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LOST IN THE TENSORS: EINSTEIN'S STRUGGLES WITH COVARIANCE PRINCIPLES 1912–1916*

Introduction

IN 1912 Einstein began to devote a major portion of his time and energy to an attempt to construct a relativistic theory of gravitation. A strong intimation of the struggle that lay ahead is contained in a letter to Arnold Sommerfeld dated October 29, 1912:

> At the moment I am working solely on the problem of gravitation and believe I will be able to overcome all difficulties with the help of a local, friendly mathematician. But one thing is certain, that I have never worked so hard in my life, and that I have been injected with a great awe of mathematics, which in my naiveté until now I only viewed as a pure luxury in its subtler forms! Compared to this problem the original theory of relativity is mere child's play.[1]

Einstein's letter contained only a perfunctory reply to a query from Sommerfeld about the Debye – Born theory of specific heats. Obviously disappointed, Sommerfeld wrote to Hilbert: 'My letter to Einstein was in vain . . . Einstein is evidently so deeply mired in gravitation that he is deaf to everything else.[2] Sommerfeld's words were more prophetic than he could possibly have known; the next three years were to see Einstein deeply mired in gravitation, sometimes seemingly hopelessly so.

In large measure, Einstein's struggle resulted from his use and his misuse, his understanding and his misunderstanding of the nature and implications of covariance principles. In brief, considerations of general covariance were bound up with Einstein's motive for seeking a 'generalized' theory of relativity; misunderstandings about the meaning and implementation of this motivation threatened to wreck the search; and in the end, the desire for general covariance helped to bring Einstein back onto the track which led to what we now recognize

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[1]A. Hermann (ed.), *Albert Einstein/Arnold Sommerfeld Briefwechsel* (Basel: Schabe and Co., 1968), p. 26. The 'friendly mathematician' was Marcel Grossmann; various aspects of the Einstein-Grossmann collaboration will be discussed below. Our translations of the Einstein–Sommerfeld correspondence are taken from Helga and Roger Stuewer's translation of this book. We are grateful to them for their permission to use their unpublished translation.

[2]*Ibid.* p. 27.

Stud. Hist. Phil. Sci., Vol. 9 (1978), No. 4. pp. 251 – 278. 0039-3681/78/1201-0251 $02.00/0

as the general theory of relativity. The purpose of this paper is to trace the influence which Einstein's changing attitudes towards covariance had on the emergence of the general theory. Though the task is basically an historical one, it is inextricably bound up with a number of issues in the foundations of physics which are still subject to lively discussion. We make no apology for our forays into these areas, for no detour around them can lead to any real appreciation of Einstein's struggles.

1. Einstein's Desire for a 'Generalized' Theory of Relativity

Einstein's motivation for the special theory of relativity derived mainly from his concern with the special principle of relativity and not with the aether hypothesis or the constitution of matter, the problems which occupied Lorentz and Poincaré.[3] To a significant degree, a similar concern with an extended version of the principle of relativity motivated the search for a theory of gravitation.

The reasons for Einstein's desire to generalize relativity, and the importance of covariance to that project, are to be found in Einstein's earliest writings on gravitation. In 1907 Einstein published a survey and discussion of experimental and theoretical work on relativity. The fifth and final part of this essay contains a discussion of the connection between gravitation and the principle of relativity, and it begins with a question:

> We have thus far taken the Principle of Relativity — the principle of the independence of natural laws from the state of motion of reference frames — only for reference frames free of acceleration. Can it be thought that the Principle of Relativity also holds for systems which are accelerated with respect to one another?[4]

Einstein's answer is in effect negative. The remainder of the section develops the principle of equivalence for the first time, and explores some of its implications. This principle was the rock upon which Einstein built his gravitation theory: in nine years of work on gravitation between 1907 and 1916, the principle of equivalence was, above all, the idea Einstein refused to give up, and yet in his very first statement of the principle he finds it inconsistent with the constancy of the velocity of light and with any generalization of the principle of relativity. The velocity of light cannot be the same in accelerated and unaccelerated frames; the laws in an unaccelerated frame may have to incorporate a homogeneous gravitational field which would vanish in an accelerated frame.

Einstein published nothing more on gravitation for three years, and when in

[3]See T. Hirosige, 'The Ether Problem, the Mechanistic Worldview, and the Origins of the Theory of Relativity', *Historical Studies in the Physical Sciences,* Vol. 7 (Princeton: Princeton University Press, 1976), pp. 3–82.

[4]A. Einstein, 'Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen', *Jahrbuch der Radioaktivität und Elekronik* 4 (1907), p. 454.

1911 his next paper[5] on the subject appeared, it went little beyond his 1907 discussion in reconciling the principles of equivalence and relativity. Einstein did, however, suggest a sense in which his ideas on gravitation generalized the theory of relativity. For if an unaccelerated frame of reference subject to a homogeneous gravitational field is physically equivalent to an accelerated frame, then, at least when there is gravitation, it seemed to Einstein that there is no such thing as absolute acceleration. Just as the special theory eliminates absolute velocity, its generalization through the principle of equivalence eliminates absolute acceleration. This way of viewing the relation between relativity and the principle of equivalence was bound to lead Einstein to try to embody certain of Mach's ideas within gravitational theory, and Machian themes were to emerge explicitly in Einstein's 1913 theory. But in 1911, the principle of relativity — meaning the invariance of the form of physical laws in all appropriate reference frames — and the principle of equivalence were still at odds. Indeed, as Einstein made his ideas on gravitation more explicit, the contradiction seemed more and more irreconcilable. Einstein's 1911 and 1912 papers on gravitation develop a scalar theory of static gravitational fields in which the velocity of light plays the role of the gravitational potential.[6] Accordingly, the theory could not be invariant under Lorentz transformations. The principle of equivalence was in outright contradiction with the special theory of relativity, and no generalization seemed at hand which would save the principles of the latter. When in 1912 Einstein undertook to master the tensor calculus with the help of his friend Marcel Grossmann, it may very well have been exactly because the new mathematical apparatus appeared to offer a means to save gravitational theory, the principle of equivalence and the principle of relativity all at once. The 'Absolute Differential Calculus' would permit the representation of quantities as geometrical objects, and would further permit the statement of relations between such quantities in such a way that they would hold in every frame of reference if they held in any. The principle of covariance would thus bring with it a generalization of the principle of relativity, and the question Einstein asked in 1907 would be answered in the affirmative.

Einstein's belief that the final form of his gravitational theory, as it emerged in late 1915 and early 1916, embodied a general principle of relativity is stated explicitly:

> *The general laws of nature are to be expressed by equations which hold good for all systems of coordinates, that is, are covariant with respect to any substitutions whatever (generally covariant).*
>
> It is clear that a physical theory which satisfies this postulate will also be suitable for a general postulate of relativity. For the sum of *all* substitutions in any case

[5]A. Einstein, 'Über den Einfluss der Schwerkraft auf die Ausbreitung des Lichtes', *Annalen der Physik* **35** (1911), 898–908.

[6]Ref. 5 and 'Lichtgeschwindigkeit und Statik des Gravitationsfeldes', *Annalen der Physik* **38** (1912), 355–369, and 'Zur Theorie des statischen Gravitationsfelds', *ibid.* 433–458.

> includes those which correspond to all relative motions of three dimensional systems of coordinates . . . this requirement of general covariance . . . takes away from space and time the last remnant of physical objectivity. . .[7]

This passage displays at once Einstein's tendency during this period to blur each of three interrelated distinctions: reference frames vs coordinate systems; point transformations vs coordinate transformations; and relativity or invariance principles vs covariance principles. The quoted passage shows that Einstein believed that because the laws of his 'general theory' were generally covariant, they automatically satisfied a general principle of relativity. The reason given for this belief shows the confusion of point and coordinate transformation and of reference and coordinate systems. Sorting out all the strands of this tangle would be a Herculean task, calling for one or more book length monographs, and we do not propose to undertake such a labor here. But a few preliminary remarks about the core of the tangle are needed to set up our discussion of Einstein's struggles with gravitational theory.

If we understand a reference frame to be defined by a congruence of timelike curves (each of which is to be considered the world line of a reference point of the frame) and understand a coordinate system $\{x^i\}$, $i = 1, 2, 3, 4$, to be adapted to the frame just in case each curve of the congruence satisfies x^α = constant ($\alpha = 1, 2, 3$), then frames of reference are related by point rather than by coordinate transformations; and the fact that laws are covariant, holding in the coordinates adapted to one frame of reference if they hold in the coordinates adapted to another, signifies by itself nothing about the status of the respective frames.[8] This is but one version of Einstein's practice in this period of seeing in the abolition of coordinate dependence for physical laws also the abolition of space-time structure.

In the sections below we will follow the thread which is most central to Einstein's progress, or lack of it, towards a theory of gravitation between 1912 and 1916; we will concentrate first on the considerations which led Einstein to abandon the requirement of general covariance in 1913, and we will then trace the reasons which led him to reaffirm it at the end of 1915.

2. The Einstein–Grossmann Theory

That Einstein saw in covariance a solution to the tension between his two

[7]A. Einstein, 'Die Grundlage der allgemeinen Relativitätstheorie', *Annalen der Physik* **49** (1916), 770–822; translation from W. Perrett and G. B. Jeffrey, *The Principle of Relativity* (New York: Dover, 1923), p. 117. The italics are Einstein's.

[8]Of course, a reference frame can be represented by a maximal class of adapted coordinate systems. Any two members of such a class are related by a coordinate transformation of the form $x'^\alpha = x'^\alpha(x^\beta)$, $x'^4 = x'^4(x^\alpha, x^4)$. Thus, coordinate transformations having this form can be thought of as giving a recoordinatization of the frame while ones not of this form can be thought of as giving a change of frame. But such a coordinate representation can easily lead to a blurring of the crucial distinctions mentioned above. For more on these matters, see Sections 3–6 below; see also J. Earman, 'Covariance, Invariance, and the Equivalence of Frames', *Foundations of Physics*, (1974), 267–289.

chief principles is evidenced in his contribution to the paper he published jointly with Grossmann in 1913, 'Entwurf einer verallgemeinerten Relativitätstheorie', the first of his papers in which the apparatus of the tensor calculus is used.[9] After explaining that the principle of equivalence requires that the velocity of light not be constant, Einstein wrote:

> With the introduction of a spatial variation in the quantity c we have gone beyond what is presently understood by 'relativity theory'; for the expression of ds is no longer invariant under linear orthogonal coordinate transformations. Suppose now — what cannot be in doubt — that the principle of relativity is to be preserved; then we must generalize relativity theory so that the significant elements of the foregoing theory of static gravitational fields are included as a special case.[10]

Einstein then introduced an arbitrary coordinate transformation and required that the motion of material points in arbitrary coordinates satisfy

$$\delta \int ds = 0 \tag{1}$$

where $ds^2 = g_{ij}dx^i dx^j$.[11] He pointed out that the g_{ij} are components of a tensor — the space-time metric tensor — and remarked:

> In the case of ordinary relativity theory only linear orthogonal substitutions are permissible. It will be shown that it is possible to obtain equations for the effects of gravitational fields on material phenomena which are covariant under arbitrary substitutions.[12]

Einstein's equation of motion for material particles is simply the requirement that their trajectories be geodesics, and so is generally covariant. Einstein and Grossmann were also able to give, as Einstein promised in the passage quoted, a generally covariant equation for the motion of incoherent matter in a gravitational field. But any field theoretic treatment of gravitation requires in addition a field equation which will determine the relation between field sources and the field they generate.

[9]A. Einstein and M. Grossmann, 'Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation', *Zeitschrift für Mathematik und Physik* **62** (1913), 225 – 261. Einstein's friendship with Grossman extended back to their school days in Zürich. It was Grossmann's father-in-law who helped Einstein to get a job at the Patent Office in Berne, and it was Grossmann who helped to conduct the negotiations which brought Einstein back to Zürich as a professor of theoretical physics at the ETH. Further, Grossmann was an expert in non-Euclidean geometry and the tensor calculus — just the mathematical tools Einstein saw he needed for his work on gravitation. Thus, it was natural, if not inevitable, that Einstein would enter into a scientific collaboration with Grossmann.

[10]'Entwurf', p. 228.

[11]For sake of readability, we have modernized Einstein's notation. Latin indices run from 1 to 4 and Greek indices from 1 to 3. The Einstein summation convention on repeated indices is in effect. Ordinary derivatives are denoted by a comma and covariant derivatives by a semi-colon. In 1913 Einstein did not use the now standard convention of upper and lower indices to denote respectively contravariant and covariant components. Thus, he had to use two different symbols for contravariant and covariant tensors, a practice which makes it hard for the modern eye to scan his early papers on gravitation.

[12]'Entwurf', p. 229.

Einstein and Grossmann searched for field equations of the form

$$\Lambda_{ij} = \mathrm{K}T_{ij} \tag{2}$$

The T_{ij} are the covariant components of the energy tensor for 'matter fields'. Initially, three requirements were imposed on the Λ_{ij}: (i) they are to be components of a second rank tensor so that (2) is generally convariant, (ii) they are to be constructed from the metric potentials g_{ij} and their derivatives of first and second order, and (iii) they are to be such that in the Newtonian limit, (2) reduces to an analogue of the Poisson equation

$$\nabla^2\Phi = \text{constant}\; x\varrho \tag{3}$$

where Φ is the Newtonian scalar potential and ϱ is the mass density.

As indicated by Grossmann, a natural candidate for Λ^{ij} is the Ricci tensor R_{ij}. R_{ij} automatically satisfies (i) and (ii). And it seems promising as regards (iii). Since the g_{ij} take the place of the Newtonian scalar potential, what is wanted is an expression concocted from the g_{ij} which, in the Newtonian limit, reduces to a sum of second order derviatives of the g_{ij}. This suggests that one look for a tensor expression which involves a summation over a repeated index. For tensors, this operation amounts to contraction, and R_{ij} is the only meaningful contraction of the Riemann tensor R_{ijkl}. Einstein and Grossmann had already recognized that the Riemann tensor plays a fundamental role in gravitational theory since it vanishes if and only if the metric is pseudo-Euclidean. Grossmann, however, rejected this choice for Λ_{ij}. The fact that $R_{ij} \equiv R^{k}_{ijk}$ in itself showed, according to Grossmann's reasoning, that R_{ij} cannot have the right Newtonian limit. Grossman's idea was that as the gravational field vanishes, then so must $0R_{ijkl}$, since this is a necessary condition for flat space – time and since gravitational field is null only if space – time is flat. But since R_{ij} is a contraction of R_{ijkl}, if the latter vanishes so must the former.[13]

Grossmañ's argument does not show what it was intended to show. It does establish that when the gravitational field vaniches *completely*, R_{ij} also vanishes, but the argument does not tell one what happens in some appropriately weak but non-vanishing approximation. Grossmann could have argued validly in the opposite direction; namely, when T_{ij} vanishes, we should return to the situation of special relativity with its flat space – time. But with the choice of R_{ij} for Λ_{ij} in (2), (2) does not have this consequence since in dimension 4, $R_{ij} = 0$ does not entail $R_{ijkl} = 0$. (The entailment does hold for lower dimensions.) While this argument does have the virtue of validity, its leading idea has the physically disastrous consequence that space–time cannot be curved outside mass-energy concentrations; thus, massive bodies would not 'bend' light, the sun would not ''attract' the earth, etc.

[13]'Entwurf', p. 257.

The actual situation as regards the Newtonian limit of

$$R_{ij} = \mathrm{K}T_{ij} \tag{4}$$

is mathematically straightforward but is somewhat complicated to present because of the number of assumptions needed to guarantee the correct Newtonian limit. We demand first that (A_1) the only sources of the gravitational field are massive particles (this is already implicit in the Newtonian equation (3)). Furthermore, we may assume that (A_2) the particles are moving slowly in comparison with the speed of light. Applying these assumptions to the contravariant form of (4), it is easily seen that the only component T^{ij} which cannot be neglected to first approximation is $T_{44} \approx \varrho$, so the problem reduces to examining the equation

$$R^{44} \approx \mathrm{K}\varrho \tag{5}$$

In computing the Newtonian limit of R^{44}, we are justified in assuming that (A_3) the space–time metric differs little from the Minkowski metric, *i.e.*, that there exists a coordinate system in which (5) holds and $g_{ij} = \Delta_{ij} + \varepsilon\beta_{ij}$ where Δ_{ij} is the Minkowski matrix and ε is a sufficiently small positive number. If we finally assume that (A_4) the metric is stationary in the sense that there is a coordinate system in which (A_3) holds and in which $\partial g_{ij}/\partial x_4 = 0$, we find after neglecting all but first order items in ε that

$$R_{44} \approx \tfrac{1}{2} \sum_{\mu=1}^{3} \beta^{44}{}_{\mu,\mu} \tag{6}$$

Thus, under assumptions (A_1)–(A_4), the contravariant form of (4) reduces to

$$\sum_{\mu=1}^{3} \beta^{44}{}_{\mu,\mu} = \text{constant}\ x\varrho \tag{7}$$

which is a close an analogue of (2) as could be hoped for.

Although Grossmann's objection to (4) was not sound, there are real problems involved. In a letter to Besso dated December 10, 1915, Einstein said that he and Grossmann had rejected (4) not only because of the considerations mentioned above but also because they believed that the 'conservation law would not be satisfied'.[14] To the modern eye, the problem seems obvious; namely, the conservation equation

$$T^{ij}{}_{:j} = 0 \tag{8}$$

does not follow from (4). But this is certainly not the problem which worried Einstein in 1913, for (8) is not a consequence of the field equations in any of Einstein's early theories.

Another problem is that (8) in conjunction with (4) entails that the energy scalar $T \leftrightharpoons \Sigma T^i_i$ is a constant, a seemingly implausible restriction on matter fields. It is doubtful, however, that Einstein was aware of this consequence in 1913, for Einstein apparently did not know the key identity

[14] P. Speziali (ed.), *Albert Einstein/Michele Besso Correspondance* (Paris: Hermann, 1972), p. 60.

$$(R^{ij} - \tfrac{1}{2}g^{ij}R)_{;j} = 0$$

(where $R \equiv \Sigma R^i_i$) needed to prove the consequence that T = constant. In November of 1915, Einstein returned to the equation (4), and he did not mention this consequence, although he did go through an eleborate argument to achieve the stronger result $T = 0$.[15] And when he first proposed the final field equations

$$R_{ij} = \mathrm{K}(T^{ij} - \tfrac{1}{2}g_{ij}T)$$

which are equivalent to

$$R_{ij} - \tfrac{1}{2}g_{ij}R = \mathrm{K}T_{ij}, \tag{11}$$

the conservation law (8) was still postulated separately.[16]

Assuming that the recollection Einstein recorded in his letter to Besso is accurate, what probably worried him in 1913 was that, by his lights (8) was not a proper conservation law. For Einstein believed that an adequate conservation law would have to include the contribution of the energy-momentum of the gravitational field itself and that the conservation law should be expressed in terms of an ordinary rather than a covariant divergence so that it could be integrated to yield conserved quantities.[17] Evidently, Einstein did not see a way to achieve a conservation law of the desired form using (4).[18]

The struggle with convariance is evident throughout Einstein and Grossmann's joint paper. Based, apparently, on Grossman's argument against the Ricci tensor, Einstein says in his 'Physical Part' of the paper that it has proved impossible to find a satisfactory tensor built from the metric tensor and its derivatives up to second order. Einstein does allow that it is possible that there may be satisfactory third order tensors, but thinks there are no good physical reasons to expect such an object to be a satisfactory generalization of the Newtonian potential. In his part of the paper Grossmann concludes that

> We must therefore leave open the question as to what extent the general theory of differential tensors linked with the gravitational field is related to the problem of

[15] A. Einstein, 'Zur Relativitätstheorie (Nachtrag)', *Sitzungsberichte, der Königlichen Preussischen Akademie der Wissenschaften zu Berlin,* 1915, 799–801.

[16] A. Einstein, 'Die Feldgleichungen der Gravitation', *Sitzungsberichte, der Königlichen Preussischen Akademie der Wissenschaften zu Berlin,* 1915, 844–847. For more on this point, see our paper, 'Einstein and Hilbert: Two Months in the History of General Relativity', forthcoming in *Archive for the History of Exact Sciences.*

[17] See Section 3 below for more on Einstein's attitude towards conservation laws.

[18] E. Zahar has recently claimed that Einstein and Grossmann had another reason for abandoning the Ricci tensor. He writes: 'both Einstein and Grossmann thought that, given the appropriate boundary conditions, the ten equations $R_{ij} = 0$ would uniquely determine the ten functions g_{ij}. This means that we are not at liberty to choose an arbitrary frame of reference because the functions g_{ij} are generally altered by a change in coordinates. Thus it seems that the Relativity Principle is violated', 'Why Did Einstein's Program Succeed Lorentz's?' in C. Howson (ed.), *Method and Appraisal in the Physical Sciences* (Cambridge: Cambridge University Press, 1976). This account appears to be unfounded. The argument ascribed to Einstein and Grossmann is not given in the 1913 'Entwurf' paper, nor does it appear in any of Einstein's papers between 1913 and 1916. Zahar puts in evidence a letter from Einstein to Sommerfeld dated November 28, 1915; but the text of this letter contains nothing of the argument in question, nor anything which could plausibly be read as implying it (see *Einstein/Sommerfeld Briefwechsel,* pp. 32–36).

> the equations of gravitation. Such a connection must exist insofar as the equations allow arbitary substitutions [*i.e.*, are generally covariant]; but in this case it seems that differential equations of the second order cannot be obtained. If, on the other hand, it could be established that the equations of gravitation only admit a certain group of transformations, then one could understand why the differential tensors yielded by the general theory would not be acceptable. As mentioned in the physical part [the part written by Einstein], we are as yet not in a position to discuss this question.[19]

Einstein's own approach to the problem of gravitation had made him receptive to doubts about general covariance. As we have seen, his strategy was to seek a relativistic field equation (2) which generalizes the Newtonian equation (3). Within this strategy, a natural tactic is to try to derive Λ_{ij} by finding a four-dimensional covariant analogue ∇_c^2 of the classical ∇^2 operator and then to apply ∇_c^2 to the metric potentials g_{ij}, which Einstein had already recognized as the relativistic gravitational potentials. And the most obvious try for ∇_c^2 is $\nabla_c^2\,(\ ) \equiv [g_{mn}(\ )_{;m}]_{;n}$. However, if one assumes, as Einstein did implicitly in this period, that the affine connection of space–time is compatible with the metric, then $\nabla_c^2(g^{ij}) = 0$. After considering such results, Einstein was ready to entertain non-covariant generalizations of ∇^2. And in fact, the field equations which he gave in the 'Entwurf' paper are not generally covariant. For later reference, they are

$$\begin{aligned} \Lambda ij &= \mathrm{K} T_{ij} \\ \Lambda ij &= 1/\sqrt{-g}\,(g^{km}\sqrt{-g}\,g^{ij}{}_{,m})_{,k} - g^{km}g_{nr}g^{in}{}_{,k}g^{jr}{}_{,m} \\ &\quad + \tfrac{1}{2}g^{ki}g^{mj}g_{nr,k}g^{nr}{}_{,m} - \tfrac{1}{4}g^{ij}g^{km}g_{nr,k}g^{nr}{}_{,m} \end{aligned} \tag{12}$$

where $g \equiv \det\,(g_{ij})$.[20]

In this climate, Grossmann's rejection of (4) and his expression of doubts about the possibility of finding a suitable tensor expression for Λ_{ij} provided all that was needed to make the seed of Einstein's own doubts bloom. The unfortunate harvest was given in an appendix attached to the main paper. The appendix is signed with the initials 'A.E.', and no one who is familiar with Einstein's style of argument can doubt that it is pure Einstein. Two arguments are given for abandoning the now suspect requirement of general covariance. But before turning to a discussion of these arguments, we should make a few brief remarks about Einstein's perception of Grossmann's contribution to their joint effort of 1913.

In a letter which has been lost, Sommerfeld apparently advised Einstein to be cautious in crediting Grossmann with too large a role in the development of the theory of gravitation. Einstein's reply, dated July 15, 1915, was that

[19]'Entwurf', p. 257.

[20]For ease of reference, we will sometimes call these equations the Einstein – Grossmann equations. In all probability, however, they are Einstein's invention. The 'Entwurf' paper consists of two main sections, a 'Mathematical Part' by Grossmann and a 'Physical Part' by Einstein; the field equations are given in the latter. See also the remarks at the end of this section.

> Grossmann will never claim to be codiscoverer. He only helped me orient myself in the mathematical literature, but contributed nothing materially to the results.[21]

From the point of view of hindsight, Einstein's remarks seem at once overly kind and ungenerous. While Grossmann did help to orient Einstein in the mathematical literature, he also helped to disorient him in its interpretation. On the other hand, Grossmann did alert Einstein to the possible importance of the Ricci tensor, an idea that came to Einstein's rescue at the end of 1915. But writing in July, Einstein was aware of neither his disorientation nor his salvation.

3. Einstein's Arguments Against General Covariance

Nothing is easier for a first-rate mind than to form plausible arguments that what it cannot do cannot be done. In the course of writing the joint paper with Grossmann, Einstein constructed two arguments that seemed to establish that satisfactory generally covariant field equations are impossible. The main argument, whose style (though not its invalidity) is absolutely characteristic of Einstein, is short, simple and lucid. It is also pure hokum.

Einstein's main argument against general covariance is based on the idea that the field equations should satisfy the causality requirement that the distribution of mass-energy uniquely determines the space–time metric and, hence, the gravitational field. Einstein's argument is supposed to show that causality in this form will be violated for generally covariant equations. Towards this end, we are asked to consider a domain D of space–time which is devoid of mass-energy. Further, we are asked to imagine two coordinate systems $\{x^k\}$ and $\{\bar{x}^k\}$ which agree outside D but which diverge inside D. The components of the energy-momentum tensor in these two coordinate systems will agree everywhere since inside D we have $\overline{T^{ij}} = T^{ij} = 0$ and outside D we also have $\overline{T^{ij}} = T^{ij}$ because $\bar{x}^k = x^k$ there. But inside D, $\bar{g}_{ij} \neq g_{ij}$. Assuming general covariance, however, the $\overline{g_{ij}}$ must satisfy the field equations if the g_{ij} do. Hence, Causality fails.

Any student of modern differential geometry will immediately put his finger on the flaw in Einstein's argument: by construction, the $\overline{g_{ij}}$ and the g_{ij} are simply different coordinate representations of the same intrinsic object $\mathbf{g}$, the metric tensor. There is not the least reason we cannot have equations that remain true under a coordinate transformation that changes the components of the metric tensor but not the components of the stress-energy tensor. Thus, the argument cannot possibly show that $\mathbf{g}$ is underdetermined.[22]

As a matter of mathematical and physical fact, however, there is a good deal of truth in Einstein's conclusion. But a proper understanding of the basis of the kernel of truth reveals that general covariance is in no way impugned. The

[21] *Einstein/Sommerfeld Briefwechsel*, p. 30.

[22] See B. Hoffmann, 'Einstein and Tensors', *Tensor* **26** (1972), 157–162.

mathematics of the situation is straightforward, but the treatment it receives in most standard physics textbooks is still sufficiently non-perspicuous as to make a brief discussion appropriate here. This discussion will also facilitate an understanding of a further evolution of Einstein's argument.

Let us suppose that the field equation for gravitation is local in the sense that if $(M, \mathbf{g}, \mathbf{T})$ (where M is a four-dimensional differentiable manifold, $\mathbf{g}$ is Lorentz signature metric for M, and $\mathbf{T}$ is an energy momentum tensor) is a solution and $D \subset M$ is an open subset of M, then $(D, \mathbf{g}|_D, \mathbf{T}|_D)$ is also a solution. (Most of the theories of gravitation which, at one time or another, have been under serious consideration do have this property.) This property allows one to concentrate on the source-free field equations when we are in an Einstein domain D where $\mathbf{T}|_D = 0$. The source free field equation will be represented symbolically as

$$L_{sf}(g_{ij}) = 0 \tag{13}$$

To establish Einstein's conclusion, it must be shown that if (13) is generally covariant, then there are two solutions $(M, \mathbf{g}_1)$ and $M, \mathbf{g}_2)$ where $\mathbf{g}_1 \neq \mathbf{g}_2$. This is easy to do. Let θ by any diffeomorphism of M onto itself. Then by general covariance, if $(M, \mathbf{g})$ is a solution, so is $(M\ \theta^*\mathbf{g})$ where $\theta^*\mathbf{g}$ denotes the 'dragging along' of $\mathbf{g}$ by θ. By choosing a map which is not an isometry of g, i.e., $\theta^*\mathbf{g} = \mathbf{g}$, we have a case in point. (Note: general covariance is *sufficient* for this result; but as will be discussed later, it is not *necessary*.)

It must be emphasized, however, that this kind of freedom to perform diffeomorphisms is trivial in two senses. First, although $\mathbf{g}$ and $\theta^*\mathbf{g}$ may be different metrics, they are of the same type, *e.g.*, if $\mathbf{g}$ is flat or pseudo-Euclidean, then so is $\theta^*\mathbf{g}$. Second, this freedom has nothing to do with general relativity theory *per se*, for a similar sort of freedom extends to any generally covariant theory as long as all of the quantities of the theory are simultaneously determined via field equations. As a simple illustration, consider the scalar wave equation

$$\partial^2\Phi/\partial x^2 + \partial_2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2 - \partial^2\Phi/\partial t^2 = 0 \qquad (c \equiv 1) \tag{14}$$

encountered in special relativity theory. To construct a generally covariant version, we postulate

$$R_{ijkl}(g_{mn}) = 0 \tag{15}$$

and

$$g^{mn}\Phi_{;m\,;n} = 0 \tag{16}$$

Equation (15) says that the metric is flat, so together with the side condition that the manifold is R^4, it specifies that the space–time is a copy of Minkowski space–time. Then in an inertial coordinate system, (16) reduces to the more familiar form given in (14). Obvious, if (R^4, $\mathbf{g}$, Φ) solves (15)–(16), then so does (R^4, $\theta^*\mathbf{g}$, $\theta^*\Phi$) for any diffeomorphism θ of R^4. Note that (15) makes the space–time metric an 'absolute object' in the sense that for any two solutions (R^4, $\mathbf{g}_1$) and (R^4, $\mathbf{g}_2$) of (15), there is a diffeomorphism θ of R^4 such that $\theta^*\mathbf{g}_1 = \mathbf{g}_2$. If one treats this absolute object not as being determined by the field equations but as being given once and for all by God, then the freedom to perform diffeomorphisms is curtailed — then θ must be an isometry of the fixed metric — in this case, a Poincaré (point) transformation. Parallel examples can also be constructed for Newtonian theories, so again nothing hinges on features peculiar to relativistic physics.

In general relativistic theories the freedom to perform diffeomorphisms also occurs in the initial value problem. For the sake of concreteness, consider the source-free initial value problem for the field equation (10–(11) of the usual version of general relativity. An initial value set is a triple (S, $\mathbf{h}$, χ) where S is a three manifold, and h and χ are tensor fields on S which are interpreted respectively as the first and second fundamental forms of S (intuitively, S is an instantaneous time slice of space–time, h characterizes the intrinsic spatial geometry of S, and χ determines how S is to be imbedded into space–time). By a solution to the initial value problem is meant an object ((M, $\mathbf{g}$), Ψ) where (M, $\mathbf{g}$) is a solution to the (source free) field equations and $\Psi : S \rightarrow M$ is an embedding such that $\Psi(S)$ is a spacelike hypersurface of (M, $\mathbf{g}$) whose first and second fundamental forms are $\mathbf{h}$ and χ respectively. Again, if ((M, $\mathbf{g}$), Ψ) is a solution and θ is a diffeomorphism of M, then ((M,$\theta^*\mathbf{g}$), $\theta\Psi$) is also a solution. If S is given *a priori* or is somehow fixed observationally, then θ is constrained to preserve S pointwise, *i.e.*, $\Psi\theta\Psi^{-1} = id$; but for points in M that are not in S, θ may be chosen arbitrarily. A similar result holds for the initial value problem with sources and for an initial value problem in which the initial data are specified on a finite 'sandwich' of space–time rather than an instantaneous slice. Thus, one cannot hope to have causality in the sense that the laws determine a unique extension of the initial value set; the best one can hope for is uniqueness up to a diffeomorphism.[23]

The situation is not quite the same for theories with an absolute space–time structure since this structure can be used to 'line-up' the corresponding solutions. In the example of (15)–(16), we know that for any two solutions (R^4, $\mathbf{g}_1$, Φ_1) and (R^4, $\mathbf{g}_2$, Φ_2), there is a diffeomorphism θ of R_4 such that $\theta^*\mathbf{g}_1 = \mathbf{g}^2$. If $\theta^*\Phi^1 = \Phi_2$ on a finite sandwich of space–time, then $\theta^*\Phi_1 = \Phi_2$ everywhere. In this sense, the scaffolding of the absolute space–time background permits one to come closer to a unique projection of the initial data than is possible in the case of general relativity.

[23]This is the geometric origin of the 'four undetermined functions' in the initial value problem of general relativity. Contrary to the impression sometimes given, these functions do not arise simply because of general covariance; see the discussion below.

It is clear then that it is not general covariance itself which is responsible for the freedom to perform diffeomorphisms in the initial value problem of general relativity; rather, it is the combination of general covariance and the absence of any absolute objects. And further, although this freedom does entail a change in the formulation of the classical principle of Laplacian causality, it does not extract the teeth of the principle.

The discussion so far has admittedly not done complete justice to Einstein's idea that the distribution of mass energy should uniquely determine the metric. But justice is hard to do here since it is difficult to formulate Einstein's idea in a coherent and precise fashion. In some cases — for example, when matter exerts a pressure — T^{ij} will itself explicitly contain the metric potentials. Even when T^{ij} does not explicitly depend upon the metric, its interpretation does. For example, the energy density as measured by an observer with four-velocity V is $T_{ij}V^iV^j$. But the accuracy of this measure presupposes that V is normalized, which in turn assumes the metric. Still, let us attempt to test Einstein's idea in the following way. Let

$$L_{ws}(g_{ij}, T^{ij}) = 0 \tag{17}$$

represent the field equation with sources. For fixed $\mathbf{T}$, specified somehow or other as a tensor field on a manifold M, can there be more than one $\mathbf{g}$ which solves (17)? If (17) is generally covariant, we know that when $(M, \mathbf{g}, \mathbf{T})$ is a solution, so is $(M, \theta^*\mathbf{g}, \theta^*\mathbf{T})$ for any diffeomorphism θ of M. But by fixing $\mathbf{T}$, θ is required to be a symmetry of $\mathbf{T}$, i.e., $\theta^*\mathbf{T} = \mathbf{T}$, and in special cases there may be no non-trivial solutions of this condition. Of course, if (17) is local and if $\mathbf{T}$ vanishes on some open set D, we return to the situation discussed above. To obtain any additional information we must therefore consider specific instances. For the particular case of the field equations (10)–(11), the free-space solutions include ones which are more than mere diffeomorphic variants of one another — *e.g.*, they include both flat and curved metrics. A similar result obtains when a 'cosmological constant' is added to the field equations of general relativity.

In addition to his causality argument, Einstein offered a second reason for abandoning general covariance. He had shown that as a consequence of equation (8) and the Einstein–Grossman field equation, a conservation law for the combined matter and gravitational fields holds in the form

$$[\sqrt{-g}\,(T_i^j + t_i^j)]_{,j} = 0 \tag{18}$$

where t_i^j is the object Einstein interpreted as characterizing the energy content of the gravitational field. In his appendix, Einstein stated that (18) is covariant only for linear coordinate transformations. But once again the reasoning was unsound. *If* t_i^j transforms like a (1, 1) tensor, then in general (18) would be covariant only under linear transformations. But in general t_i^j transforms like a tensor only for linear coordinate transformations. Since (18) is a consequence of generally covariant equation (8) and the Einstein–Grossmann field equations, its covariance is just as wide as the covariance of the field equations. The point should have been intuitively evident to Einstein. The principle of equivalence implies that it is always possible to 'transform away' the gravitational field at any point; thus, whatever the state of the field, one should always be able to find a coordinate system such that $t_i^j = 0$ at the chosen point.[24] Thus, t_i^j cannot behave like a tensor under non-linear transformations.

Once Einstein recognized his mistake, he set to work to find the extent to which the Einstein–Grossmann field equations are covariant. This matter will be discussed below in Sec. 5, but before turning to it we will take up two further developments in Einstein's argument against general covariance.

4. Further Developments in Einstein's Arguments Against General Covariance

Einstein had some reasons to be pleased with his new theory of gravitation; it did incorporate a version of the principle of equivalence, and although it was not generally covariant, he believed it to be covariant under linear transformations and thus to satisfy a restricted principle of relativity. Furthermore, the new theory appeared to make the inertia of a masspoint depend on neighboring masses, and Einstein saw this result as an appealing realization of Mach's ideas. But the theory was not especially well-received by Einstein's contemporaries. The discussion of Einstein's paper at the Viennese Naturforscherversammlung was desultory at best, and dominated by Gustav Mie, who chided Einstein for failing to discuss Mie's own theory, and then proceeded to describe it at length.[25] Einstein wrote to Besso shortly afterwards complaining that

> The physicists have a negative attitude towards my work on gravitation . . . A free and impartial vision is scarcely proper for the adult German. (Blinkers!).[26]

Even a free and impartial vision, however, might have had very serious reservations about the Einstein–Grossmann theory. It explained no known anomalies of Newtonian gravitational theory,[27] the principle of equivalence was

[24]An explicit demonstration that this is indeed the case was given by E. Schrödinger, 'Die Energiekomponenten des Gravitationsfeldes', *Physikalische Zeitschrift* **19** (1918), 4 – 7.

[25]See A. Einstein, 'Zum gegenwärtigen Stande des Gravitationsproblems', *Physikalische Zeitschrift* **14** (1913), 1249–1266, and the discussion following Einstein's lecture on pp. 1262–1266. The reason given in the body of Einstein's paper for limiting the requirement of covariance to linear transformations was that the validity of the conservation law in the form (18) is not compatible with any wider covariance. But in a footnote (see p. 1257) which was probably added in proof, Einstein states that he has discovered 'in the last days' a demonstration showing that generally covariant field equations cannot be used.

[26]*Einstein/Besso Correspondance*, p. 50.

[27]Besides the anomalous advance of the perihelion of Mercury, other anomalies had been established by the end of the 19th century, the most notable one being the movement of the nodes of Venus. See H. Jeffreys, 'The Secular Perturbations of the Four Inner Planets', *Monthly Notices of the Royal Astronomical Society*, **77** (1916), pp. 112–118. See also P. Cohen, *Relativity and the Excess Advances of Perihelia in Planetary Orbits* (University of Pennsylvania M. A. Thesis, 1971).

embodied in a strong form without any argument, and, finally, though the setting of the theory made general covariance almost irresistible, the field equations and the conservation laws were not generally covariant. Einstein did not help matters as he elaborated his views on covariance. In 1914 Mie published a critical review article[28] in which he attacked the Einstein–Grossmann theory for, among other things, failing to generalize the principle of relativity to accelerated reference frames. Einstein replied that he had been misunderstood, and rather than counterattacking, attempted to lay out more clearly the reasoning that led him to put credence in his theory.[29] He found it unbelievable that the velocity of light should be entirely unaffected by any other physical phenomena, he wrote. Dropping the requirement that the velocity of light be constant permits one to use arbitrary coordinates; the equation of motion of a mass point is the usual Hamiltonian equation (1). Now however, the principle of equivalence seems to require that the metric components g_{ij} also be the components of the gravitational field. One can, furthermore, give a generally covariant equation specifying the effects of the gravitational field on physical phenomena. 'It is important for the theory', Einstein wrote, 'that equation [(1)] is invariant under arbitrary transformations'.[30] What more is required is a generalization of Poisson's equation, that is, a field equation, and that, too, should be generally covariant.

Einstein's entire approach to the theory of gravity, then, demands a generally covariant theory, for it is exactly the coupling of the admissibility of arbitrary coordinates with the equivalence principle that leads Einstein to a tensor theory. Yet the field equations of the theory are not generally covariant, Einstein admits; still, that is not so disastrous as it seems. Here is Einstein's argument. Suppose a set of equations fail to be generally covariant; then one of two cases must obtain. Either there are generally covariant equations which in appropriate coordinates reduce to the original equations, or there are not. In the former case, there is no harm in using the non-covariant equations. In the latter case, however, the non-covariant equations must lack any physical significance whatsoever and can only be a complex way of specifying coordinate systems. But no one can doubt that the Einstein–Grossmann theory is physically significant.[31]

The thrust of this argument seems plainly to be that it is of no consequence that the Einstein–Grossmann field equations are not covariant so long as there are covariant equations that reduce to them; and, furthermore, that the Einstein–Grossmann equations satisfy a sufficient condition for being special cases of

[28]G. Mie, 'Bemerkungen zu der Einsteinschen Gravitationstheorie', *Physikalische Zeitschrift* **14** (1914), 115–122, and 169–176.

[29]A. Einstein, 'Prinzipielles zur verallgemeinerten Relativitätstheorie und Gravitationstheorie', *Physikalische Zeitschrift* **15** (1914), 176–180.

[30]*Ibid.* p. 177.

[31]*Ibid.* p. 177–178.

covariant relations. After writing down the field equations Einstein continued:

> It is indubitable that these equations conform with one, even perhaps a small number of generally covariant equations, the statement of which is neither of logical nor of physical interest . . .[32]

Einstein's argument is both puzzling and fascinating, and for several reasons. In the first place, the argument is given in conjunction with his earlier arguments to the conclusion that the field equations for gravitation cannot even in principle be covariant: If **T** is to determine **g** uniquely, then the equations cannot be generally covariant; further, laws which are of the form of the conservation law cannot be generaly covariant. It seems very much as though Einstein is saying both that his field equations must have and do have a covariant generalization, and furthermore that they cannot have them and do not have them. There is, perhaps, a consistent way to understand Einstein's point of view in this regard. It is that every physically significant equation which is not generally covariant must be the specialization (to special coordinates) of some generally covariant equation, but that in special coordinates a non-covariant equation must express, or somehow reflect, a feature of the world (causality or conservation, for example) that cannot be caught by any generally covariant relation.

In the second place, what reason did Einstein have for believing that every physically significant equation is equivalent, in the special coordinates used, to a generally covariant one? If we are dealing with a differential equation or system of differential equations using geometrical objects of the usual kinds, in a single coordinate system, then Einstein was quite correct. Ignoring tensor densities, one can always find corresponding covariant equations, for example, simply by introducing a flat affine connection which vanishes in the special coordinates and replacing all coordinate derivatives by covariant derivatives. But if, as is actually the case with the Einstein–Grossmann theory, one is given a set of equations which are to hold in not one but an entire collection of coordinate systems, then it is not clear under what conditions there exists a covariant equation which reduces to the original equations in *all* of the coordinate systems in the collection. The trick of introducing a flat connection will not work if, for example, some of the coordinate system in the collection are non-linearly related. An analogous trick will work provided there is an operator which is a derivation, and which is identical with coordinate derivation in all of the specified coordinates. We do not know for what group of coordinate transformations such operators exist.

In the third place, the argument Einstein gave in reply to Mie marks an important and fundamental change in his attitude toward general covariance.

[32] *Ibid.* p. 179. A similar point is made at the end of Walter Dällenbach's notes on Einstein's lectures at the ETH, presumably delivered during the winter semester, 1913–1914; see Einstein Papers, Princeton University, microfilm reel I.A.8, no. 4.

Heretofore, Einstein seems to have seen in generai covariance the perfect and complete generalization of one aspect of the principle of relativity. Now, in the face of Mie's attack, Einstein denigrates general covariance, and sees the demand for it as physically vacuous; the gravitational theory he and Grossmann put forward was an intelligible generalization of special relativity and satisfied the equivalence principle; that is what mattered. Although the arguments Einstein advanced for his new view were unsound or incomplete, the view itself was closer to the truth of the matter than those Einstein had earlier held. Thus Einstein's account of covariance in reply to Mie is essentially the one given two years later by Kretschmann[33] in criticizing Einstein's 1916 paper on the foundations of general relativity.[34] Yet, paradoxically, in forming a more accurate opinion of the physical significance of the demand for general covariance, Einstein may well have made it more difficult for himself to find a satisfactory gravitational theory. Certainly it seems that if he had held fast to general covariance at this point he might have been driven to re-evalute Grossmann's arguments against the Ricci tensor.

Though Einstein's rejection of general covariance did not waver during 1914, the argumentation did undergo a significant shift. In his communication to the Berlin Academy for October 1914,[35] Einstein's argument against general covariance appears in a revised form, reflecting a recognition of the main error of interpretation in the original form.[36] We are again asked to consider a domain D of space–time which is devoid of all material processes. Then the total state of D is completely specified once the metric potentials $g_{ij}(x^k)$ as functions of some coordinate system $\{x^k\}$ are given. Again let $\{\bar{x}^k\}$ be a second coordinate system which coincides with $\{x^k\}$ outside D but diverges inside. Einstein now correctly states that the metric potentials $\overline{g_{ij}}(\bar{x}^k)$ as functions of the new coordinates describe exactly the same physical situation as the $g_{ij}(x^k)$. Here then was an opportunity to escape from past confusions and to reinstate the requirement of general covariance. But the opportunity was not seized, and Einstein was quickly seduced by a new form of the same misunderstandings that caused the original problem.

If the field equations are generally covariant, then it follows, Einstein argues, that if the $g_{ij}(x^k)$ are solutions then so are the $\bar{g}_{ij}(x^k)$. But $\bar{g}_{ij}(x^k)$ and $g_{ij}(x^k)$ describe different gravitational fields if the barred functions are different from the unbarred ones, as can always be arranged by the choice of $\bar{x}^k$. Thus,

[33] E. Kretschmann, 'Üner den physikalischen Sinn der Relativitätspostulate, A. Einsteins neue und seine ursprüngliche Relativitätstheorie', *Annalen der Physik* **53** (1917), 575–614.

[34] Ref. 7.

[35] A. Einstein, 'Die formale Grundlage der allgemeinen Relativitätstheorie', *Sitzungsberichte, der Königlichen Preussischen Akademie der Wissenschaften zu Berlin*, (1914), 1030 – 1085.

[36] We have not been able to determine either from Einstein's published papers or his correspondence who or what was responsible for this change. This is just one of the many unsolved mysteries about Einstein's odyssey from the special to the general theory of relativity.

we arrive at the same conclusion: generally covariant field equations violate Causality. The promises of Einstein's argument are largely correct and amount to a coordinate representation of the freedom to perform diffeomorphisms discussed in Section 3, though Einstein's construction presupposes a non-standard notion of 'dragging along'. In a more standard presentation, the construction proceeds as follows. Consider two subdomains D_1 and D_2 of D, and let $\theta : D_1 \to D_2$ be a diffeomorphism. θ can be given a coordinate representation by describing it as the map which takes a point in D_1 with coordinates x^k to a point in D_2 with coordinates $f^k(x^m)$. With this point map we can associate a coordinate map to a new coordinate system $\{\bar{x}^k\}$ where $\bar{x}^k\,(\theta(p)) \equiv x^k(p)$, *i.e.*, the new coordinates at the new point are numerically equal to the old coordinates at the old point. If F^k is the inverse of f^k then the coordinate transformation is given by $\bar{x}^k = F^k(x^m)$. The metric $\theta^*\mathbf{g}$ dragged along by θ can be defined in coordinate terms by the rule $\overline{(\theta^*\mathbf{g})_{ij}}|_{\theta(p)} = g_{ij}|_p$ *i.e.*, the new metric at the new point has the same components in the new coordinate system as the old metric has in the old coordinate system at the old point.[37] If θ is an isometry of $\mathbf{g}$, then $(\theta^*\mathbf{g})_{ij}|_{\theta(p)} = \overline{g_{ij}}|_{\theta(p)}$, and combining this with the definition of $\theta^*\mathbf{g}$, we have $\overline{g_{ij}}|_{\theta(p)} = g_{ij}|_p$. Since the barred coordinates at $\theta(p)$ are numerically the same as the unbarred coordinates at p, the $\overline{g_{ij}}$ and the g_{ij} are the same functions. Conversely, if the functions are the same, then θ is an isometry, and the coordinate transformation $\bar{x}^k = F^k(x^m)$ can be said to generate an isometry.

What Einstein did not notice, however, was that dropping the requirement of general covariance need not block the unwanted consequence. Let us suppose, as Einstein did, that the source free field equation is a differential equation constructed entirely from the metric potentials and their derivatives but that the equation need not be generally covariant. As a simple example, consider the equation

$$g_{ij,k} = 0 \tag{19}$$

Suppose that the associated gravitational theory says that a metric $\mathbf{g}$ is admissible (in the absence of matter) just in case there exists a coordinate system in which (19) holds. This if $\mathbf{g}$ is admissible, so is $\theta^*\mathbf{g}$ for any diffeomorphism θ. For if $\{x^k\}$ is the coordinate system that makes $\mathbf{g}$ admissible, then $\{\bar{x}^k\}$, where $\bar{x}^k = F^k(x^m)$, is the system that makes $\theta^*\mathbf{g}$ admissible — $\theta^*\mathbf{g}^{ij,k} = 0$ in $\bar{x}^k$: Only if some particular coordinate system is singled out ahead of time will (19) uniquely determine the metric. The singling out could be done by ostention, but unless guided by a physical motivation such ostention is arbitrary. If there were additional quantities not determined through the field equations but which were given *a priori*, then the value of the coordinate components of these

[37]Einstein's construction seems to presuppose the alternative definition $(\theta^*g)ij|p \equiv \overline{gij}|\theta(p)$, *i.e.*, the components of the new metric in the old coordinate system at the old point are numerically the same as the components of the old metric in the new coordinate system at the new point.

objects could be used to pick out a preferred coordinate system. But on Einstein's own hypotheses, such quantities are not available.

To summarize, Einstein believed that general covariance led to a violation of Causality. His original argument to this effect (discussed in Section 3) was invalid. His second argument, presented in his 1914 Academy paper, was essentially correct. But this argument, like the original one, assumes a naive and ultimately untenable interpretation of Causality. Finally, Einstein did not notice that the use of non-covariant field equations did not provide any easy way to resolve his alleged problem.

5. The Return to General Covariance

In a joint 1914 paper[38] Einstein and Grossmann reported what they took to be a major advance in the understanding and application of the covariance properties of their gravitational field equations. The advance proved to be largely illusory. But this illusion, or rather the piercing of it, was later to serve as the spur to a real advance; indeed, it proved to be one of the main reasons why Einstein dropped the Einstein–Grossmann theory and returned to general covariance. In the second section of this paper, Einstein's Causality argument is repeated in the original form of the 'Entwurf' paper, indicating that Einstein had not hit upon the refinement discussed above in Section 4. But tucked away in a footnote[39] is the admission that, contrary to the claim made in the 'Entwurf' paper, the validity of the conservation law (18) does not limit covariance to linear transformations. The covariance of (18) will be just as wide as that of the Einstein–Grossmann field equations. But then what is the covariance of the latter?

By way of answering this question, Einstein and Grossmann first show that the Einstein–Grossmann field equation together with the generally covariant conservation law (8) for matter fields entails

$$B_n \equiv (\sqrt{-g}\, g^{ij} g_{nk}\, g^{km}{}_{ij} j)_{,i,m} = 0 \qquad (20)$$

A coordinate system satisfying the condition (20) is called an 'adapted' (angepasste) system and the transformation between two such systems is said to be 'justified (berechtigte).

Next, it is shown that the field equations are derivable from a variational principle.

$$\delta \int (H - 2K\sqrt{-g}\, T_{ij} g^{ij}) d^4 x = 0 \qquad (21)$$

where δ denotes functional variation of the metric potentials and the Hamil-

[38] A. Einstein and M. Grossmann, 'Kovarianzeigenschaften der Feldgleichungen der auf die verallgemeinerte Relativitätstheorie gegründeten Gravitationstheorie', *Zeitschrift für Mathematik und Physik* **63** (1914), 215–225.

[39] *Ibid.* p. 218.

tonian of the gravitational field is given by[40]

$$H = 1/2\sqrt{-g}\, g^{ij} g_{km,i} g^{km}{}_{,j} \tag{22}$$

Using (21) and (22), Einstein and Grossmann present an argument designed to show that their field equations are covariant under any justified transformation. This argument and the parallel considerations Einstein offered in his 1914 Academy paper were the subject of an intensive correspondence with Levi-Civita.[41] We will discuss this matter in another paper, and for present purposes we will concentrate on what Einstein took to be the significance of his result.

As Einstein saw it, the main advance of the 'Kovarianzeigenschaften' paper was the partial removal of the 'severe defect' — the perceived limitation on the relativity principle — which he had announced in his reply to Mie.[42] His reasoning now was that (a) justified transformations may be non-linear, and, therefore, (b) the theory allows the equivalence of accelerated and non-accelerated frames. As he put it in a letter to Besso, written in March of 1914:

> . . . *the equations of gravitation are valid for all coordinate systems satisfying these conditions* [eq (20)]. From this, it follows that there are transformations representing accelerations of a very varied nature (for example, rotations), so that the equivalence hypothesis is vouchsafed in its original form.[43]

From the importance he attached to 'Mach's Paradox' (see Section 6 below). Einstein must have been very pleased by (the illusion!) that rotations were covered.

Because Einstein blurred the distinctions between point and coordinate transformations, reference frames and coordinate systems, and relativity and covariance principles, it is difficult to put his claims in the letter to Besso to a test; but one case which comes as close to a test as can be hoped concerns rotation in an empty region of space–time. Let D be a region where $\mathbf{T} = 0$. Then in D, the Minkowski metric η is a solution of the Einstein–Grossmann field equations (12). Choose in D a reference frame K which is inertial with respect to η, and then choose an inertial coordinate system $\{X^i\} = \{X, Y, Z, T\}$ which is adapted to K, *i.e.*, in $\{X^i\}, \eta_{ij} = \mathrm{diag}(1, 1, 1, -1)$ and the world lines of the points of K have the form $X^\alpha = \text{constant}$ ($\alpha = 1, 2, 3$). Obviously, $B_n(X^i) = 0$ so the X^i is 'adapted' in the sense of Einstein and Grossmann. Next, consider a frame K' which is rotating with constant angular velocity ω with respect to K. It is not so obvious what coordinate system to choose as representative of K', but one natural choice is to use polar coordinates $\{r, \phi, z, t\}$ adapted to K'. the relation between the two coordinate systems is

[40]Interestingly, Paul Bernays is given credit for calling this fact to the attention of Einstein and Grossmann; see the footnote on p. 219. of Ref. 38.

[41]Einstein Papers, Princeton University, microfilm reel I.B.1, no. 16.

[42]See Ref. 29, p. 176 and our discussion above in Section 4.

[43]*Einstein/Besso Correspondence*, p. 53. The italics are Einstein's.

$$X = r\cos(\phi + \omega t)$$
$$Y = r\sin(\phi + \omega t)$$
$$Z = z$$
$$T = t$$

The non-vanishing metric components in the new coordinate system are

$$\eta_{11} = 1, \eta_{22} = r^2, \eta_{33} = 1$$
$$\eta_{24} = \eta_{42} = r_2, \eta_{44} = (1 - \omega^2 r^2)$$
$$(c \equiv 1)$$

But as can easily be checked, the condition (20) does not hold for the new coordinate system. A similar result follows for other natural choices for a coordinate representative of K'.

It is quite likely that considerations of just this kind were in large part responsible for Einstein's subsequent loss of confidence in both the methods and results of the Einstein–Grossmann theory. In a letter to Sommerfeld, dated November 28, 1915, Einstein listed three reasons for having abandoned his previous theory. The first on the list is that 'I proved that the gravitational field for a uniformly rotating system does not satisfy the field equations'.[44] The same point is also made in a letter to Lorentz written on the first day of January, 1916.[45] Learning that the 'severe defect' had not after all been removed must surely have been a serious blow; when added to the other two reasons — the failure of the Einstein–Grossmann field equations to yield the observed value of the advance of perihelion of mercury and the collapse of an attempt to justify the choice of H in (22) — it was sufficient to undermine whatever confidence Einstein still had in the Einstein–Grossman theory.[46]

We have described elsewhere[47] the final months of Einstein's struggle towards the covariant form of the general theory of relativity, and we will mention here only two episodes. Einstein did not immediately return to general covariant field equations; rather, he initially considered equations covariant for transformations for which the Jacobian is unity.[48] In his next attempt,[49] Einstein proposed the equations (4) which he and Grossmann had considered much earlier and whose rejection had set Einstein off on his painful odyssey. In November of 1915, however, Einstein's desire for general covariance became so strong that he was willing to tolerate the consequence that T = constant

[44] *Einstein/Sommerfeld Briefwechsel*, p. 32.

[45] Einstein Papers, Princeton University, Microfilm reel I.B.1, no. 16. In this letter the same three reasons are given, but the consideration about rotating systems appears second on the list.

[46] For more discussion of this matter, see our paper 'Einstein and Hilbert: Two Months in the History of General Relativity', *op. cit.* (Ref. 16).

[47] *Ibid.*

[48] A. Einstein, 'Zur allgemeinen Relativitätstheorie', *Sitzungsberichte, der Königlichen Preussischen Akademie der Wissenschaften zu Berlin* 1915, 778–786.

[49] Ref. 15.

and the even worse result that $T = 0$ and to support these consequences with a highly speculative hypothesis about the constitution of matter.

But this makes the obvious question even more puzzling: How could Einstein return to general convariance when he had 'proved' that generally covariant field equations are physically unacceptable? This is the puzzle we will try to resolve in the next section.

6. From Causality to Covariance

Physicists seldom seem content unless they have shown that what is possible is also necessary. Having obtained satisfactory covariant field equations late in 1915, Einstein proceeded to adduce arguments to show that satisfactory field equations *must* be covariant. These arguments are given both in his famous 1916 *Annalen der Physik* paper[50] and in his correspondence with Ehrenfest[51] and others[52] late in 1915 and early in 1916.

In his 1916 paper on the foundations of general relativity, Einstein presents a variety of arguments for general covariance. There is, in the first place, the consideration which we have earlier suggested attracted Einstein to covariant methods originally. Einstein first presents an epistemological argument, which he attributes to Mach, against privileged reference frames. Two fluid bodies at a great distance from each other and from all other masses are in rotation, each with respect to the other. Suppose measurements of each of the two bodies are carried out by means of instruments at rest with respect to the corresponding body, with the result that one body is an ellipsoid and the other a sphere. What is the cause of this asymmetry? The anwer given by Newtonian mechanics and special relativity theory — that one of the bodies is really accelerated with respect to a privileged frame of reference (actually a class of privileged frames, the inertial ones) — is unsatisfactory because the cause is not an 'observable fact of experience'. The only acceptable answer must be that the differing behavior is caused by the distant masses. Similarly, any privileged class of reference frames is open to the same epistemological objections; thus: 'The laws of physics must be of such a nature that they apply to systems of reference in any kind of motion'.[53]

Just what Einstein meant by this 'extended principle of relativity' is unclear. His formulation of it suggests that he meant that the laws of physics must be formulated so that they *hold* in any system of reference but his argument sug-

[50]Ref. 7.

[51]Einstein Papers, Princeton University, microfilm reel I.B.1, no. 9. See especially the letters dated December 26, 1915, December 29, 1915, and January 5, 1916.

[52]See *Einstein/Besso Correspondence,* pp. 63–64.

[53]Ref. 7; translation from Perrett and Jeffrey, p. 113. For earlier expressions of Einstein's interest in 'Mach's paradox', see A. Einstein, 'Bases Physique d'une Theorie de la Gravitation', *Archieves des Sciences Physiques et Naturelles* **37**, (1914), 5–12; 'Zum Relativitäts-Problem', *Scientia* **15** (1914), 337–348; and Ref. 5, pp. 1031–1032.

gests instead that the laws must not require the *existence* of a privileged class of reference systems, privileged, that is, in the sense that motion with respect to this class of frames plays a special causal role. Quite likely, Einstein believed that these two formulations of the principle of relativity are equivalent, and he certainly implied, quite erroneously, that general covariance guarantees that the two formulations of the principle of relativity are satisfied.[54] Einstein's argument, against Mie, that general covariance is physically vacuous is now utterly forgotten; general covariance not only guarantees that the laws 'apply' in every system of coordinates, it also abolishes the structure of space and time. Einstein's new discussion is in almost every way less accurate and more confused than his earlier discussion, given at a time when he thought general covariance could not be satisfied by gravitational theory.

Einstein had another, very different argument for general covariance. It is given in the 1916 *Annalen* paper, but its point is first made, so far as we can determine, in a letter to Ehrenfest, dated December 26, 1915.[55]

> Section 12 of my last year's paper [the October 1914 Academy paper] is entirely correct (in the first 3 paragraphs) up to what is in italics at the end of the third paragraph. [This is the argument examined in detail in Section 4 above.] There is no incoherence to be derived from the fact that both systems $G(x)$ and $G'(x)$ [$G(x)$ is the symbol Einstein used to denote the totality of the functions $g_{ij}(x^k)$], which refer to equivalent reference frames, satisfy the conditions on the gravitational field. The apparent force of this consideration is immediately dashed if one realizes that
> (1) the coordinate system signifies nothing real
> (2) the nature of the theory makes possible the simultaneous realization of two (distinct) g-systems in the same region of the continuum. [Ehrenfest has a question mark in the margin here.]
>
> The following consideration must take the place of Section 12. The physically real events in the world (in contrast to those that depend on the choice of coordinates) are *space-time coincidences.*

And since (continuous one-one) coordinate transformations preserve all such space–time coincidences, they do not change the physical situation. Essentially the same paragraph is contained in a letter to Besso written about a week after the letter to Ehrenfest.[56]

From these two letters, the main outlines of Einstein's new strategy is clear: (i) keep the requirement of causality, but (ii) despite the previous argument, deny that causality is violated by generally covariant field equations. This is to be accomplished by (iii) maintaining that physical reality consists of space–time coincidences and (iv) observing that coincidences are preserved by con-

[54]Ref. 7, Section 3; see especially the passage quoted above in Section 1 above.
[55]Einstein Papers, Princeton University, microfilm reel I.B.1, no. 9.
[56]The letter to Besso is dated January 3, 1916; see *Einstein/Besso Correspondence,* pp. 63–64.

tinuous transformations. What is not so clear is how well Einstein strategy succeeds; in particular, one wants to know how the crucial premise (iii) is to be justified and how (iii) and (iv) support (ii).

One of the circumstances in which Einstein thought that (iii) would be justified is mentioned briefly in the letter to Ehrenfest and in more detail in the letter to Besso:

> If, for example, physical phenomena were built up from the movements of material points alone, then the encounters of these points, that is to say the intersections of their world lines, would be the only physical reality, the only observable.[57]

But even if we grant for the sake of argument what Einstein says regarding the case of point particles, how does a similar conclusion follow for the case at issue — the gravitational field? Einstein answered this question indirectly by arguing that (iii) must be true for physical reality in general and, therefore, for the gravitational field in particular. From the equation of 'reality' with the 'observable' one can guess how the argument goes. In his 1916 paper on the foundations of general relativity, Einstein proceeds from the premise that 'all our space–time verifications invariably amount to a determination of space–time coincidences' to 'all our physical experience can ultimately be reduced to such coincidences' and thence to the tacitly understood conclusion that all of physical reality can be so reduced.[58]

The influence of Mach, who is explicitly mentioned in the introduction of the paper, is evident here in the emphasis on the means of verification and the use of epistemological considerations to establish constraints on ontology. But although Mach's searching skepticism and his critique of Newtonian mechanics had a deep influence on the early Einstein, orthodox Machian positivism was never Einstein's way. Einstein never saw the Machian atom of experience, the individual sensation, as the basic building block for nature or for scientific theories. And Einstein accepted Mach's insistence on the primacy of epistemological questions only when there was doubt or uncertainty; when doubt and uncertainty faded for Einstein, so did the importance of epistemological questions. Writing to Besso in 1914 he says:

> I do not doubt any more the correctness of the whole system [the theory of gravitation] may the observation of the eclipse succeed or not. The sense of the thing is too evident.[59]

And to a large extent, it was the general theory of relativity which turned Einstein consciously against Mach's philosophy of science. In the Herbert Spencer lecture at Oxford in 1933, Einstein stated that the general theory brought the realization that the fundamental concepts of physics cannot be

[57]*Einstein/Besso Correspondence,* p. 64.
[58]Ref. 7, Section 3.
[59]*Einstein/Besso Correspondence,* p. 53.

arrived at by 'abstraction' from experience.

> Scientists . . . [of earlier times] were for the most part convinced that the basic concepts and laws of physics were not in a logical sense free inventions of the human mind, but rather that they were derivable by abstraction, *i.e.*, by a logical process, from experiments. It was the General Theory of Relativity which showed in a convincing manner the incorrectness of this view.[60]

Moreover, Einstein's growing hypostatization of space–time was directly contrary to Mach's picture of a world composed of sensations in ordinary space and time; indeed, one supposes that Mach would have perceived the space–time of general relativity as just another form of the 'metaphysical monster' he had detected in Newton's absolute space and time.[61]

Thus, the irony is doubly heavy: while the general theory of relativity led Einstein to an anti-Machian outlook, the theory itself was motivated in part by specific ideas and a general philosophical orientation derived from Mach, and at the crucial juncture in 1915, Machian positivism was used to legitimize the dual embrace of causality and general covariance.

The second notable feature of Einstein's strategy concerns a failure to distinguish sufficiently clearly between coordinate and point transformations. Because the strategy is most directly concerned with covariance, (iv) must be interpreted as referring to coordinate transformations. But then (iv) is trivially true, and it is true not just for an impoverished physical reality characterized by simple point coincidences of particles but equally for a richer reality characterized by any set of intrinsic geometric object fields, no matter how complicated or varied. On the other hand, the correct kernel of Einstein's causality argument concerns the freedom to perform diffeomorphisms — *point* transformations, not coordinate transformations, though of course the point mapping can be given a coordinate representation as was done above in Section 4 — and these mappings, do, *prima facie*, change the physical situation, be it simple or complicated.

One way to fill in the lacuna in Einstein's attempted reconciliation of causality and covariance is suggested by his remark to Ehrenfest that a coordinate system signifies 'nothing real'. If this attitude is extended to the underlying manifold of which the coordinate systems are representatives, then the following position can be entertained. The fact that generally covariant field equations determine the metric field only up to an arbitrary diffeomorphism (when $T = 0$) does not show that the equations fail to uniquely fix the total physical situation. The contrary impression is fostered by taking too literally the mathematical apparatus of an underlying differentiable manifold and various object fields on this manifold. For the space–time manifold is not

[60] A. Einstein, *On the Method of Theoretical Physics,* (Oxford University Press, (1933), p. 11.

[61] For more on Einstein and Mach, see G. Holton, *Thematic Origins of Scientific Thought* (Cambridge: Harvard University Press, 1973).

something over and above the collection of all physical events; rather, the mathematician's M is only a formal device which allows a convenient description of physical events. Thus, two diffeomorphic variants — $(M, \mathbf{g})$ and $(M, \theta^*\mathbf{g})$ — are simply different but equivalent ways of describing the same physical content. As it stands, this interpretation is highly programmatic, and one naturally wants to know how the details are to be filled in before passing judgment. But at least the program does hold out the hope of a coherent means for implementing Einstein's strategy.[62]

Unfortunately, the suggested implementation appears to be incompatible with Einstein's further remark to Ehrenfest to the effect that two distinct '*g*-systems' can be 'equally realized' in the same region of the continuum. Not surprisingly, Ehrenfest has put a question mark in the margin beside this sentence of Einstein's letter, and again, not surprisingly, the correspondence on this topic continued with Einstein trying, both by means of abstract arguments and concrete examples, to win Ehrenfest over to his interpretation of general covariance. While Einstein was tenacious as always, he did concede in a letter dated January 5, 1916, that

> I cannot blame you for not understanding the permissibility of generally covariant equations, because I myself needed so long to become clear about this point.[63]

But much more time was needed before the dawning of any geniune clarity.[64]

7. Conclusion

Einstein's struggles with general covariance were both magnificent and paradoxical. They began, we believe, with an idea at once both right and wrong: that a generally covariant formulation of gravitational theory would permit the statement of laws that hold in every frame of reference, and that such a statement would be a physically important generalization of the principle of relativity, consistent with the principle of equivalence. From then on, Einstein produced physical and conceptual analyses of the requirement of covariance to fit his technical needs. Thus, convinced by Grossman that no second order

[62]For some remarks on how a similar strategy might be used to explicate Leibniz's views on space and time, see J. Earman, 'Leibnizian Algebras and Leibnizian Space-Times', in R. Butts and J. Hintikka (eds.), *Proceedings of the Fifth International Congress of Logic, Methodology, and Philosophy of Science* (Dordrecht-Holland: D. Reidel, 1977).

[63]Einstein Papers, Princeton University, microfilm reel I.B.1, no. 9.

[64]The second part of David Hilbert's paper, 'Die Grundlagen der Physik', *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen,* 1917, 53–76, contains a discussion of covariance and causality which bears a certain resemblance to Einstein's treatment. Hilbert says that causality is not violated by generally covariant equations because from the appropriate initial data one can infer 'necessarily and unambiguously' all those propositions about the future behavior of the system which 'have physical meaning' (p. 61). Physically meaningful propositions are taken to be ones about invariants, *e.g.*, coincidences. Hilbert makes no reference to Einstein's 1916 paper of the foundations of general relativity theory, and we have not found any evidence of a direct link between Hilbert's views and Einstein's views on this matter.

tensor expression was available to generalize Poisson's equation, Einstein produced arguments to show that general covariance was impossible for gravitational theory. Pressed by Mie about the lack of general covariance for his 1913 field equations, he produced an argument to show that covariance was of no physical importance. When after a month's furious work, Einstein finally hit on satisfactory generally covariant equations in November of 1915, he rapidly produced arguments to show that covariance is essential. Almost none of these arguments are, from either a modern or a contemporary standpoint, satisfactory; typically they confuse coordinate systems and reference frames, coordinate transformations and point transformations, covariance and space–time structure. But they are also paradoxical. Einstein gave Machian arguments for the very theory which was to lead him to a more realist view, and he seems best to have understood the physical significance of covariance exactly when he was most in error about what could and could not be done within tensor calculus.

Shortly after Einstein's 1916 *Annalen* paper, Kretschmann wrote a paper on the principle of relativity,[65] claiming that general covariance was of no physical significance and that any equation could be made to conform to a generally covariant one in a given coordinate system. The physical significance and precise sense of the principle of relativity had to lie elsewhere. Although Kretschmann had no better arguments than Einstein had for the same thesis two years before, he was correct enough. Einstein replied that general covariance has nonetheless a great heuristic value,[66] and he was exactly right.[67] While there is little evidence that the arguments that Einstein gave for and against the possibility or necessity of generally covariant laws actually convinced him or anyone else of their conclusions, there is abundant evidence that general covariance was the key factor in Einstein's approaches to and recessions from an adequate gravitational theory. Both the principle of equivalence and the demand that gravitational theory generalize special relativity remained from 1907 on; what changed between 1912 and 1916 was Einstein's conviction regarding whether a generally covariant theory was possible at all. When he believed it was, given his other principles, he was bound to come close to the theory that appeared in late 1915.

The magnificence of Einstein's intellectual odyssey lies not only in the grandeur of its conclusion, but also in its chaos, in the indirectness of the paths

[65]Ref. 33.

[66]A. Einstein, 'Prinzipielles zur allgemeinen Relativitätstcheorie', *Annalen der Physik* **55**, (1918), 241–244.

[67]However, in his reply to Kretschmann, Einstein did not mention his earlier correct contention that the requirement of general covariance when combined with other requirements, *e.g.*, that the field equations be second order and derivable from an action principle, does have bite and may lead to a quite definite result. See A. Einstein, 'Hamiltonsches Prinzip und allgemeinen Relativitätstheorie', *Sitzungsberichte, der Königlichen Preussischen Akademie der Wissenschaften zu Berlin*, 1916, 1111–1116. We plan to discuss these matters in more detail in another paper.

that led to home. One cannot read this history without amazement at Einstein's intellect; for much of the period between 1912 and 1916 he was truly lost in the tensors, quite completely on the wrong path, accompanied by erroneous reasons he claimed to be fundamental. And yet, quite singularly, in the course of a month be abondoned his errors and their justifications. The moral, perhaps, is that a certain fickleness is more conducive to theoretical progress than is any abundance of conceptual clarity — at least if one is Einstein.