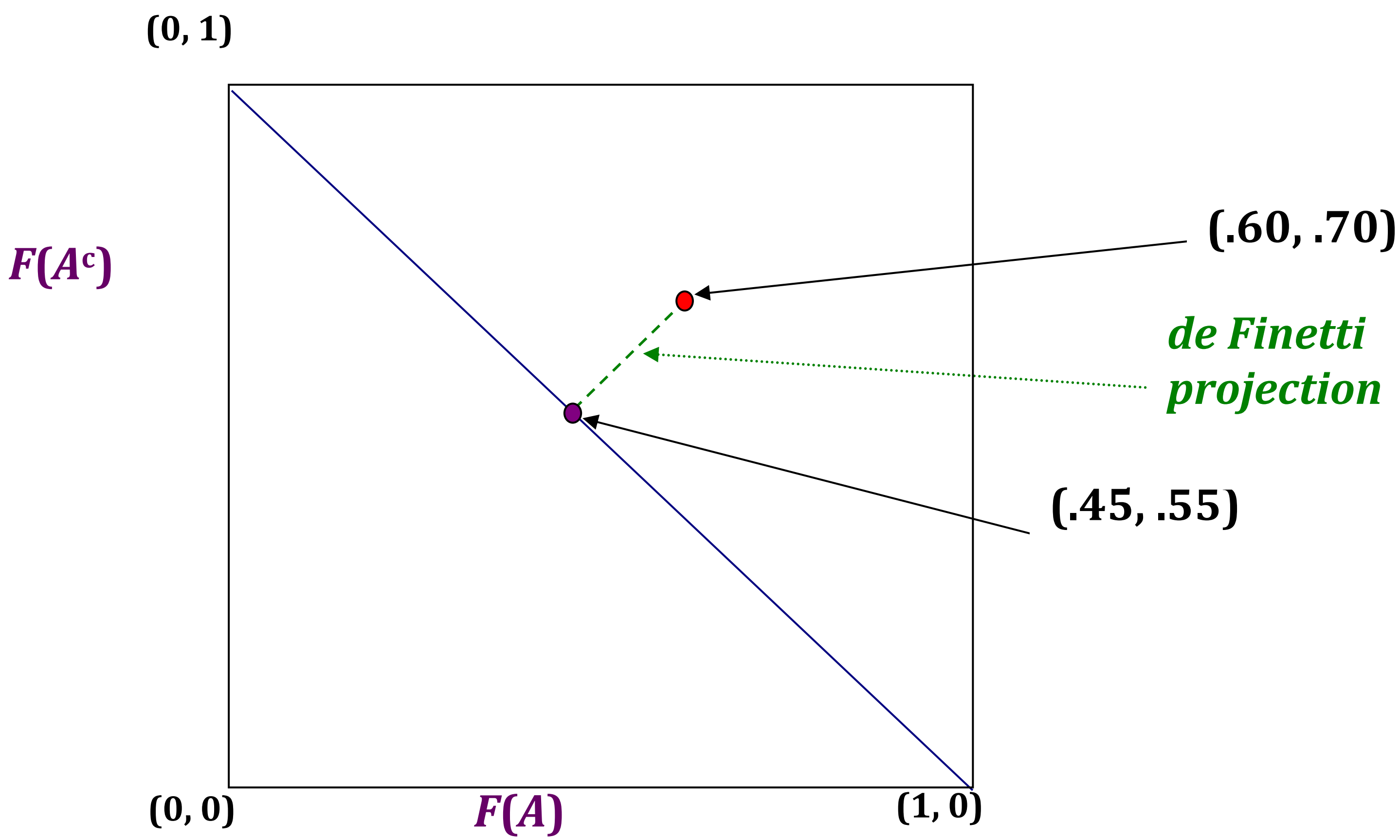
***A Bookie*'s two previsions, {$P(A)=.6$; $P(A^c)=.7$}, are incoherent$_1$**

The *Bookie* has overpriced the two variables.

A *Book* is achieved against these previsions with the *Gambler*'s strategy $\alpha_A = \alpha_{A^c} = 1$, requiring the *Bookie* to buy each variable at the announced price.

The net payoff to the *Bookie* is -0.3 regardless which state ω obtains.

In order to see that these are also *incoherent*$_2$ forecasts, review the following diagram, which follows de Finetti's reasoning (1974, §3.4.1).



(0, 1)
F(A^c)
(.60, .70)
de Finetti
projection
(.45, .55)
(0, 0)
F(A)
(1, 0)

If the forecast previsions are not coherent$_1$, they lie outside the probability simplex. Project these incoherent$_1$ forecasts into the simplex. As in the *Example*, (.60, .70) projects onto the coherent$_1$ previsions depicted by the point (.45, .55). By elementary properties of Euclidean projection, the resulting coherent$_1$ forecasts are closer to each endpoint of the simplex. Thus, the projected forecasts have a dominating (smaller) Brier score regardless which state obtains. This establishes that the initial forecasts are incoherent$_2$. Since no coherent$_1$ forecast set can be so dominated, we have coherence$_1$ of the previsions if and only coherence$_2$ of the corresponding forecasts.

De Finetti's interest in *coherence*$_2$, avoiding dominated forecasts under squared-error loss (Brier-score), was prompted by an observation due to G.W.Brier (1950).

Theorem **(Brier, 1950) A SEU forecaster whose forecasts are scored by the (finite) sum of squared error losses in utility units, uniquely maximizes expected utility by announcing her/his expected value for each variable.**

- **Brier Score is a (*strictly*) *proper scoring rule*.**

Recall: The expected value of the indicator *A* is the probability P(A).

That is, squared error loss provides the incentives for an SEU forecaster to be entirely straightforward with her/his forecasts.

As we saw, wagering (as in the *Prevision Game*) does *not* ensure the right incentives are present for the *Bookie* always to announce her/his expected $E_P(X)$ value as the "fair price" $P(X)$ for variable X.

By contrast, according to Brier's observation,
a strictly proper scoring rule incentivizes straightforward forecasting!

So, de Finetti thought that playing the *Forecasting Game* with a strictly proper scoring rule that fixes losses (e.g. Brier score).

- **preserved the central theme that coherent play requires playing in accord with the theory of Subjective expectations, and**
- **sidestepped the concerns about strategic play in the *Prevision Game*.**

Part 3: The role of the *numeraire* in these games.

Begin with a result about *equivalent* SEU representations.

Suppose an SEU agent's $\succ$ preferences over acts on $\Omega = \{\omega_1, \ldots, \omega_n\}$ is represented by prob/*state-dependent* utility pair $(P;\, U_j\!: j = 1, \ldots, n)$.

	ω_1	ω_2	ω_3	…	ω_n
A_1	o_{11}	o_{12}	o_{13}	…	o_{1n}
A_2	o_{21}	o_{22}	o_{23}	…	o_{2n}

$$A_2 \succ A_1 \quad \textit{if and only if} \quad \textstyle\sum_j P(\omega_j)U_j(o_{2j}) > \sum_j P(\omega_j)U_j(o_{1j}).$$

Let Q be a probability on Ω that agrees with P on null events:

$$P(\omega) = 0 \textit{ if and only if } Q(\omega) = 0.$$

Let V_j be defined as c_jU_j, where $c_j = P(\omega_j)/Q(\omega_j)$.

Then, trivially, we have the following – a variant of *Radon-Nikodem Thrm*.

***Basic Proposition*:**

$$(P;\, U_j) \text{ represents } \succ \quad \textit{if and only if} \quad (Q;\, V_j) \text{ represents } \succ.$$

In words: Coherent preferences <u>underdetermine</u> the separation of credences and values when *state-dependent* utilities are entertained.

***Example* 4: The role of a *numeraire* in pricing random variables.**

Let the state-space have three points $\Omega = \{\omega_1, \omega_2, \omega_3\}$.

Consider two currencies, $ US dollars and € EU euros.

Suppose that (*the agent believes*) the state-dependent exchange rates are:

In state	ω_1	ω_2	ω_3
	$1 ≡ €(2/3)	**$1 ≡ €1**	**$1 ≡ €2**

Let <*x*, *y*, *z*> represent a gamble that rewards

***x* in state ω_1, *y* in state ω_2, and *z* in state ω_3.**

Suppose that the agent is indifferent among these three dollar gambles,

$$< \$1, \$0, \$0 > \sim < \$0, \$1, \$0 > \sim < \$0, \$0, \$1 >.$$

These are the indicator functions for the three states, using dollars as the unit for monetizing the random variables.

In the *Prevision Game*, these indifferences compel the coherent pricing

$$P_{\$}(\omega_1) = P_{\$}(\omega_2) = P_{\$}(\omega_3) = 1/3$$

when random variables are monetized in dollars.

The agent judges those three dollar gambles equivalent, respectively, to these three Euro gambles, which then are indifferent under the agent's preferences:

$$< €(2/3), €0, €0 > \sim < €0, €1, €0 > \sim < €0, €0, €2 >.$$

The indifferences among these three gambles requires the following coherent pricing in the *Prevision Game* when random variables are monetized in Euros,

$$P_{€}(\omega_1) = 1/2 \quad P_{€}(\omega_2) = 1/3 \quad P_{€}(\omega_3) = 1/6.$$

By the *Basic Proposition*, ($P_{\$}$;$U_j$) is SEU equivalent to ($P_{€}$;$V_j$)

where U_j treats dollars as state-independent in value but not euros,
and V_j treats euros as state-independent in value but not dollars.

- **The marginal exchange rate is equal between the two currencies!**

$1 = < $1, $1, $1 > ~ < €1 , €1, € 1 > = €1

This is easy to verify in either of two ways.

1. **Write the constant €1 gamble in dollars as**

€1 ~ <$1.50, $1.00, $0.50>

and note that this random variable in dollars, has a dollar subjective expected value of $1.00 = (1/3)[$1.50 + $1.00 + $0.50].

2. **We get the same exchange rate if the constant $1 gamble is written Euros:**

$1 ~ < €(2/3), €1, €2 >

whose euro subjective expected value is

(1/2)€(2/3) + (1/3)€1 + (1/6)€2 = €1.

If a gamble has a $P_{\$}$-expected value of $*k*, it has a $P_{\text{€}}$-expected value of €*k*.

- **Note well that (straightforward) *fair*-pricing in the *Prevision Game* makes each contract indifferent to the status-quo, regardless which currency is used.**
- **Hence all *fair*-contracts are indifferent to each other, regardless the currency used for pricing.**

That is, in the *Prevision Game*, with straightforward pricing, there is no strategic incentive to use one currency over another!

As we see, next, the same is <u>not</u> true in the *Forecasting Game*.

***Part* 4: Suppose the agent is asked to forecast each of these three states, $F(\omega_i)$ for $\{\omega_i\}$, $i = 1, 2, 3$, subject to Brier score.**

Monetized in dollars, the expected Brier-score loss for each forecast is 2/9.

To see why, recall $1/3 = F_{\$}(\omega_i) = P_{\$}(\omega_i) = 1/3$ for $i = 1, 2, 3$.

So, the expected dollar loss for each forecast is:

$$(2/3)\$(0 - 1/3)^2 + (1/3)\$(1 - 1/3)^2$$

$$= (2/3)\$(1/9) + (1/3)\$(4/9)$$

$$= \$(2/9).$$

Monetized in Euros, the expected Brier score loss for forecast $F_{€}(\omega_i)$ is:

for $F_{€}(\omega_1) = 1/2$

$(1/2)€(0-1/2)^2 + (1/2)€(1-1/2)^2$ = **expected loss €(1/4) > expected loss $(2/9);**

for $F_{€}(\omega_2) = 1/3$

$(2/3)€(0-1/3)^2 + (1/3)€(1-1/3)^2$ = **expected loss €(2/9) = expected loss $(2/9);**

and for $F_{€}(\omega_3) = 1/6$

$(5/6)€(0-1/6)^2 + (1/6)€(1-1/6)^2$ = **expected loss €(5/36) < expected loss $(2/9)**

The agent strictly prefers forecasting ω_1 in dollars rather than in euros;

is indifferent between the two currencies for forecasting ω_2;

and strictly prefers forecasting ω_3 in euros, rather than in dollars.

The strategic forecasts, thus are

$$F_{\$}(\omega_1) = 1/3 \qquad F_{\$}(\omega_2) = 1/3 = F_{€}(\omega_2) \qquad F_{€}(\omega_3) = 1/6.$$

These forecast numbers < 1/3, 1/3, 1/6 > seem *incoherent*$_2$,

But they are *coherent*$_2$, as the first one is monetized in a different currency than the third.

Suppose the agent may choose only one currency to make all 3 forecasts:

The expected sum of the three dollar Brier-score losses is 3×\$(2/9) = \$(2/3).
The expected sum of the three euro Brier-score losses is

$$€(1/4 + 2/9 + 5/36) = €(11/18),$$

which is 1/18 euro less than the (expected) dollar Brier score loss.

Since the (*ex ante*) marginal exchange rate is 1:1 between the two currencies, these inequalities indicate a strict preference in the choice of currencies to be used for making the three forecasts.

The upshot is strategic forecasting:

The agent strictly prefers forecasting the three states with losses in Euros,

forecasts < 1/2, 1/3, 1/6 >

rather than forecasting with losses in Dollars,

forecasts < 1/3, 1/3, 1/3 >

even though the two schemes are (*ex ante*) SEU equivalent representations of preferences over all *equivalent* monetized random variables.

- **This result obtains even if there is some extraneous method of determining which one of the equivalent SEU state-dependent utility representations uses the agent's "straightforward" subjective credence.**

Part 5: *Concluding thoughts*.

We see that forecasting events with a strictly proper scoring rule (e.g., Brier-score loss) opens the door to a strategic choice of currencies for making those forecasts.

A popular theme in contemporary Formal Epistemology is to propose scoring rules as indices of an epistemological goal, *accuracy*.

Assess *forecasts* by their cognitive merits, where the magnitude of the loss is an index of the *inaccuracy* of the forecast.

This approach is offered in contrast with a merely (so-called) "pragmatic" assessment of *gambles*.

***Gambles* are assessed by appeal to the desirability of practical outcomes, which values reflect non-cognitive goals, e.g., wealth.**

But what are the *units* of epistemic accuracy?

Are there counterparts to rival currencies when fixing units of accuracy?

- **The *Basic Proposition* answers that question.**

Suppose we are assessing the accuracy of credences for events in the algebra generated by the partition $\Omega = \{\omega_1, \ldots, \omega_n\}$.

We use Brier-score to assess the inaccuracy of a forecast, as before.

If $F(X)$ is the forecast for the random variable X, then

in state ω, the *Forecaster* is penalized $-[X(\omega) - F(X)]^2$.

Suppose we agree on some state-independent unit U for indexing the cardinal utility of epistemic accuracy:

$$U_j(-[X(\omega_j) - F(X)]^2) = U(-[X(\omega_j) - F(X)]^2) = -[X(\omega_j) - F(X)]^2$$

By de Finetti's theorem, provided the *Forecaster* is coherent$_2$, there exists a finitely additive probability P on Ω such that these forecast quantities are the P-Expected values of the corresponding variables:

$$E_P[X] = F(X).$$

Let Q be a probability on Ω that agrees with P on null events:

$$P(\omega) = 0 \text{ if and only if } Q(\omega) = 0.$$

Let V_j be defined as $c_j U$, where $c_j = P(\omega_j)/Q(\omega_j)$.

<u>Basic Proposition</u>* – a variant of the *Radon-Nikodem Theorem

$$(P; U) \text{ represents } \succ \text{ if and only if } (Q; V_j) \text{ represents } \succ$$

over all decision problems, regardless whether the utilities reflect economic or epistemic outcomes.

- **Define a rival *state-independent* epistemic accuracy in terms of V_j units.**

That is, use the Basic Proposition to define a rival "epistemic currency" that has state-independent V utilities.

$$V_j(-[X(\omega_j) - F(X)]^2) = V(-[X(\omega_j) - F(X)]^2) = -[X(\omega_j) - F(X)]^2$$

and where coherent$_2$ forecasts satisfy: $F(X) = E_Q[X]$.

Then (P; U) and (Q; V) are equivalent representations of the Forecaster's expected accuracies, using rival epistemic "currencies."

Different units of accuracy are matched with different credences over Ω.

***Concluding question*:**

How does the interpretation of a loss function (e.g. Brier-score) as quantifying epistemic goals (e.g., inaccuracy of forecasts) avoid the problem of the strategic choice of units of accuracy?

- ***Recall*: Even if by some method extraneous to the preference relations we could identify agent's *straightforward* credence function, the issue of a strategic choice of units of accuracy remains.**

- **The issue applies, equally, to IP decision theories that generalize SEU.**

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