Three degrees of Imprecise Probability [IP] Theory

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Here, I rely freely on enjoyable collaborations/discussions with:
Mark Schervish, Jay Kadane, Fabio Cozman, Erik Quaeghebeur, and Matthias Troffaes;
Each of them has the good sense and keen judgment to disagree on some particulars in what follows!

Consider, for example, these five canonical "Bayesian" theories:

- **F. Ramsey (1931)**
- **B. de Finetti (1931, 1937, 1974)**
- **L. J. Savage (1954, 1971)**
- **F.J. Anscombe and R.J. Aumann (1963)**
- **M. DeGroot (1970)**

In these theories, *rational preference* is a *binary relation* over *options* represented by inequalities in Subjective Expected Utility [SEU].
In these theories, a decision maker's uncertainty is represented by a single *subjective probability*, which is an ingredient in SEU.

Two related themes cut through all these theories:

- ***Rational preference* satisfies a dominance (*coherence*) principle.**
- ***Degrees of belief* are identified with a precise *subjective probability*, which may be revealed (*elicited*) through *rational preference*, which in turn is revealed through choice behavior.**

<u>QUESTION</u>: ***What becomes of these two themes within IP theories*?**

Outline

1. **(de Finetti's) Coherence criteria and related dominance rules**

    Coherence: Fair-prices (*previsions*) that avoid a sure-loss

2. **Three degrees of *IP* theory relating to (de Finetti's) *coherence* criteria**

    ***Fundamental Theorem*: *imprecise* vs. *indeterminate* previsions**

    Using binary comparisons for *elicitation* with IP-sets.

    Choice functions for solving the limitations of binary comparisons

1. de Finetti's criterion of coherence

- **Choices are *coherent* when they respect a (restricted) *dominance* in outcomes relative to a partition, Ω.**

Fix a partition $\pi = \{\omega_1, \ldots, \omega_n, \ldots\}$, which might be infinite.

Consider a pair of acts where each *Act* can be formulated as a function from π to a set of outcomes $\mathcal{O}$.

Assume that outcomes may be compared by a preference relation, at least within the same state.

	ω_1	ω_2	ω_3	…	ω_n	…
Act_1	$o_{1,1}$	$o_{1,2}$	$o_{1,3}$	…	$o_{1,n}$	…
Act_2	$o_{2,1}$	$o_{2,2}$	$o_{2,3}$	…	$o_{2,n}$	…

Act_2 is strictly preferred to Act_1 – so Act_1 is rejected whenever Act_2 is available

if:

<u>de Finetti's Uniform dominance</u> **(for infinite partitions):**

- **There is a positive amount $\varepsilon > 0$, and for each state ω_i**

  $o_{2,i}$ is strictly preferred to $o_{1,i}$ by at least ε.

Here are two dominance criteria that are not satisfied in de Finetti's theory:

<u>Strict dominance</u> **(Shimony, 1955);** ***<u>Admissibility</u>*** **(Wald, 1950):**

- **For each state, ω_i $o_{2,i}$ is weakly preferred to $o_{1,i}$**

and for some state, ω_j $o_{2,j}$ is strictly preferred to $o_{1,j}$.

<u>Simple dominance</u>**:**

- **For each state ω_i outcome $o_{2,i}$ is strictly preferred to outcome $o_{1,i}$.**

- **Review of de Finetti's coherence for pricing random variables.**

Let χ = {X_i: $\Omega \rightarrow \Re$; i = 1, …} be a class of real-valued (bounded) variables measurable with respect to algebra $\mathcal{B}$ over Ω.

Coherence **(de Finetti):**

an arbitrary partition of states, $\Omega = \{\omega_i : i \in I\}$, and

an arbitrary collection of real random variables, $\chi = \{X_j : j \in J\}$, defined on Ω.

fair prices (*previsions*) for each element of χ.

Dominance is with respect to outcomes formulated over Ω. The comparison is between buying/selling variables at their *fair prices* (*previsions*) and *abstaining* .

For each random variable, $X \in \chi$, the agent has a *<u>two-sided</u> prevision* P(X),

which is to be interpreted as a *fair* price, for *buying* and *selling*.

For all real $\beta > 0$, small enough so that the agent is willing to pay the possible losses, the agent is prepared

to pay $\beta P(X)$ in order *to buy* (i.e., to receive) $\beta X(\omega)$ in return.

and, is willing to accept $\beta P(X)$ in order *to sell* (i.e., to pay) $\beta X(\omega)$ in return.

In symbols, *ex ante*, the agent will accept the gamble that, in state ω, pays

$$\beta[X(\omega) - P(X)]$$

as a change in fortune, for all sufficiently small (positive or negative) β.

The agent is required to accept all finite sums of gambles.

That is, for all finite *n* and all small $\beta_1, \ldots, \beta_n$ and all $X_1, \ldots, X_n \in \chi$,

The agent will accept the linear combination of gambles

$$\sum_{i=1}^{n} \beta_i[X_i(\omega) - P(X_i)].$$

Where β_i is positive, the agent buys β_i-units of X_i for a price of $\beta_i P(X_i)$.

Where it is negative, the agent sells $|\beta_i|$-units of X_i for a price of $\beta_i P(X_i)$.

The previsions are *incoherent* if there is an acceptable

***finite* combination of gambles with *uniformly negative* net-payoff.**

Incoherent prices: **There is a finite set $\{\beta_i \neq 0\}$ and $\varepsilon > 0$ so that for each $\omega \in \Omega$,**

$$\sum_{i=1}^{n} \beta_i [X_i(\omega) - P(X_i)] < -\varepsilon .$$

Otherwise the agent's previsions are *coherent*.

Coherence:

Respect uniform dominance in Ω with respect to the alternative "status-quo" – *abstaining* from the market.

***Abstaining* is represented by the constant outcome, 0, in each state.**

COHERENCE as *elicitation*

A 2-person, 0-sum sequential *prevision* game

The *Bookie* moves first and sets fair (buy/sell) prices for each X $\in \chi$, *P*(*X*).

The *Gambler* acts on the *Bookie*'s offers.

The *Gambler* – may make *finitely many* (non-trivial) contracts at the

**   *Bookie*'s announced prices.**

For each *X*, *Gambler* fixes a real number, β_X, which determines a contract.

In state ω, a contract has an *outcome* to the *Bookie* (with negative outcome to the *Gambler*) of $\beta_X[X(\omega) - P(X)]$.

The *Bookie*'s net *outcome* in state ω is the sum of the payoffs from finitely many non-zero contracts: $\sum_{X\in\chi} \beta_X[X(\omega) - P(X)]$.

Basic Theorem for Coherence

Theorem **(de Finetti, 1974):**

A set of previsions {*P*(*X*)} is *coherent*

if and only if

There exists a (finitely additive) probability *P* on Ω such that these quantities are the *P*-Expected values of the corresponding variables

$$\mathcal{E}_P[X] = P(X).$$

Contrasts between this *COHERENCE*

and the three different senses of dominance

Coherence fails <u>*strict dominance*</u> (Shimony, 1955)

***Example*: Let $\Omega = \{\omega_1, \omega_2\}$ with $P(\{\omega_1\}) = 1$. The *prevision* $\mathrm{P}(\{\omega_1\}) = 1$ fails to respect strict dominance. The agent judges the contract $1(I_{\omega_1}(\omega) - 1)$ "fair" – indifferent to *Abstaining* – even though:**

	ω_1	ω_2
$\mathrm{P}(\{\omega_1\}) = 1$; $\beta = 1$	0	-1
Abstain	0	0

- **Shimony (1955): Previsions are strictly coherent$_1$ *iff regular*.**

$\mathrm{P}(X) = 0$ if and only if $X(\omega) = 0$ for each $\omega \in \Omega$.

Coherence fails simple dominance (de Finetti, 1974)

***Example* – Let Ω be countably infinite $\Omega = \{\omega_1, \omega_2, \ldots\}$ and let $X(\omega_n) = n$. Use a (strongly finitely additive) probability $P(\{\omega\}) = 0$.**

Then $\mathcal{E}_P(X) = 0$. With prevision $P(X) = 0$ and $\beta = -1$ the "fair" payoffs from $-(X(\omega) - P(X)) = -X(\omega)$, are simply dominated by *abstaining*.

	ω_1	ω_2	ω_3	…	ω_n	…
$P(X) = 0, \beta = -1$	-1	-1/2	-1/3	…	-1/n	…
Abstain	0	0	0	…	0	…

<u>NOTE</u> de Finetti (1974): Coherence satisfies simple dominance if P is σ-additive.

(de Finetti, 1972, p. 91) A set of coherent previsions may be uniformly dominated by *abstaining* if countably many contracts are combined.

Example **(continued): Let Ω be countably infinite $\Omega = \{\omega_1, \omega_2, \ldots\}$.**

Consider the countably many indicator functions for the elements of Ω

$$I_n(\omega) = 1 \text{ if } \omega = \omega_n \text{ and } I_n(\omega) = 0 \text{ if } \omega \neq \omega_n.$$

Use a (strongly finitely additive) probability $P(\{\omega\}) = 0$.

Then $\mathcal{E}_P(I_n) = 0$. With prevision $P(I_n) = 0$ and $\beta = -1$ the "fair" payoffs from combining all (infinitely) many contracts is $\sum_n -(I_n(\omega) - 0) = -1$,

independent of ω, which is simply dominated by *abstaining*.

	ω_1	ω_2	ω_3	…	ω_n	…
$\sum_n P(I_n) = 0, \beta = -1$	-1	-1	-1	…	-1	…
Abstain	0	0	0	…	0	…

2. Imprecise Probabilities [*IP Theory*]

2.1 Weak IP theory and *Coherence*

- **Incomplete elicitation – de Finetti's *Fundamental Theorem of Previsions***

Suppose coherent (2-sided) previsions are given for each variable in a set χ,

Let Y be a real-valued function defined with respect to Ω but not in χ.

Define: $\underline{A}$ **= {X: X(ω) ≤ Y(ω) and X is in the linear span of χ}**

$\overline{A}$ **= {X: X(ω) ≥ Y(ω) and X is in the linear span of χ}**

Let $\underline{P}(Y) = \sup_{X \in \underline{A}} P(X)$ **and** $\overline{P}(Y) = \inf_{X \in \overline{A}} P(X)$

<u>Then</u> the 2-sided prevision, P(Y), may be a number from $\underline{P}(Y)$ to $\overline{P}(Y)$

and the resulting enlarged set of previsions is coherent.

Outside this interval, the enlarged set of previsions is incoherent.

The *Fundamental Theorem* provides an early instance of *IP*-theory where,

in Levi's (1980) terms (relating to I.J. Good's "Black Box" Theory)

the '*I*' in *IP* stands for an *Imprecise* prevision,

rather than an *Indeterminate* prevision.

The interval for a new prevision [$\underline{P}$ (Y) $\overline{P}$ (Y)] from the *Fundamental Theorem* constrains a new, 2-sided prevision for a variable, Y $\notin \chi$, while preserving coherence of the 2-sided previsions already assigned to X $\in \chi$.

- **Some observations relating to Weak IP Theory.**

Coherence for 2-sided previsions does not require the rational person identify precise previsions beyond the linear span of the variables in the set χ.

Specifically, the rational agent is not required by *coherence* to have determinate probabilities defined on an (even finite) algebra of events, let alone on a power-set of events.

It is sufficient to have probabilities defined *as-needed* for the arbitrary set χ, as might arise in a particular decision problem.

- **See, e.g., F. Lad's (1996) book for interesting applications of this result.**

- *Toy Example 1.1* – Ω = {1, 2, 3, 4, 5, 6} the outcome of rolling an ordinary die.
  χ is the set of indicator functions for the following four events

$$\chi = \{ \{1\}, \{3,6\}, \{1,2,3\}, \{1,2,4\} \}$$

Suppose 2-sided previsions for these four events are given, and agree with the judgment that the die is "fair."

$$P(\{1\}) = 1/6;\ \ P(\{3,6\}) = 1/3;\ \ P(\{1,2,3\}) = P(\{1,2,4\}) = 1/2.$$

The set of events for which precise 2-sided previsions follow from precise previsions for these four events is given by the *Fundamental Theorem*.

- That set does ***not*** form an algebra. Only 24 of 64 events have precise previsions.

  For instance, by the Fundamental Theorem,

$$\underline{P}(\{6\}) = 0 < \overline{P}(\{6\}) = 1/3;$$

likewise $\underline{P}(\{4\}) = 0 < \overline{P}(\{4\}) = 1/3;$

however, $P(\{4,6\}) = 1/3.$

- **The smallest algebra generated by these 4 events is the power set of all 64 events**

2.2 – Indeterminate Previsions based on 1-sided previsions.

In order to link de Finetti's *coherence* with IP-theory where, the 'I' stands for *indeterminate* previsions, we shift from 2-sided, to 1-sided previsions.

Then the decision maker is required only to provide a pair of (1-sided) previsions $\{\underline{P}(X_i), \overline{P}(X_i)\}$ for each random variable X_i in χ, corresponding to a largest "buy" price and smallest "sell" price for the corresponding 1-sided previsions, depending upon whether β_i is positive, or negative, respectively.

Generalizations of de Finetti's *Coherence Theorem* for 1-sided previsions have been given by *many* researchers. (C.A.B. Smith, 1961)

- **There are variations, e.g., that use only closed intervals of previsions, and others with mixed boundaries (some open, some closed) for their IP sets of probabilities.**

These generalizations of de Finetti's *coherence* all rely on *binary comparisons* between gambles, identifying those that are *favorable* versus others.

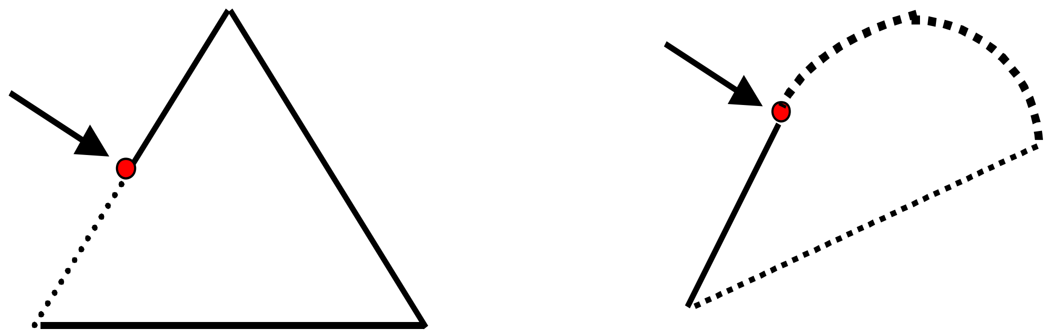
The resulting IP sets of probabilities that may be distinguished from one another are *convex*, with relatively simple boundaries – where *extreme* points of the convex set are also exposed.

- **The common technique uses one or another *Separating Hyperplane* Theorem.**

Here are two *convex sets* with some

***extreme points* – points not convex combinations of others, that are no*t* *exposed* points – can not be separated from the set by hyperplanes.**

Dotted segments are open boundaries.



- **Thus, with this IP approach, based on gambles that are favorable compared with the status-quo, only some convex sets of probabilities can be elicited/distinguished .**

- **Below, I will explain why we want to distinguish among such sets.**

2.3 A third, still stronger account of IP Theory.

Generalizations of de Finetti's *coherence* that use 1-sided previsions, or comparisons between a gamble and the status-quo as a reference point, use *binary* comparisons exclusively to determine admissible options.

- **These approaches cannot distinguish between some sets that differ only on their boundaries.**

  In higher dimensions, the dimension of the boundary may be large too!

- **Also, such approaches cannot distinguish between sets of probabilities that have the same convex hull.**

  Some cannot distinguish between sets of probabilities that have the same closed convex hull.

One way to improve IP-elicitation in accord with the central theme that

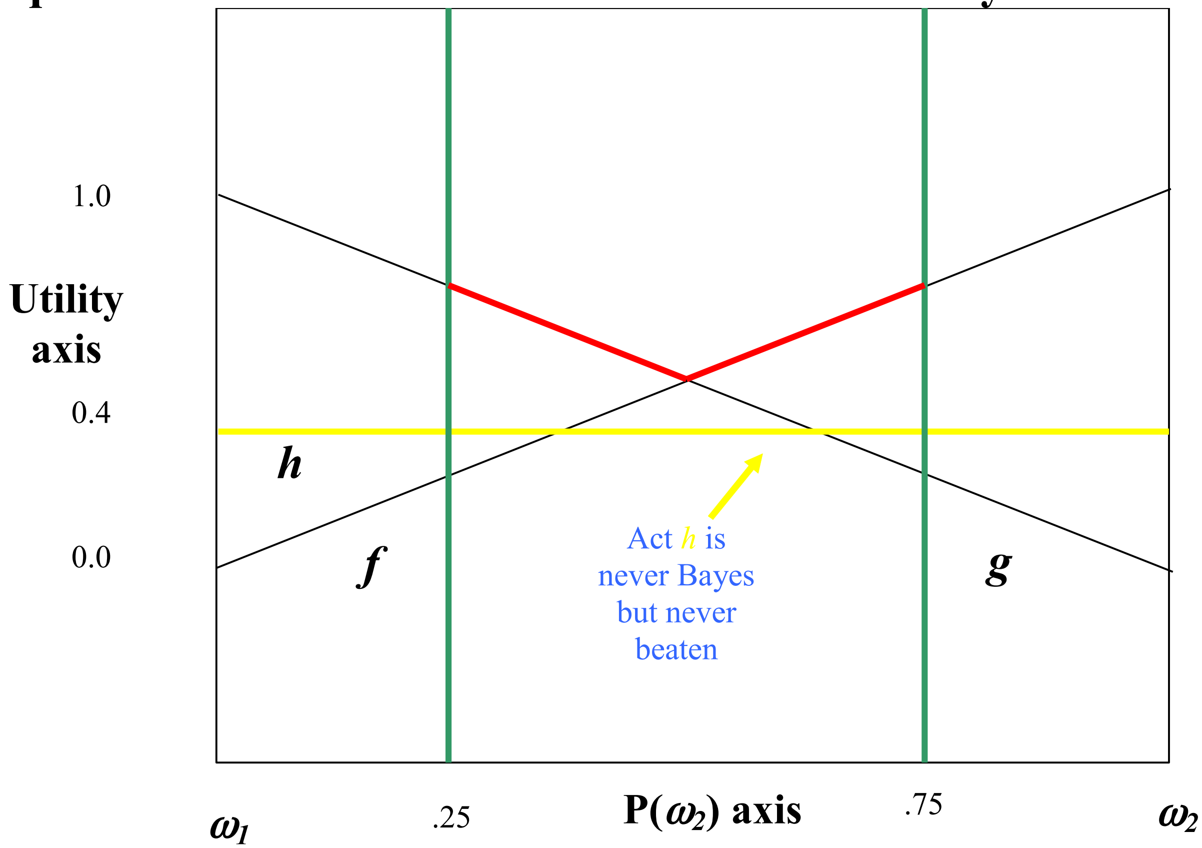
choice behavior reveals the agent's uncertainties

is to use *choice functions* rather than (binary) *preference relations*.

As we'll see, then elicitation is not restricted to convex sets.

- **Then, IP-theory can be richer than is represented by the class of convex sets of probabilities.**

Return to question the relation between IP-decision theory and dominance.



Only {*f*,*g*} are Bayes-admissible from the triple {*f*,*g*,*h*};

however, all pairs are Bayes-admissible in pairwise choices.

I. Levi calls *h* <u>*second worst*</u> in the triple {*f*,*g*,*h*}.

Contrast three *coherent* decision rules for extending Expected Utility [*EU*] theory when probability – but not cardinal utility – is indeterminate.

The decision problems involve (bounded) sets of lotteries, where the outcomes have well-defined cardinal utility but where the (act-independent) states are uncertain, represented by a *convex* set of probabilities $\mathcal{P}$.

- ***Γ-Maximin*** **(Gilboa-Schmeidler, 1989) – maximize min. expectations over $\mathcal{P}$.**
- ***Maximality*** **(Walley 1990) – admissible choices are undominated in expectations over $\mathcal{P}$ by any single alternative choice.**
- ***E-admissibility*** **(Levi/Savage) – admissible choices have Bayes' models, i.e., they maximize *EU* for some probability in the (convex) set $\mathcal{P}$.**

Each rule has *EU* Theory as a special case when probability is determinate, i.e., when $\mathcal{P}$ is comprised by a single probability distribution.

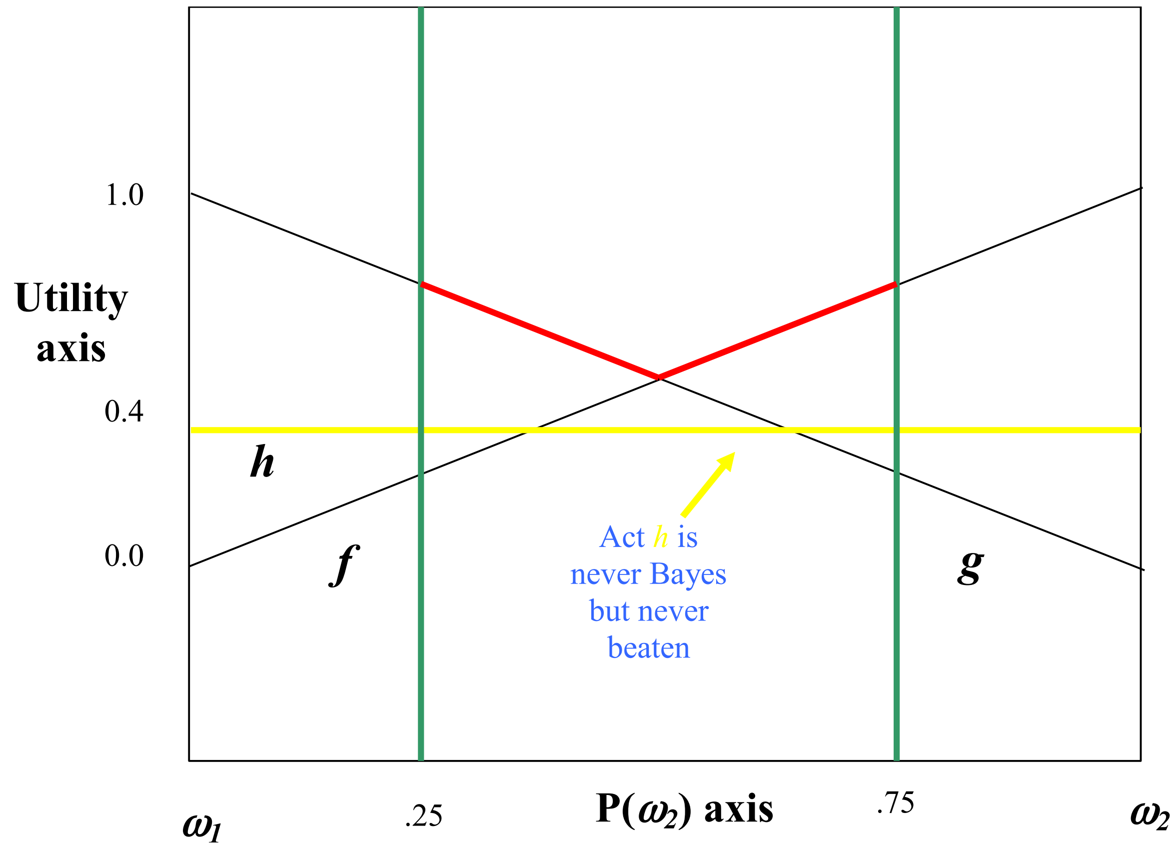
And each rule is *coherent* in the sense that sure loss (*Book*) is not possible.

The three rules are chosen to reflect the following progression, where each rule relaxes more of the ordering assumption than does its predecessors:

- ***Γ-Maximin*** **produces a (real-valued) ordering of options; hence, defined by binary comparisons – but it fails** ***Independence*****.**

- ***Maximality*** **does not generate an ordering of options; however, it is given by binary comparisons.**

- ***E-admissibility*** **does not generate an ordering, nor is it given by binary comparisons.**



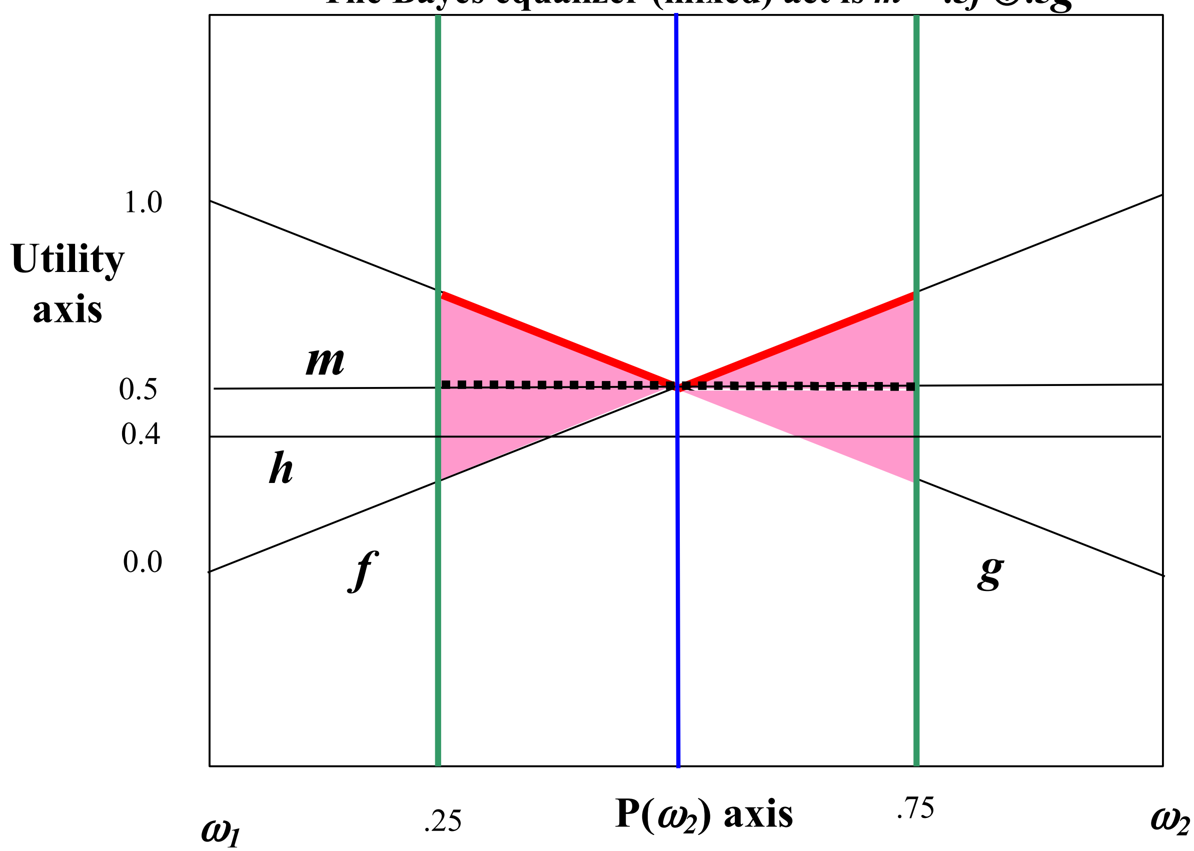
- ***The Γ-Maximin*** **solution is {*h*}.**
- **The *E-admissible* solution set is {*f*, *g*}.**
- **And *Maximality* finds all three options admissible, {*f*, *g*, *h*}.**

Thus, each rule gives a different set of admissible options in this problem.

Create a convex option space by allowing mixed strategies.

Expected Utility for the (Bayes) mixed options $\alpha f \oplus (1-\alpha) g$ is in pink; they maximize EU at $p(\omega_2) = .5$ (blue)

The Bayes equalizer (mixed) act is $m = .5f \oplus .5g$



- ***The Γ-Maximin*** **solution is the EU-*equalizer* {*m*}.**
- **The *E-admissible* and *Maximally admissible* options are the same set of *Bayes* solutions (pink).**

Agreement of the 3 decision rules on Bayes solutions then is no accident:

- ***Walley*** **(Theorem 3.9.5, 1990) establishes that when the option set is** ***convex*** **and the (convex) set of probabilities** $\mathcal{P}$ **is** ***closed*****,**

or (SSKL) if **the set** $\mathcal{P}$ **is open and the option set is finitely generated,**

then ***E-admissibility*** **and** ***Maximality*** **give the same solution sets.**

Their admissible sets are precisely the Bayes-admissible options.

- **And then it also follows that the** ***Γ-Maximin*** **admissible acts are a (proper) subset of the Bayes-admissible options.**
- ***Under these conditions, pairwise comparisons of acts suffice to determine the set of Bayes-admissible choices and from them we can elicit the IP-model.***
- ***However, otherwise options that are admissible by Maximality may be E-inadmissible.*** **(SSKL, 2003).**

D.Pearce (1984), reports a result which is important for understanding the underlying connection between *dominance* and *Bayes-admissibility*.

Theorem **(Pearce, 1984): In a decision problem under uncertainty,**

- **with finitely many states and finitely generated option set *O*,**
- **with utility of outcomes determinate – cardinal utilities,**

if an option $o \in O$ fails to be Bayes-admissible,

then *o* is (uniformly) dominated by a finite mixture from *O*.

Aside: This result can be extended to some infinite decision problems, where the option set is not finite and utilities are bounded.

In this sense, incoherent choices suffer de Finetti's penalty – being (uniformly) dominated by a mixed option – within the decision at hand – and not merely for the prevision game, which is a specialized decision.

In accord with Pearce's Theorem, in the example above,
the mixed act $m = .5f \oplus .5g$ strictly dominates h.

***Definition*: Given a (closed) set *O* of feasible options, a <u>*choice function*</u> *C* identifies the set *A* of <u>*acceptable options*</u> *C*[*O*] = *A*, for a non-empty subset $A \subseteq O$.**

Aside: There may be no acceptable option if the option set is not closed,

e.g., there is no "best" option from the continuum of utility values in [0, 1).

Definition: **Option $o \in O$ *<u>has a local Bayes model</u> P* if**

***o* maximizes the *P*-expected utility over the options in *O*.**

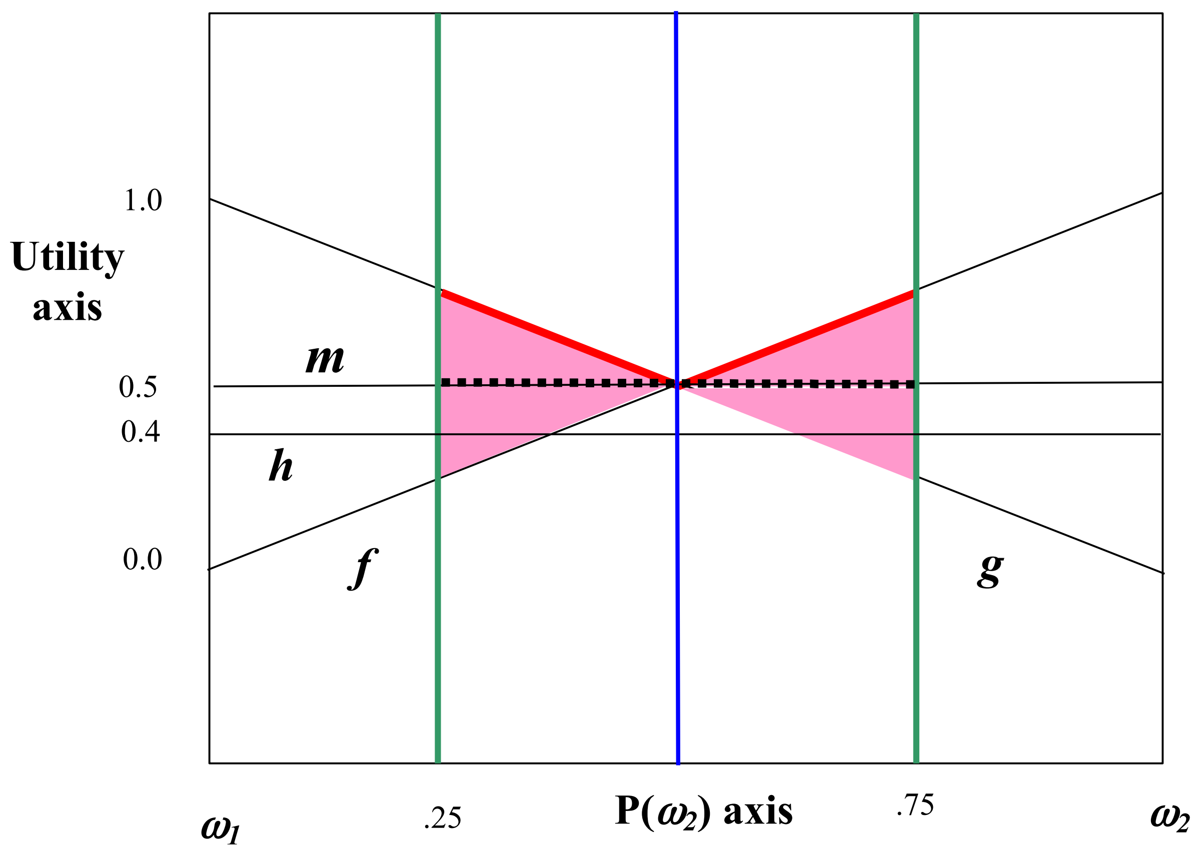
***Theorem* (Pearce, 1984 for finite state spaces): If an option $o \in O$ fails to have a local Bayes model then it is (uniformly) dominated by a finite mixture of options already available from *O*.**

So – at least when the option space is closed under (finite) mixtures –

(uniform) dominance assures that admissible options are locally coherent.

That is, then the choice function needs to be *locally coherent* at least.

- ***Definition*: A choice function C is *<u>coherent</u>* if there exists an IP-set $\mathcal{P}$ of probabilities such that the acceptable options under C are precisely those that maximize expected P-utility for some $P \in \mathcal{P}$.**



- **Note that option *m* has a local Bayes model from the choice set {*f*, *g*, *m*}**

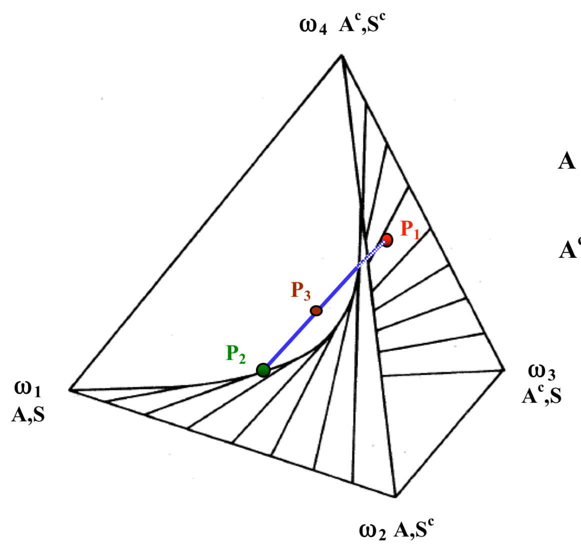
if and only if **P(ω_2) = .5 belongs to the IP set $\mathcal{P}$.**

This observation about the admissibility of a mixed option generalizes to allow very fine IP elicitations using coherent choice functions.

- **Each (arbitrary) IP set has its own distinct coherent choice function.**
- **For each two different sets of distributions there is a (finite) decision problem where they have distinct coherent choices.**

Application

We can represent the IP set of probability distributions that make two events independent, since convexity of the IP set is not required with coherent choice functions.



	S	S^c
A	ω_1	ω_2
A^c	ω_3	ω_4

This leads to different admissible options in a simple (normal form) decision problem than results when the IP set is, instead, the convex hull formed with extreme points satisfying independence between two events.

Coherent choice functions may be characterized by axioms on admissible sets that parallel familiar axioms for coherent preferences over horse-lotteries.

Coherent Preference **<**

Axiom$_1$ **<** ***is a weak order***

Axiom$_2$ **<** ***obeys Independence***

$o_1 < o_2$ ***iff***

$x o_1 \oplus (1-x) o_3 < x o_1 \oplus (1-x) o_3$

Axiom$_3$ ***Archimedes***

If $o_1 < o_2 < o_3$, **then** $\exists\ 0 < x, y < 1$

$x o_1 \oplus (1-x) o_3 < o_2 < y o_1 \oplus (1-y) o_3$

Axiom$_4$ **State-independent Utilities**

Preference over constant acts reproduces within each non-null state.

Coherent Choice Functions

Axiom$_{1a}$ **Sen's Property** α:

An inadmissible option remains so upon addition of other options.

Axiom$_{1b}$ **Aizerman condition, *almost*:**

Deleting an inadmissible option does not promote another inadmissible options.

These two axioms determine a strict partial order, $O_1 \ll O_2$ **on option *sets*:**

O_1 **contains no admissible options from among the choice of** $O_1 \cup O_2$.

The 3 remaining pairs of axioms are expressed using the partial order « and parallel the axioms for coherent preference.

<u>Representation Theorem (SSK 2007)</u>

- **The 4-pairs of axioms are necessary for a choice function to be coherent.**

- **The axioms suffice for representing a choice function with the Bayes-styled decision rule:**

  An option is admissible from a menu if
  it is maximizes Expected Utility over the menu for some probability in $\mathcal{P}$
  using a set of *Probability/Almost-state-independent utility* pairs.

- **We offer a sufficient condition where the representation uses a single, state-independent utility on rewards.**

Summary

1. (de Finetti's) Coherence criteria and related dominance rules

2. *IP* theory relating to (de Finetti's) *coherence* criteria

2.1 IP as incomplete elicitation of a precise prevision:

***Fundamental Theorem*: *imprecise* vs. *indeterminate* previsions**

2.2 IP based on lower and upper 1-sided previsions

Limitations using binary comparisons for *elicitation* with IP-sets

2.3 IP based on Coherent Choice Functions

Characterizing each set of (precise) probabilities

Eliciting IP sets using choice functions

Axiomatizing coherent choice functions.

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