SECOND EDITION

Engineering Economics for Capital Investment Analysis

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Preface

The objective of this book is to present the basic concepts and the analytical techniques for the economic evaluation of engineering projects in both the public and private sectors. It covers the subject in sufficient detail to develop a thorough understanding of the decision-making process in capital investment planning with a minimum amount of descriptive material on the decision environment of the public or private institutions. It emphasizes the application of analytical techniques to a great variety of problems that may confront engineers and managers who are concerned with capital budgeting.

As an introductory text on engineering economic analysis, the book concentrates on the principles that provide a solid foundation in the pursuit of more advanced topics such as benefit-cost analysis and financial management. It is self-contained in offering a proper perspective that is essential to the understanding of capital investment decisions. One of the most important features of this book is the separation of the profit measures of investment decisions from the methods of analysis. Hence, the criteria for the acceptance of independent projects and for the selection of mutually exclusive projects are clearly defined before the various methods of analysis are introduced.

The general approach of this book is to begin with analytical techniques based on simplifying assumptions and to relax these restrictions gradually in order to introduce the more complicated elements reflecting professional practices. Although the net present value method is recognized as the most direct and unambiguous basis in the evaluation of capital projects, the benefit-cost ratio method and the internal rate of return method are also treated thoroughly because of their extensive uses in the public and private sectors, respectively. The implications of capital rationing in economic evaluations are also discussed.

In addition to the coverage of replacement analysis, price level changes, depreciation, and corporation taxes, this book provides a chapter on production cost and benefit estimation to cover some elements of microeconomics and another chapter on financial statements pertinent to decision makers who are not familiar with financial accounting. The chapter on uncertainty and risk covers the broad issues of operating risk and financial risk associated with economic evaluation of capital projects. Finally, the basic issues affecting the selection of the minimum attractive rates of return for capital investments in the private and public sectors are discussed respectively in the last two chapters.

This book has provided more than sufficient material for a one-semester course so that an instructor may have the flexibility to exercise discretion in the selection of topics and illustrative examples. From the experience of various instructors using the last edition of this book, several alternatives for the course contents are suggested to meet the needs of individual instructors:

1. For a basic one-semester undergraduate course, cover Chapters 1 through 13, except that for Chapters 7 and 8, only the first three sections of each chapter are included. The course may also include the first nine sections of Chapter 15.
2. For an introductory undergraduate course with emphasis on investment decisions without considering uncertainty for the public sector, cover Chapters 1 through 10 plus Chapter 17.
3. For a two-quarter or two-semester sequence in the undergraduate curriculum, cover all chapters in the book.
4. For a fast-paced one-semester advanced undergraduate or graduate course, cover the contents of the entire book but assign the elementary chapters to students for review or self-study.

This book is a joint undertaking of the authors to capture the forefront of development in engineering economic analysis. Chapters 1 through 10 and Chapter 17 were written by the first author and Chapters 11 through 14 by the second author. Chapters 15 and 16 were written jointly. The manuscript has been checked by both authors for continuity and consistency. Ample examples are completely worked out to illustrate the methods of approach for a great variety of problems with varying degrees of difficulty. The opinions expressed in this book are our own and not those of the institutions with which we are affiliated in our respective capacities.

We are indebted to Drs. Chris Hendrickson and Martin Wohl for their many suggestions and discussions in the development of the first edition and the subsequent revision for the second edition of this book. We thank others who offered many suggestions at various stages of development of the manuscript, particularly Drs. Robert G. Beaves, Bruce N. Janson, Harold E. Marshall, Francis C.

McMichael, Sue McNeil, and James Saklas. We also thank the following reviewers of the second edition who acted on behalf of the publisher in reviewing the manuscript and providing helpful comments and constructive criticisms: Dr. Paul D. Cribbins, North Carolina State University at Raleigh; Dr. Stuart E. Dreyfus, University of California at Berkeley; Dr. Michael D. Meyer, Georgia Institute of Technology; and Dr. John L. Richards, University of Pittsburgh. Appreciation is due to many others whose work has been cited in the appropriate references. While we are indebted to contributions that have improved our efforts, we are solely responsible for any errors or omissions.

Special thanks are also due to Helen Pusty and Yolande Petersen for typing the manuscript of the first edition and to Shirley Knapp for typing the revisions in the second edition. We are indebted to many users of the first edition, particularly the students at Carnegie Mellon University who made numerous comments and suggestions for improvement.

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Thomas P. Au

CHAPTER

1 Introduction

1.1 ENGINEERING PROJECT DEVELOPMENT

Engineering facilities and products are constructed or manufactured to meet human or societal needs and desires. An engineering facility or product is made up of many diverse components serving a common purpose and is often referred to as an engineering system. The entire process of planning, design, construction, operation, and maintenance of engineering systems, such as interstate highways or industrial plants, entails many important and often complex decisions. Both technological and economic considerations are essential in undertaking any new project requiring modern technology and involving a heavy commitment of resources.

The initiation of a new project follows the perception of human needs and desires in the society. An engineering system developed to satisfy such needs must pass tests of technological and economic feasibility under the prevailing institutional, social, and political conditions. First, an engineering system must be proved technologically feasible before anyone will commit resources to build it. Obviously, it would be a waste of time, effort, and resources to try to build a perpetual motion machine. Second, even if an engineering project is technologically feasible, it may not be worth building if the cost is prohibitively high compared to the benefit derived from the project. Because the advancement of technology is dynamic, the cost of a new system can be greatly reduced through continual technological improvements, and the cost relative to benefit may change rapidly. Furthermore, old technologies often become obsolete economically as well as technologically with the passage of time. Generally, economic consideration of an engineering project is based on the expected financial return on the investment. However, other institutional, social, and political forces as well as technological developments will tend to influence and modify the economic considerations. Thus, the goal of promoting economic efficiency is often tempered by other goals such as social equity, quality of life, or preservation of the environment.

Apart from the bare minimum requirements for human survival, societal needs are not necessarily based on the demand in a competitive market, but on value judgments. What are considered needs in one society may be considered luxuries in another. Hence, goals and constraints as well as needs and requirements are value-laden. For example, in some developing countries, the emphasis on improving the national welfare through increasing economical efficiency is so great that the only hindrances to an engineering project are technological and economic constraints. Goals such as comfortable working conditions, health and safety standards, and environmental quality, while desirable in themselves, are often compromised in the interest of technological developments. In developed countries, such actions will be seriously challenged by the general public as well as those whose personal lives and fortunes are affected.

In the private sector, the failure of a new venture to satisfy perceived needs and desires will result in a loss on the investment and its ultimate disappearance from the marketplace. In the public sector, however, the failure of a new project to meet the societal needs is less evident since public welfare cannot be conveniently measured, and the project may turn out to be a liability that cannot easily be disposed of. Hence, in the development of an engineering project, it is important to assess its benefits and costs before making a choice. Although the motives in the initiation of a project are different in private and public sectors, there are enough similarities in the approaches for assessing the merit of a project that are applicable to both. Consequently, we shall examine the basic principles that will guide us to make better choices in the development of engineering projects.

1.2 THE DECISION-MAKING PROCESS

In general, the decision-making process for engineering project development includes the following basic steps:

1. Defining the problem
2. Establishing objectives and criteria
3. Generating alternative plans and designs
4. Evaluation and choice
5. Implementation and control

Although these steps are often carried out in sequential order, backtracking and iteration are necessary when more detailed investigation in a subsequent step leads to new insight and results in modification of actions taken in previous steps. Thus, the delineation of this sequence and the grouping of actions for various steps are not rigid or unalterable.

Before developing a definitive statement of a problem, we must first recognize and understand the environment in which the problem exists. This environment includes the available technology, economic requirements, and social values of the time and locale, among other factors. Then and only then can we set realistic goals and constraints for initiating a new engineering project. It is important to distinguish the causes, rather than the symptoms, of a problem that we wish to solve. For example, if the problem is the high cost of energy, we should strive to find and develop new energy sources or make more efficient use of available energy. To introduce price controls alone would not only fail to solve the problem but would aggravate the problem by maintaining an artificially high demand for scarce energy. To avoid self-deception, we must collect information related to the problem and examine the issues to determine which factors or parameters are most relevant. It is important to realize that in defining a problem, decision makers inject value judgments into the perception of the environment, including a propensity or aversion to taking risks in the face of uncertainty. Value judgments are also exercised in assessing the "needs" and "requirements" according to the prevalent trend of the society, which is forever changing at a rapid pace. Thus, when we formulate the problem in a rational framework for analysis, we must recognize the limitations of our knowledge and understanding of the situation.

Since the goals of a proposed project are often broadly stated, they must be translated into specific objectives and operational criteria that can be measured in specific terms. It is important to note the hierarchical nature of the decision making in an organization. The goals at the highest level are reiterated as specific objectives and operational criteria as guideposts for setting goals at the next lower level. For example, a company may have several divisions that operate independently but must follow goals that are compatible with those of the company. Each division, in turn, may consist of several departments that must work toward a common goal. Depending on the major function of an organization, the information flows through various levels of the hierarchy in influencing the goals, objectives, and criteria may vary considerably.

The generation of alternative plans to meet the criteria for a proposed project involves the inductive process of synthesizing various solutions to the problem. The activities in synthesis are often described as an art rather than a science and are regarded as more akin to creativity than to knowledge, or to judgment than to methodology. The decision maker attempts to anticipate the consequences of various solutions and to make trade-offs among such factors as safety, economy, and aesthetics. The decision maker must search extensively and select judiciously among numerous possible alternative plans those that are most promising for further consideration.

Moreover, the technological and economical feasibility of each of the promising alternative plans must be scrutinized, and the infeasible ones are thus eliminated. In general, the conceptual models of proposed alternative engineering systems are analyzed and evaluated with regard to the characteristics of the relevant parameters and variables, and, if necessary, are modified and developed to optimize their performance with respect to cost. Useful as the quantitative relationships between performance and cost are in making a choice between alternative plans, they are not necessarily the only basis for comparison since they often do not include intangible factors, such as aesthetics, and social benefits and costs that cannot be readily quantified. In the final analysis, the decision must be based on the qualitative evaluation of intangible factors as well as on the quantitative evaluation of quantifiable parameters.

After a choice is made, the implementation of the proposed project can commence upon the commitment of adequate resources. In the process of implementation, it is necessary to exercise control to ensure that the project is carried out as planned and to provide feedback to the decision maker if adjustments in the original plan are necessary. For example, in the case of constructed facilities, not only must the detailed plans of the facilities be faithfully executed during the construction, but also the operating and maintenance procedures for the completed facilities must be followed.

This description of the decision-making process is intended to illustrate the complexity of many problems confronting a decision maker. There are, of course, different degrees of complexity in the problems at different levels of the hierarchy within an organization, or at different times or locales. In the final analysis, a decision about initiating an engineering project must be based on as complete a consideration of all relevant factors as the decision maker can comprehend and on as detailed an analysis of the problem as the benefit from the analysis can justify its cost.

1.3 RESOURCE ALLOCATION DECISIONS FOR PUBLIC PROJECTS

A commitment to a public engineering project usually represents the allocation of a substantial amount of societal resources for some expected social benefits. It is difficult to measure social benefits against social costs, much less express the social value as a function of technological performance. For example, when the United States decided to land a person on the moon, technological feasibility was the major consideration; there was no attempt to evaluate its social benefits in terms of national pride and the uplift of the human spirit, although the latter must have been weighed in the decision. Even in allocation of resources to civilian projects, the consideration of social benefits and social costs is far from simple.

One of the difficulties in project evaluation is that the stated goals for such projects are often broad and evasive. For example, one of the historic pieces of legislation involving the government in large-scale civilian projects, the Flood Control Act passed by the U.S. Congress in 1936, states in part[1]:

[1] See *United States Code*, p. 2964. Washington, DC: Government Printing Office, 1940.

> [T]he Federal Government should improve or participate in the improvement of navigable waters or their tributaries, including watersheds thereof, for flood control purposes if the benefits to *whomsoever they may accrue* are in excess of the estimated costs, and if the lives and social security of people are not otherwise adversely affected. [Italics added]

The difficulties in implementing this national goal were succinctly pointed out by Marglin:[2]

> The prime objective of public water resource development is often stated as the maximization of national welfare. That this is a goal to be desired, few would question; that it cannot be translated directly into operational criteria for system design, few would deny. Translation would require not only agreement on a definition for the deceptively simple phrase national welfare, but also some assurance that the defined concept is measurable.
>
> One possibility is to define national welfare as national income. The objective of system design then becomes maximization of the contribution of the system to national income. This definition is measurable, but it has implications for the meaning of national welfare that make us unwilling to accept it as a complete expression of the broad objective. Identifying national welfare with the size of the national income not only excludes noneconomic dimensions of welfare but also implies that society is totally indifferent as to the recipient of the income generated by river-development systems, or that a desirable distribution of gains will be made by measure unrelated to the manner in which the system is designed.

Depending on the nature of a public project, one of the following specific objectives is often suggested:

1. Maximization of national income
2. Maintaining a balance between income generation and distribution
3. Encouraging regional development and environmental quality improvement

For each of these objectives, the proposed plans for the project will be considerably different. Obviously, how the problem is defined affects the ultimate solution of the problem.

While it is not possible to examine all relevant factors at this stage, a simplified example is introduced to illustrate some aspects of the decision-making process.

Example 1.1

Towns A and B are located near each other in the tributaries of a river basin as shown in Fig. 1.1. Over a number of years Town A experienced water shortages in dry seasons, and Town B was subjected to flooding at times when the snow accumulated on the mountains in the watershed melted too suddenly. The

[2] See Ref. 1.2, p. 17, for a more complete discussion following this statement.

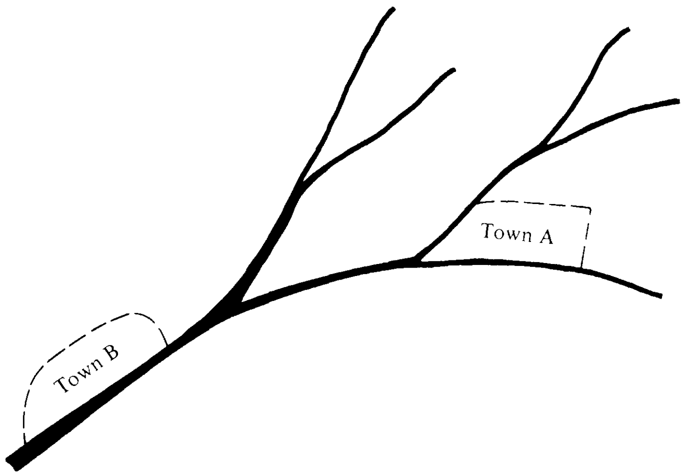

Figure 1.1 River basin under study for water control projects.

people in both towns are discussing their problems and suggesting that water conservation and flood control projects should be undertaken individually or jointly to improve their situations. How can they achieve their goals?

Traditionally, the responsibility for maintenance and improvement of navigable rivers has been delegated to the U.S. Army Corps of Engineers. However, the Corps of Engineers cannot initiate any project unless it is authorized to do so by the U.S. Congress. Consequently, the problem must first be brought to the attention of the congress members representing the districts in which the towns are located. If the problems are deemed sufficiently important to warrant intervention by the federal government, a bill must be passed by the Congress to authorize a feasibility study of initiating a project or projects to alleviate the problems.

After receiving an authorization from the Congress, the Corps of Engineers will then go through an entire planning process. It will communicate with the local people and assess their needs, and then define the objectives of the project or projects within the context of the national goal. Next, alternative plans will be studied. For example, a small dam may be built to provide a reservoir near Town A and a dike may be built near Town B to prevent flooding. On the other hand, a larger dam may be built near the confluent just upstream from Town B so that it may alleviate both the problems of water shortages in Town A and of flooding in Town B. However, as a part of the authorized feasibility study, the Corps of Engineers may also investigate the opportunity to create recreational areas for swimming and boating near the reservoir. Hence, specific objectives of the project must be clearly defined. Should the Congress authorize a study to solve only the pressing local problems at the lowest cost? Should it expand the project to benefit more people in the region even though greater expenditures may also be incurred, as in the case of creating recreational areas? Who will be benefited the most and who will have to make the most sacrifices in making the project possible? How should benefits and costs, particularly social benefits and costs, be identified and measured?

Local citizens have increasingly demanded direct participation in this phase of decision making pertaining to value judgments, particularly in times of changing values. Consequently, communications with various segments of the communities are important in developing the criteria for judging the merits of alternative plans. After the consequences of each plan in terms of benefits and costs are determined, the alternative plans then can be evaluated on the basis of the same set of criteria. Finally, the recommendations of the Corps of Engineers will be reported to the Congress, which will select the best plan, if any, according to its collective judgment, and make the necessary appropriations for constructing the project.

It should be noted that in planning, designing, and constructing a public project, the professional staff only makes decisions at the level of its professional competence. It must assess the technological problems in site selection and project construction, the economic efficiency in terms of monetary gains and losses, or the environmental impacts of alternative plans. However, the ultimate decision of what to build or whether to build at all rests with public officials who often view a decision as an attempt to resolve conflicts and interests of various groups affected by the decision.

1.4 CAPITAL INVESTMENT DECISIONS IN PRIVATE VENTURES

Although business enterprises are organized with profit motives, the goals of a private corporation may also be vague. In large corporations, most stockholders can effectively exercise their powers only to the extent of selling their stock when they lose confidence in the management. Consequently, the officers of the corporation will set goals that may include prestige, security, power, and continuity of the organization.

However, the most common specific objectives guiding the transactions of a corporation are:

1. Maximizing profits
2. Maintaining its market position
3. Stabilizing company structure with respect to assets and liabilities

Usually, these objectives are pursued within the constraints of government regulations and tax laws. Before making a substantial commitment of resources to a new venture, the dominant factors that may affect the marketing of the product must be considered.

For example, the use of solar energy for home heating and cooling may become a viable alternative if the costs of conventional fuels are expected to rise steadily. Even so, private investors are still cautious in embarking on new ventures in the solar energy field without some form of government commitment and tax incentives. Thus, the recent federal legislation on solar energy development enacted by the U.S. Congress represents only a first step to bring the developing technology into the marketplace. It will increase the prospects of making profits and decrease the degree of risks for private ventures.

Generally, all ventures involve taking risks. Executives of private corporations who authorize major investment in physical plant expansion must base their decisions on future outcomes that are far from certain. What will be the trends of technology? Will the supply of natural resources including energy and materials be adequate? Will the governmental regulations and tax policies become more restrictive? What will be the social values that may influence the manufacturing process of a new product as well as its market demand? However, long-range forecasting is inherently difficult. It is never advisable to extrapolate the trends of the next few years into the next few decades. Attempts to discern the general direction of social changes are generally more fruitful in long-range investment planning. No company wants to tie up all its assets in some marginally productive enterprises while missing more profitable investment opportunities.

While the relevant factors influencing the commitment of resources for private investment can be very complex, an example is given to illustrate some aspects of the decision-making process.

Example 1.2

A company that manufactures home heating and cooling equipment (furnaces, air conditioners, etc.) would like very much to develop a new product line of units that can utilize solar energy if necessary. Because of the anticipated shortage of conventional fuels, the company feels that there will be a demand for such units in the near future. How does the company go about making a decision?

First, the company collects relevant information regarding the technology of producing solar energy for home heating or cooling. Next it determines which type of equipment is most suitable for the intended purpose. It is found that there are no technological barriers for home heating and cooling with solar energy, at least as a supplementary source of energy for new construction. The significant obstacles to widespread use of solar energy are high initial costs, lack of support from utility companies, rigid building code requirements, no guarantee of sun rights, and uncertainty of public acceptance. Consequently, the company wants to determine if the latest technology available may possibly remove some of these obstacles.

In a quick review of the climatic conditions in the United States, it appears that in most regions, solar energy can best be used as a supplementary energy source for home heating and cooling. Even though recent research reports indicate that new solar cells for converting a high proportion of sunlight to electricity are technologically feasible, the company does not feel that its own capabilities and resources should be diverted to a product line which is completely unrelated to its current activities. Instead, it is interested only in the development of capacities for producing solar panels and accessories that collect, store, and supply solar energy to supplement conventional fuels. Thus, the company concentrates its initial investigation on the types of solar panels and accessories that are necessary and determines whether some available in the market are compatible with its own existing equipment and whether they are cost-effective.

However, the company must also look at the broader picture of home construction which provides supplementary solar energy. Usually, the additional construction cost of outfitting a home for solar heating and cooling is substantial. Even if the long-run cost of using solar energy is lower, it is necessary for the home builders to make arrangements to finance the additional costs for home construction and for the purchase of the solar panels and accessories for solar heating and cooling. In addition, the company is concerned about the lack of national standards governing products for solar home heating and cooling. Consequently, manufacturers must tailor their products to the most stringent local standards currently in force, which will increase the cost of production and thus reduce the potential sales volume. All things considered, the company regards this potential investment opportunity as rather risky at this time and decides to postpone introducing this new product line.

1.5 THE ROLE OF ENGINEERING ECONOMIC ANALYSIS

Technological developments for satisfying societal needs have been accepted as a way of life since the Industrial Revolution. Technological considerations establishing the relationship between the expected performance of a new development and the required investment of societal resources are a traditional part of all engineering planning and design. Thus, the planning and design of an engineering facility or product entails a variety of activities such as preliminary investigation, feasibility studies, detailed analyses, and careful specification of the process of construction or manufacturing.

A broad view of engineering was incorporated in 1828 by the Institution of Civil Engineers (London) in its charter which defined engineering as "the art of directing the great sources of power in nature for the use and convenience of man."[3] Hence, the responsibility of engineers was linked to the management and allocation of resources within the context of nature, people, and technology. With the subsequent rapid expansion of technologies, engineers were very much preoccupied with technological development and gave scant attention to the impact of technology on human beings and nature. The danger of concentrating on the development of "the

[3] See the description in "The Institution—Its Origin and Progress." *Journal of Institution of Civil Engineers (London)* 1:4 (1935).

great sources of power in nature" to the neglect of "the use and convenience" of humankind was recognized by people of vision, as forewarned in 1887 by Wellington[4]:

> [T]he distorted pre-eminence given by engineers, and by those who teach them and employ them to the pettiest details of *how* to build the separate works which make a railway, to the neglect of the larger questions of where to build and when to build, and whether to build them at all, has in it something at once astounding and discouraging.

In this age of high-tech industries and keen international competition, the questions of where to build and when to build, and whether to build at all, become more important than ever. A good understanding of the broad issues concerning the efficient utilization of resources and the impact of new technologies on society is essential in the strategic planning of engineering projects. Since it is generally impossible to quantify all relevant factors in the decision-making process, more attention will be given to economic analysis of engineering projects than other considerations. However, this emphasis on economic analysis should not detract from the qualitative evaluation of other relevant factors in reaching the final decision.

The purpose of engineering economic analysis is therefore to provide a basic understanding of the probable answers from an economic viewpoint to the questions of what to build, where to build, and when to build, or whether to build at all. As an introduction, it deals primarily with concepts and methods of analysis related to the study of economic efficiency of engineering projects in terms of their benefits over costs to the society. Other factors influencing decisions in engineering project investments are discussed only briefly. Since the evaluation of such investments based on economic analysis must be reconciled with other considerations, each problem must ultimately be dealt with in the context of its real environment. However, engineering economic analysis is an important first step toward broadening the decision maker's perspective, enabling her or him to understand the consequences of decisions with greater insight and confidence.

1.6 SUMMARY AND STUDY GUIDE

In this chapter, we have highlighted the decision-making process in the development of engineering projects. We have also examined briefly the application of such a process within the context of developing public projects and private ventures. Although the economic efficiency of such projects in terms of their benefits over costs to the society has been emphasized, it has been pointed out that other factors, such as equitable distribution of income, stabilization of the economy, and preservation of the quality of life valued by the society, also play a significant role in the decision-making process. All these objectives must be pursued within the existing political and institutional constraints.

[4] See Ref. 1.4, p. 7.

While acknowledging various social and political concerns in the development and selection of engineering projects, the primary objective of this book is to provide a systematic treatment of the principles of economic analysis leading to the acceptance and selection of economically efficient projects for investment. In subsequent chapters, the concepts and methods of analysis will be discussed in depth, but the institutional environments associated with the application of these concepts and methods will receive only cursory reference. Since most activities related to engineering construction or manufacturing processes as well as the transactions in business and financial worlds are too complex for detailed description in an introductory course in engineering economic analysis, the examples used in this text are often simplified models of reality, focusing the attention only on the specific topics under consideration. The students are expected to grasp firmly the basic principles of making investment decisions and acquire the proficiency in applying these principles correctly.

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PROBLEMS

P1.1 In the 1960s and 1970s, many older cities in the United States were encouraged and heavily subsidized by the federal government to redevelop or renew the blighted urban areas. The results were mixed. If you were given the opportunity to advise the Congress for a new legislation for the redevelopment of the blighted urban areas now, what would you consider as the major objectives of the program?

P1.2 In recent years, many industries have considered using robots to take over some of the jobs now performed by human workers. If you were a production manager responsible for making a decision of introducing robots into your own plant, what would be the most important factors that would influence your decision?

CHAPTER

2 Basic Concepts of Capital Investment

2.1 THE NATURE OF INVESTMENT

Investment may be defined as the commitment of resources to some economic activity in anticipation of greater returns or benefits in the future. Thus, investment clearly implies foregoing consumption now with the expectation of more consumption at a later time. In a strict economic sense, investment takes place only when natural resources are converted into real assets such as the plant and equipment. Thus, an investment occurs when a user of capital, such as a private corporation or a public agency, takes the savings resulting from foregoing present consumption out of the capital or financial market and spends them on the acquisition of new assets.

In the private sector, savings may flow into investments through the creation of equity or debt. *Equity* refers to the value of the stock of a corporation, and new assets may be added either by issuing new shares of stocks or by ploughing back corporate earnings. *Debt* may be incurred either by taking loans from banks or by issuing bonds to the public which are redeemed at a later date. Individual savings are channeled into investments through intermediary financial institutions which provide a market mechanism to establish the relative values of various investment opportunities. For example, when a person deposits his or her personal savings in a bank, no real investment is made until someone borrows the money from the bank and spends it for home building, plant expansion, and similar activities.

In the public sector, present consumption may be deferred involuntarily through taxation by the government for the purpose of investment. The investments in public projects may also be financed by public debt in order to achieve socially desired goals. In each case, the market mechanism for capital formation is either replaced or influenced by the action of the government.

Investment in new real assets, whether in the private or public sector, is referred to as *capital investment*. An investment project of this nature requires a long-term commitment of resources with returns to be realized over the life of the real asset. The purpose of an economic evaluation of an investment in a project is to determine whether the benefits will outweigh the costs. What constitutes a satisfactory return depends on investor attitudes and on the resources available. On the other hand, the commitment of savings into investment through various means provided by financial institutions is referred to as *financing*. Stocks, bonds, and bank notes representing claims on real assets which are financed by the issuance of these papers are called *financial assets* or *securities*. The purchase of such assets or securities for investment is referred to as *financial investment*.

It is therefore important to distinguish between an economic evaluation of a project and a feasibility study of its financing. The former refers to an analysis to determine whether a capital investment is economically worthwhile, assuming that the resources are available. The latter refers to an investigation to find the means of obtaining necessary funds to acquire a specified capital asset. These two types of analysis are different and they will be treated separately.

Example 2.1

Mr. Wilbur recently bought 100 shares of existing stock of Michigan Mining Company in the stock market. In a strict economic sense, is this purchase a new investment?

Mr. Wilbur merely acquired the ownership of the shares that were relinquished by someone else. The purchase is a financial investment and not a capital investment. In a strict economic sense, it is not a new investment.

Example 2.2

The Manor County Sanitary Authority plans to construct a new sewage treatment plant which will cost $5 million. It has engaged the professional services of a consultant to investigate the possibility of raising the money for construction through issuing public bonds. After a careful analysis of the bond market, the consultant recommends an annual interest rate of 6% for the bonds in order to attract enough buyers. Is this analysis an economic evaluation?

The consultant has analyzed the feasibility of financing the construction by borrowing. It has nothing to do with the question of whether the project is worthwhile in terms of future returns of the proposed investment. Therefore, this analysis is not an economic evaluation.

2.2 TIME PREFERENCE

Although we are primarily interested in the analysis of capital investments in a strict economic sense, it is often easier to illustrate the basic principles by using simple examples in personal financial transactions familiar to most people. We shall therefore discuss the time preference of investors in this context.

The time period to which an investor wishes to look ahead is called the *planning horizon*. A person planning to make an investment is interested in the return that will produce the greatest satisfaction. Generally speaking, one can be induced to postpone consumption beyond basic necessities if one is promised a greater amount of money at some future time which will produce greater satisfaction. The preference between consumption in different periods is measured by the *rate of time preference*. For example, an investor who is indifferent regarding either the prospect of receiving \$100 now or receiving \$110 a year later is said to have a rate of time preference of 10% per year. In general, if an investor has a rate of time preference of i per time period, then he or she is indifferent toward either the prospect of consuming P units now or consuming $P(1 + i)$ units at the end of the period. Put differently, the rate of time preference of an investor is reflected by the interest rate that the investor is willing to accept for the period in lieu of immediate consumption.

Consider the simplest situation of a single sum P at the beginning of a time period and a single sum F at the end of this period. If i is the *interest rate* for this period, then the amount F is related to the amount P in this cash flow profile as follows:

$$F = P + Pi = P(1 + i) \tag{2.1}$$

in which Pi represents the interest accrued during this period. Conversely,

$$P = F(1 + i)^{-1} \tag{2.2}$$

where i is sometimes referred to as the *discount rate* because the present sum P may be regarded as a discounted value of the future sum F. Furthermore, the interest rate or discount rate i may be obtained as follows:

$$i = \frac{F - P}{P} \tag{2.3}$$

For people of limited means, the most obvious way to save for future consumption is to deposit the money in a savings account. The interest rate offered for savings accounts by a bank must be at least as high as the rate of time preference of the investor if the investor is to commit money to a savings account. Then, the interest rate offered by the bank may be regarded as the *minimum attractive rate of return* of the investor. In reality, the situation is far more complex than this simplified explanation. For example, the investor may prefer depositing the money in a savings account over other investment opportunities because it is risk-free, it can easily be liquidated, or it is simply more convenient. In general, the minimum attractive rate of return of an investor depends on a number of factors. If a sum of money is committed to a particular investment, the same sum cannot be invested in other opportunities to earn a return. The cost of foregoing other investment opportunities is called the *opportunity cost* and can be expressed in terms of the rate that the best foregone opportunity will earn. By saving or borrowing against future returns, an investor will be able to select the pattern of consumption that he or she enjoys most by making proper choices of investment and financing.

2.3 A SIMPLIFIED VIEW OF THE MARKET ECONOMY

An important aspect of economics that concerns the evaluation of capital investment is the study of the allocation of scarce resources among alternative uses. The allocation process is said to be *economically efficient* when the total amount of benefits received by members of society from the consumption of all commodities is maximized under the prevailing income distribution. In a production and exchange economy, the economic efficiency is dependent on the following premises:

1. The objective of production is the satisfaction of individual wants so that the goods and services desired by the members of society are produced.
2. An interconnected market system sets the prices of goods and services according to individual preferences and productive technology.

The actual functioning of a production and exchange economy is extremely complex, and the possibilities of market failure will not be considered here. However, it is possible to examine a highly simplified view of the market system in order to understand the role of the financial institutions as intermediaries for production and exchange. The demand of investment capital can be described by a *demand curve or schedule* which represents the relationship between the interest rate for borrowing and the amount of capital demanded. Similarly, the supply of investment capital can be described by a *supply curve or schedule* which represents the relationship between the interest rate for lending and the amount of capital supplied. These curves are shown in Fig. 2.1. The willingness of the users of capital to borrow at certain interest rates depends on the rate of productivity in their potential uses of the

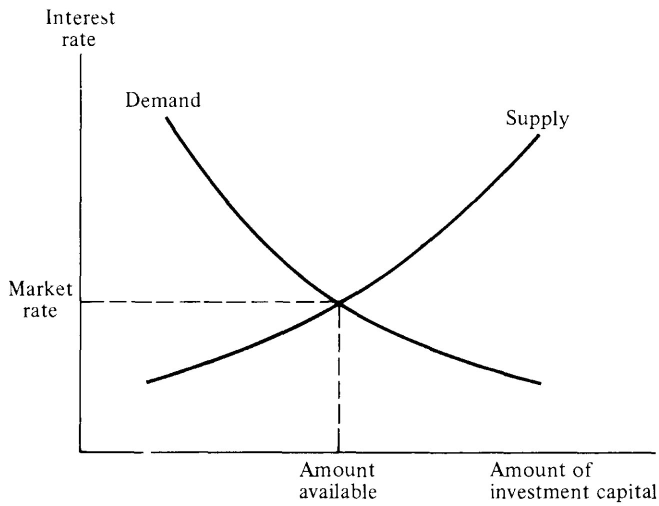

Figure 2.1 Aggregate demand and supply curves for investment capital.

capital. As the interest rate for borrowing decreases, more users of capital are willing to borrow and thus the aggregate demand for capital increases. Conversely, the willingness of the suppliers of capital to lend at certain interest rates depends on their rates of time preference. As the interest rate for lending increases, more suppliers of capital are willing to lend and thus the aggregate supply of capital increases. Since the users of capital must bid for what they want and the suppliers of capital will try to maximize their satisfaction, the equilibrium of supply and demand is reached when the amount of capital supplied and the amount demanded are equal. Hence, under the conditions of perfect competition, the point of equilibrium must be at the intersection of the supply and demand curves in Fig. 2.1. The interest rate corresponding to the point of equilibrium in a perfectly competitive market is referred to as the *market interest rate*.

In a perfectly competitive market, investors can borrow or lend freely at the market interest rate. However, investors will borrow only if investment opportunities exist that will earn a higher return than the market interest rate. Similarly, they will loan the money to others only if they cannot receive a greater satisfaction in their time preferences than the market interest rate. Hence, the market interest rate may be regarded as the minimum attractive rate of return for an investor.

In reality, the market is imperfect and different interest rates may exist because of transaction costs in borrowing and lending. For example, investors may not be able to find willing lenders for worthwhile projects if they are already heavily committed to other projects, or a lender may impose a higher interest rate for financing projects involving greater risks for fear of the bankruptcy of the investor. Such conditions which prevent the free flow of investment capital are referred to as *capital rationing*. These conditions may cause complications in economic evaluation. Nevertheless, they can be taken into consideration in the analysis.

2.4 THE CLASSIFICATION OF INVESTMENT PROJECTS

Before a new capital investment is undertaken, an extensive search is usually made to spot potential investment opportunities. It is important to recognize the characteristics of investment projects that appear to be attractive, particularly their interrelationships with other potential projects. An investment project may be economically independent of or dependent on other projects. An investment project is said to be *economically independent* if it is technically feasible to undertake this project alone and if its expected profit or net benefit will not be affected favorably or adversely by the acceptance or rejection of any other projects; otherwise an investment project is said to be *economically dependent*.

For example, replacing a small, deteriorating bridge with a new one may be regarded as an economically independent project if no other alternative or supporting actions related to the replacement are anticipated. In reality, a project is seldom totally economically independent; however, if the effects from other possibilities are small, then for all practical purposes the project is considered independent. On the other hand, a proposed bridge for river crossing at a given site may be economically dependent on a network of access roads to the site which is not presently in existence, since without the network of access roads, the net benefit of the bridge cannot be realized. Alternatively, if an underwater tunnel instead of a bridge is proposed near the same site, then neither one is economically independent of the other since the construction of one will affect the expected net benefit of the other.

When two projects are dependent on each other, they may either *complement* each other or *substitute* for each other to some degree. For example, a proposed bridge and a proposed network of access roads complement each other. Suppose that the network of access roads can serve the community near the bridge site regardless of the acceptance or rejection of the bridge proposal, but the bridge will be inaccessible without the road network. Then the proposed network of access roads is said to be the *prerequisite* of the proposed bridge. On the other hand, a proposed bridge and a proposed underwater tunnel near the same site of the river crossing may have a substituting effect for each other for lack of sufficient traffic volume to justify the construction of both. In fact, if the construction of one makes it technically infeasible to construct the other or if the net benefit expected from one proposed project will be completely eliminated by the acceptance of the other, then they are said to be *mutually exclusive*.

In economic evaluation, we arrange the potential projects in the form of a set of investment proposals so that they are not economically interdependent. If the investment proposals are independent, we can analyze each proposal on its own merit since its net benefit will not be affected by the acceptance or rejection of other proposals. If all investment proposals are mutually exclusive, the acceptance of one investment will automatically lead to the rejection of all others. Hence, we can rank the merits of various investment proposals and select the best among them.

To appreciate the importance of preparing independent or mutually exclusive investment proposals for evaluation, we need to understand the decision structure of the organization which will finance the investment. In most organizations, the funds available for investment are limited by the management. All capital investment proposals above a certain spending level are subject to review and approval by higher levels of management. If the set of investment proposals submitted for review and approval contains some dependent projects with complementary effects but only one of them is selected, the net benefit expected of the selected project cannot be realized because of the rejection of the remaining dependent projects. On the other hand, if the set of investment projects consists of some dependent projects with substituting effects and more than one of them is selected, the net benefit of each of the selected projects will be adversely affected by the acceptance of dependent projects. By presenting investment proposals as a set of projects that are not interdependent, we can make the appropriate choice according to the specified criteria of selection. This approach is applicable whether the decision is made by the person who conducts the analysis or by higher levels of management in the organization.[1]

[1] See Ref. 2.1 for further discussion.

2.5 THE CASH FLOW PROFILE OF AN INVESTMENT PROPOSAL

An investment proposal can be described by the amount and timing of expected costs and benefits in the planning horizon. Here, the terms *benefits* and *costs* are used in a broad sense to denote receipts and disbursements, respectively. The term *net benefit* is usually associated with public projects encompassing all social benefits less costs, while the term *profit* is used to denote receipts less disbursements in the private sector. Generally, the costs refer to the expected outlays and benefits to the expected proceeds over the life of an investment. When the outlays are paid in cash and the proceeds are also received in cash, an investment proposal can be represented by a stream of cash disbursements and receipts over time. For example, in the acquisition of a new machine, the typical costs are determined by the cash disbursements for the initial purchase price and the annual expenditure for operating and maintenance. The salvage value of the machine, which represents its market price at the time when it is disposed of, may be regarded either as a benefit or as a *negative* cost. The benefits may be measured by the cash value of the amount of labor saved. The investment proposal will thus be converted into a series of cash flows.

The stream of disbursements and receipts for an investment proposal over the planning horizon is said to be the *cash flow profile* of the investment. For private corporations, the cash flows may be estimated from expected *gross revenues* and *expenses*. Since some of the cash receipts are subject to taxation, the series of cash flows representing each investment alternative refers to after-tax values. In public investment projects, estimated benefits include only those that are quantifiable in monetary terms, and estimated costs represent the price to the society for obtaining the desired benefits. The series of cash flows representing such an investment alternative refers to the before-tax values since government agencies are tax-exempt.

Although the amount and timing of the cash flows associated with each investment proposal can only be estimated in advance, they are generally assumed to be known with certainty for analysis. In reality, the future is uncertain, and various factors affecting the cash flows, such as inflation, change in tax rate, etc., must also be considered in making an investment. It is sufficient to point out at this time that an investment proposal can be appropriately represented by its cash flows at regular time periods, say years. Then, the cash flow at each time period is said to have a time value corresponding to the timing of its receipt or disbursement. The subject of the time value of money will be treated in detail in Chapter 3.

Example 2.3

Office copying equipment that costs $6,000 now is expected to be kept for 5 years. At the end of 5 years, it will have a salvage value of $800. The annual operating and maintenance cost is $1,000 per year, and the annual benefit generated by the equipment is $3,000 per year. Assuming that the receipts and disbursements are made at the end of the year, except the initial cost which is paid at present (end of year 0), describe this investment proposal in terms of its cash flows.

The series of cash flows describing this investment proposal may be represented by the annual net benefits, i.e., annual benefits in excess of annual costs as shown in Table 2.1. Note that at the end of year 5, the annual benefit is \$3,800 because the salvage value of \$800 is regarded as a benefit such that 3,000 + 800 = 3,800.

TABLE 2.1 CASH FLOWS OF THE INVESTMENT

Year	Annual cost	Annual benefit	Annual net benefit
0	\$6,000	\$ 0	−\$6,000
1	1,000	3,000	+ 2,000
2	1,000	3,000	+ 2,000
3	1,000	3,000	+ 2,000
4	1,000	3,000	+ 2,000
5	1,000	3,800	+ 2,800

Example 2.4

Suppose that the salvage value of \$800 in Example 2.3 is treated as a negative cost instead of an additional benefit. What is the cash flow at the end of year 5?

In this case, the cost will be 1,000 + (−800) = 200 and the benefit will be 3,000 at the end of year 5. Hence, the net benefit for the year will be 3,000 − 200 = 2,800, which is the same as that in Example 2.3.

2.6 OBJECTIVE OF CAPITAL INVESTMENT

The primary objective of capital investment is to maximize profit or net benefit within a planning horizon. The term *profit* is used to denote net gain from investment in a private firm, whereas in the public sector the term *net benefit* is used in the sense of social benefit less social cost. Given the difficulty and uncertainty involved in estimating benefits and costs, and in choosing an appropriate time frame and a minimum attractive rate of return, factors other than maximization of profit or net profit must be considered in the selection of capital projects. Social, political, and environmental concerns as well as economic factors may be significant in influencing project selection.

Once the objective of profit maximization is accepted, imperfect though it may be, a decision criterion reflecting the stated objective may be established for the economic evaluation of proposed capital projects. An investment decision criterion consists of two elements: a merit measure and a set of decision rules associated with this measure. Historically, many merit measures have been introduced, each requiring a set of decision rules to implement the stated objective. We consider here only elementary examples to illustrate two basic types of merit measures: the net future value and the internal rate of return in the context of one-period investments. In this section, we examine the nature of these two merit measures and their associated decision rules to accept or reject a proposed project. In the following two sections, we consider separately the decision rules related to these two merit measures for the selection of the best among a group of mutually exclusive proposals.

It is important to realize that real assets are size-dependent and hence require investments of different magnitudes. Since the concept of profit maximization is based on the assumption of perfect capital markets in which an investor can lend or borrow freely, the size of investment on a proposed project is not at issue in establishing the decision criteria. In other words, as long as an investor can borrow or lend at the minimum attractive rate of return (MARR), it is worthwhile to invest in a proposed project that will generate a profit greater than the MARR regardless of the size of the investment.

The net future value (NFV) is a *direct* measure of the *size* of profit or net benefit at the end of the investment period that the investor would have gained by having invested in the proposed project instead of investing in the foregone opportunity. Hence, the comparison between the gross profit resulting from a proposed project and that obtained by investing at the MARR has already been taken into consideration in this measure. In that sense, the NFV is a *relative* measure of profit dependent on the MARR.

Suppose that an investor invests a sum P_0 in a proposed project now and expects to receive a sum F_1 a year later, and that the minimum attractive rate of return of the investor is i^* per year. Then, by letting $i = i^*$ in Eq. (2.1), we find F^* at the end of 1 year as follows:

$$F^* = P_0(1 + i^*)$$

where F^* is the amount that the investor could have obtained by investing in the foregone opportunity. The difference between F_1 and F^* is referred to as the *net future value* (NFV):

$$\text{NFV} = F_1 - F^* = F_1 - P_0(1 + i^*) \tag{2.4}$$

For example, consider a sum of \$1,000 banked at an interest rate of 6% per year. At the end of a year, your bank account will grow to an amount of $F^* = (1{,}000)(1 + 0.06) = \$1{,}060$. Now suppose a friend has invented a gadget and asked you to invest \$1,000 to help bring the gadget to market. Suppose that she first offers to pay you back \$1,050 a year later but when you demur, she raises the offer to \$1,100. Clearly when you compare $F_1 = \$1{,}050$ in the first offer with the \$1,060 that you could have received from the bank, you wonder if you could afford to make the sacrifice for a friend. However, when the offer is raised to $F_1 = \$1{,}100$, you are better off financially by accepting it.

In general, a positive NFV indicates a net gain at the end of 1 year, and a negative NFV indicates a net loss relative to the foregone opportunity. The case of NFV $= 0$ is neutral but may be treated as the limiting condition for accepting a proposed project. Using the NFV as the merit measure, the decision rule for accepting or rejecting a proposed project can be stated as follows: "Accept the project if NFV ≥ 0; reject it otherwise."

The internal rate of return (IRR) is an *indirect* measure of profit or net benefit because it indicates the *percentage rate* rather than the *size* of the profit at the end of the investment period. However, the IRR is an *absolute* measure in the sense that it is independent of the foregone opportunity. The IRR measure by itself does not indicate whether or not a proposed project is worthwhile. A comparison with the minimum attractive rate of return (MARR) is necessary in making that decision.

Consider again that you have deposited \$1,000 in a bank that pays an interest rate of 6% per year. Your parents have a larger sum deposited in a different type of account that pays an interest rate of 10% per year. An entrepreneurial classmate who knows your family well comes up with a new invention and promises to repay the principal plus 8% per year of dividend for whatever amount either you or your parents may invest in his invention. One can expect different reactions from you and your parents because of different foregone opportunities. In your case, a rate of return $i' = 8\%$ is better than your MARR of $i^* = 6\%$, but for your parents $i' = 8\%$ is lower than their MARR of $i^* = 10\%$.

According to Eq. (2.3), the rate of return i' for 1 year from an investment P_0 which yields an amount F_1 at the end of the year is

$$i' = \frac{F_1 - P_0}{P_0} \tag{2.5}$$

More specifically, the quantity i' represents the rate of return from investing in the proposed project internally without considering any external opportunity foregone, and is therefore referred to as the *internal rate of return.*

If IRR is used as a merit measure, the decision rule for accepting or rejecting a proposed project in the one-period investment becomes: "Accept the investment project if IRR $\geq$ MARR; reject it otherwise." The case of IRR = MARR is included as the limiting condition for accepting a proposed project.

Example 2.5

The City of Blacksboro owns a vacant lot and has agreed to sell it to a builder a year later for development. In the interim period, the city can spend \$10,000 for improving the ground and rent it to a parking lot operator for a rent of \$12,100 to be paid at the end of the year. The city requires a minimum attractive rate of return of 10% per annum, which represents the foregone opportunity. Is it worthwhile to improve the vacant lot?

The net benefit of the proposed project at the end of the year is represented by the net future value, which can be obtained from Eq. (2.4):

$$\text{NFV} = 12{,}100 - 10{,}000(1 + 0.10) = \$1{,}100$$

Thus, the city will gain an amount of \$1,100 at the end of the year by investing \$10,000 in parking lot improvement instead of investing in the foregone opportunity which would yield a return at the minimum attractive rate of return at 10%. Hence, the NFV merit measure indicates that the proposed project is worthwhile.

On the other hand, the rate of return from the proposed project can be obtained from Eq. (2.5):

$$i' = \frac{12{,}100 - 10{,}000}{10{,}000} = 0.21 = 21\%$$

The internal rate of return of 21% is clearly higher than the minimum attractive rate of return of 10%. Hence, the IRR merit measure also indicates that the project is worthwhile in comparison with the opportunity foregone.

2.7 DECISION CRITERIA FOR DIRECT MERIT MEASURES

The net future value (NFV) is not the only direct merit measure regarding profit or net benefit. However, since it represents the net amount that an investor would gain by having invested in the proposed project over the foregone opportunity at the end of the investment period, it is most easily understood. The NFV decision rule for ranking the merits of a set of mutually exclusive proposals will simply be based on the ranks of a direct merit measure since NFV is a measure of the size of net profit relative to that of the foregone opportunity at the end of the investment period. This decision rule, which is applied to one-period investments here, is applicable to investments involving multiperiod cash flows without modification. The application of this concept can also be extended to other direct merit measures introduced in later chapters.

If an organization can obtain additional funds or invest its excess funds in the capital market at a market interest rate, then the investment decision criterion for accepting noncompeting or independent investment proposals is to accept each independent project that produces an overall net benefit or profit. Under the same conditions, the investment decision criterion for selecting the best project among a set of mutually exclusive alternatives is to select the project with the highest overall net benefit or profit. When a spending limit or borrowing limit is imposed for whatever reasons, under capital rationing, each problem must be treated on the basis of additional information available.

Let N denote the net benefit or profit of an investment, B denote the total benefit realized, and C denote the total cost incurred, all of which are based on the same point in time, i.e., expressed in future values. The decision criterion for accepting an independent project without capital rationing is that the net future value N must be nonnegative. That is,

$$N = B - C \geq 0 \tag{2.6}$$

For mutually exclusive proposals, the objective is to maximize profit potential by accepting only the best of all proposals. Then, the decision criterion for profit maximization without capital rationing is to select the alternative with the highest nonnegative net future value. That is, if B, C, and N are again expressed in future values which satisfy Eq. (2.6), the criterion becomes

$$\text{Maximize } N = B - C \tag{2.7}$$

Example 2.6

Suppose that three mutually exclusive proposals are examined for improving the vacant lot in Example 2.5. Depending on the extent of grading, each proposal requires a different initial cost and yields a different rent. Their cash flow profiles are given below.

Time	Proposal 1	Proposal 2	Proposal 3
0	−\$10,000	−\$20,000	−\$14,800
1 year	+\$12,100	+\$23,210	+\$17,600

Note that time 0 denotes the present or the beginning of the time horizon and 1 year denotes the end of the investment period. The city requires a minimum attractive rate of return (MARR) of 10% per annum. Which proposal should be selected?

It is important to emphasize that the proposed projects for utilizing the vacant lot are capacity dependent. While proposal 1 will cost half as much as proposal 2 because of less grading, it is not possible to invest in two parking lots similar to proposal 1. Otherwise, doubling the investment of proposal 1 at a cost of \$20,000 would result in a return of \$24,200 at the end of the year, and such an action would clearly be superior to proposal 2. Under the assumption of a perfect capital market, the city can lend or borrow money *freely* at a MARR of 10%, and can therefore select any proposal irrespective of the size of the investment. Using Eq. (2.4), it can be found that the net future values of the three proposals are, respectively,

$$[\text{NFV}]_1 = 12{,}100 - (10{,}000)(1 + 0.1) = \$1{,}100$$

$$[\text{NFV}]_2 = 23{,}210 - (20{,}000)(1 + 0.1) = \$1{,}210$$

$$[\text{NFV}]_3 = 17{,}600 - (14{,}800)(1 + 0.1) = \$1{,}320$$

The NFV of each of these proposals is nonnegative, and each of them is acceptable. However, the NFV of proposal 3 is highest, and according to the decision criterion in Eq. (2.7), proposal 3 is the best.

If the city imposes capital rationing in the form of a budget constraint, the decision criterion can easily be modified to meet the constraint. For example, if the budget constraint is \$15,000, then the city should select the proposal with the highest NFV, provided that the initial investment is not above \$15,000. Under such a constraint, proposal 2 will be eliminated from consideration, but proposal 3 will still be selected.

2.8 DECISION CRITERIA FOR INDIRECT MERIT MEASURES

If the objective of profit maximization is accepted as the basis for investment decision, the decision criterion associated with an *indirect* merit measure must lead to the same conclusion reached by using an appropriate decision criterion associated with a *direct* merit measure. More specifically, a decision criterion associated with an IRR merit measure must yield the same conclusion as that reached by the decision criterion associated with a *direct* merit measure. Although a number of indirect merit measures will also be introduced in later chapters, the IRR merit measure is distinguished among others by its independence of the MARR. We confine our discussion here to the decision criterion associated with IRR to illustrate its application.

The internal rates of return of the three mutually exclusive proposals in Example 2.6 may be computed as follows:

$$i_1' = \frac{12{,}100 - 10{,}000}{10{,}000} = 21\%$$

$$i_2' = \frac{23{,}210 - 20{,}000}{20{,}000} = 16.05\%$$

$$i_3' = \frac{17{,}600 - 14{,}800}{14{,}800} = 18.92\%$$

Does this mean that proposal 1 is the best choice among the three proposals because it has the highest IRR? No, that is generally not true. In this case, such an inference is plainly incorrect because it is in conflict with the conclusion reached by the net future value criterion in Example 2.6. Although proposal 1 has the highest IRR, the size of the investment is smallest. Under the assumption of a perfect capital market, the city can lend or borrow money freely at a MARR of 10%. Hence, it pays to increase the size of investment in improving the vacant lot as long as the IRR from additional investment exceeds the MARR.

Initially, it appears logical to calculate the return of the proposal with the lowest initial cost. If the IRR for this investment is less than the MARR, then reject this proposal and try the proposal with the next lowest cost until a proposal is found for which the IRR is greater than the MARR. The basic objective of this approach is to spend as little money as possible but to insure that it will receive at least a satisfactory return. Once this minimum requirement is satisfied, the city wants to find out if it can get a larger return by investing in another proposal with higher initial cost. Let proposal No. 0 be the status quo of doing nothing, which incurs no cost and generates no benefit. If proposal Nos. 1, 2, 3, . . . are arranged in the ascending order of their initial costs, then successive pairwise comparisons of the proposals can be carried out, as shown in Fig. 2.2, by using a rational procedure for making choice.

At the beginning of the process, the pairwise comparison of the lowest cost proposal to the status quo is simply to find the internal rate of return of the lowest cost proposal. In each subsequent comparison, let the proposal tentatively accepted from the previous step be identified as y and the next higher cost proposal as x. The

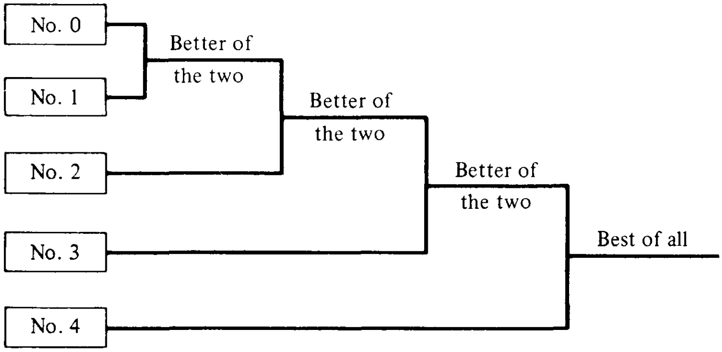

Figure 2.2 Ordering projects for pairwise comparison.

incremental costs and benefits resulting from selecting x instead of y are given respectively by the following relations:

$$\Delta C_{x-y} = C_x - C_y \tag{2.8}$$

$$\Delta B_{x-y} = B_x - B_y \tag{2.9}$$

The IRR resulting from each additional investment or increment for larger real assets is referred to as the *incremental internal rate of return* (IIRR). Thus, the incremental rate of return resulting from an increase in investment from lower cost proposal y to higher cost proposal x is

$$i'_{x-y} = \frac{\Delta B_{x-y} - \Delta C_{x-y}}{\Delta C_{x-y}} \tag{2.10}$$

If i'_{x-y} is greater than MARR, the additional investment of ΔC_{x-y} is better than the alternative of investing in the foregone opportunity; otherwise the additional investment is not justified. We must make such comparisons by pairs of proposals starting from the one with the lowest initial cost and reaching the one with the highest cost before we can decide which proposal will produce the maximum return.

A set of decision rules can be established for the IRR decision criterion involving a set of mutually exclusive proposals for one-period investments:

1. List the mutually exclusive proposals in an ascending order of the initial costs.
2. Screen the proposals in ascending order and find the first proposal for which the IRR is greater than or equal to the MARR. Accept this proposal as a tentative choice and reject prior proposals on the list, if any, for which the IRR is less than the MARR.
3. Starting with the proposal having the lowest initial cost as a tentative choice, find the increments of the initial costs and the anticipated benefits resulting from investing in the proposal with the next higher initial cost. If the IIRR obtained from this incremental analysis of costs and benefits is better than or equal to the MARR, accept the proposal with the next higher initial cost as a

better tentative choice; otherwise, retain the original tentative choice and reject the proposal with the higher initial cost.

4. Repeat the pairwise incremental analysis of costs and benefits between the current tentative choice and the next higher cost proposal until the list of all proposals in the ascending order of initial costs has been exhausted. The last tentative choice resulting from successive pairwise incremental analyses is the best proposal.

This set of decision rules involving one-period investments is illustrated schematically in Fig 2.3. This set of rules may seem complicated, but the complete set of rules for investments needed to provide an ironclad procedure for analyzing cases in-

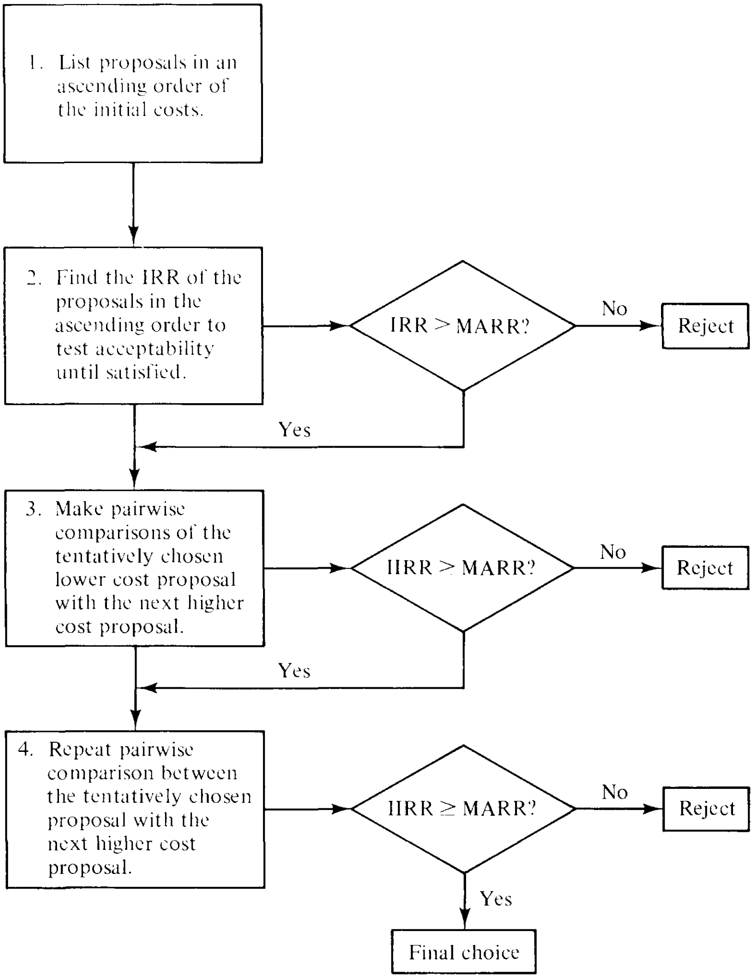

Figure 2.3 Algorithm for incremental analysis.

volving multiperiod cash flows would require far greater complexity. The causes and consequences of these decision rules are discussed in later chapters.

Example 2.7

For the data given in Example 2.6, select the best proposal on the basis of IRR decision criterion.

The three mutually exclusive proposals are first arranged in the ascending order of their initial costs. Thus, proposal 1 is the first to be considered, proposal 3 is the next, and proposal 2 is the last. The IRR for proposal 1 has been found to be 21%, which is greater than the MARR of 10%. Hence, proposal 1 is the tentative choice.

By a pairwise comparison for the increments of costs and benefits between proposal 1 and proposal 3, we find from Eqs. (2.8) and (2.9),

$$\Delta C_{3-1} = 14{,}800 - 10{,}000 = 4{,}800$$

$$\Delta B_{3-1} = 17{,}600 - 12{,}100 = 5{,}500$$

Then, according to Eq. (2.10), the incremental rate of return (IIRR) is given by

$$i'_{3-1} = \frac{5{,}500 - 4{,}800}{4{,}800} = \frac{700}{4{,}800} = 14.58\%$$

This result indicates that the additional investment of \$4,800 will yield an IIRR of 14.58%, which is still greater than 10%. Hence, proposal 3 is superior to proposal 1 since the city can earn more at the end of the year by investing in proposal 3.

Similarly, by comparing the increments of costs and benefits between proposal 3 and proposal 2, we find from Eqs. (2.8) and (2.9)

$$\Delta C_{2-3} = 20{,}000 - 14{,}800 = 5{,}200$$

$$\Delta B_{2-3} = 23{,}210 - 17{,}600 = 5{,}610$$

Then, according to Eq. (2.10), the IIRR is given by

$$i'_{2-3} = \frac{5{,}610 - 5{,}200}{5{,}200} = \frac{410}{5{,}200} = 7.88\%$$

This result indicates that the additional amount of \$5,200 will yield an IIRR of only 7.88%, which is less than 10%. Hence, proposal 2 is inferior to proposal 3 and should be rejected. After completing successive pairwise incremental analyses, proposal 3 is found to be the best choice. The same conclusion was reached in Example 2.6 on the basis of NFV decision criterion.

Example 2.8

Suppose that proposal 3 in Example 2.6 is eliminated from consideration and a choice is made between proposals 1 and 2, plus a new proposal 4 as shown below. Determine the best proposal for investment.

Time	Proposal 1	Proposal 2	Proposal 4
0	−$10,000	−$20,000	−$8,000
1 year	+ 12,100	+$23,210	+$8,500

If the NFV decision criterion is used, we can compute the NFV for proposal 4 as follows:

$$[\mathrm{NFV}]_4 = 8{,}500 - (8{,}000)(1 + 0.1) = -\$300$$

With this additional information and the results in Example 2.6 for proposals 1 and 2, a choice can be made on the basis of the NFV. Proposal 2 was found to have a NFV of $1,210, which is greater than $1,100 for proposal 1. Since the NFV of proposal 4 is negative, proposal 2 is the best solution.

However, if the IRR decision criterion is used, the computation in Example 2.7 will not help at all. We must rearrange the proposals in ascending order of their initial costs. Since proposal 4 has the lowest initial cost, compute the IRR for proposal 4 as follows:

$$i'_4 = \frac{8{,}500 - 8{,}000}{8{,}000} = 6.25\%$$

Since this IRR is less than the MARR of 10%, we reject proposal 4 and move on to proposal 1, which has the next higher initial cost. After verifying that the IRR for proposal 1 is 21%, which is greater than the MARR of 10%, we make the incremental analysis of costs and benefits between proposals 1 and 2. Thus, from Eqs. (2.8) and (2.9)

$$\Delta C_{2-1} = 20{,}000 - 10{,}000 = 10{,}000$$

$$\Delta B_{2-1} = 23{,}210 - 12{,}100 = 11{,}110$$

Then, from Eq. (2.10), the IIRR between the two proposals can be computed:

$$i'_{2-1} = \frac{11{,}110 - 10{,}000}{10{,}000} = \frac{1{,}110}{10{,}000} = 11.1\%$$

This result indicates that the additional investment of $10,000 will yield an IIRR of 11.1%, which is still greater than the MARR of 10%. Hence, proposal 2 is superior to proposal 1 since the city can earn more at the end of the year by investing in proposal 2.

For the sake of argument, suppose that you have omitted the step of checking the IRR of the lowest cost proposal 4 and proceed directly to compute the IIRR between proposals 4 and 1. Thus, from Eqs. (2.8) and (2.9)

$$\Delta C_{1-4} = 10{,}000 - 8{,}000 = 2{,}000$$

$$\Delta B_{1-4} = 12{,}100 - 8{,}500 = 3{,}600$$

Then, from Eq. (2.10)

$$i'_{1-4} = \frac{3{,}600 - 2{,}000}{2{,}000} = 80\%$$

Because this IIRR is greater than the MARR, proposal 1 is superior to proposal 4. If the IIRR between proposals 1 and 2 is computed as before, what is the use of the information $i'_{2-1} = 11.1\%$? Well, it tells you that proposal 2 is better than proposal 1 because 11.1% is greater than the MARR of 10%. However, if you cannot show that proposal 4 is acceptable, there is no guarantee that proposal 2 is good enough for acceptance even if it is better than proposal 1, and proposal 1 is better than proposal 4.

Worse yet, suppose that you have omitted the step of arranging the proposals in the ascending order of their initial costs, but place proposal 2 first and proposal 1 second. By computing the IIRR between proposals 1 and 2 blindly, you get from Eqs. (2.8) and (2.9)

$$\Delta C_{1-2} = 10{,}000 - 20{,}000 = -10{,}000$$

$$\Delta B_{1-2} = 12{,}100 - 23{,}210 = -11{,}110$$

Then, from Eq. (2.10),

$$i'_{1-2} = \frac{(-11{,}110) - (-10{,}000)}{-10{,}000} = \frac{-1{,}110}{-10{,}000} = 11.1\%$$

What does the information $i'_{1-2} = 11.1\%$ tell you? Nothing! Certainly it does not mean that proposal 1 is better than proposal 2 because you have violated the concept of *increasing* the size of investment incrementally when you omit the step of arranging the proposals properly according to the ascending order of initial costs. This step is necessary even when you have only two proposals under consideration. That is why it is so important to provide robust or ironclad decision rules when you use incremental analysis for the IRR.

2.9 COMPARISON OF MERIT MEASURES FOR ECONOMIC ANALYSIS

A precise definition of various merit measures is important to avoid misunderstanding the process of profit maximization. Whereas the NFV is a *direct* measure of the size of profit *relative* to investing at the MARR, the IRR is an *indirect* measure of the *absolute* percentage rate rather than the size of the profit and is independent of the MARR. Consequently, different decision criteria are applied to these merit measures in accepting a noncompeting or independent project and in ranking a group of competing or mutually exclusive proposals.

The relationship between the NFV and IRR measures can easily be explained for the case of one-period investment. Let IRR be denoted by i' and MARR be denoted by i^*. Then, their difference $i_d = i' - i^*$ represents the *excess* rate of return, which is a *relative* measure of the percentage rate of profit that the investor would have gained by having invested in the proposed project instead of investing in the opportunity foregone. To relate the relative measure of *percentage rate* to the relative measure of the *size* of net profit, let the initial amount of investment be denoted by P_0. Then,

$$\text{NFV} = P_0 i_d = P_0(i' - i^*) \tag{2.11}$$

For example, the net future values of the mutually exclusive projects in Example 2.6 and their respective IRR have been obtained as follows:

Measure	Proposal 1	Proposal 2	Proposal 3
NFV	$1,100	$1,210	$1,320
IRR	21%	16.05%	18.92%

Noting $i^* = 10\%$ and the initial investments P_0 given for various proposals in the example, it is easy to verify the relationship between NFV and IRR by using Eq (2.11). Thus,

$$[\text{NFV}]_1 = (10{,}000)(21\% - 10\%) = 1{,}100$$

$$[\text{NFV}]_2 = (20{,}000)(16.05\% - 10\%) = 1{,}210$$

$$[\text{NFV}]_3 = (14{,}800)(18.92\% - 10\%) = 1{,}320$$

Conversely, the size of profit for the one-period investment can also be expressed in an *absolute* measure. Let P_0 be the initial amount of investment and F_1 be the return at the end of the investment period. Then, their difference $D = F_1 - P_0$ represents the size of profit in the investment period that is independent of the MARR. While this absolute measure D may be useful for accounting purposes, it is not a preferred measure for investment decision because it is not related to the MARR.

The internal rate of return measure is intuitively appealing since it resembles the interest rate that most people are familiar with. Unfortunately, the decision criterion based on the IRR for profit optimization is deceptively complex. Even for the simple case in Example 2.6, we cannot rank the merits of the mutually exclusive proposals according to the ranks of the IRR for these proposals. For example, proposal 1, which has the highest IRR, is not the best proposal according to the NFV decision criterion. By applying correct decision rules based on incremental analysis of IRR, proposal 3 is found to be the best, as shown in Example 2.7.

Several observations can be drawn from Examples 2.6 and 2.7. First, the net future value measure provides a direct approach in ranking the merits of mutually exclusive proposals regardless of the number of proposals. It is straightforward and does not require incremental analysis. Second, the internal rate of return measure in itself is independent of the MARR but cannot be used to rank the merits of mutually exclusive proposals without resorting to incremental analysis in which a series of pairwise comparisons of proposals are made on the basis of the MARR. The incremental analysis of IRR requires the arrangement of the proposals according to the ascending order of the size of the initial investment as long as there are two or more mutually exclusive proposals. Third, both the NFV decision criterion and the IRR decision criterion will inevitably lead to the same conclusion for the objective of profit maximization if they are applied correctly. However, the NFV approach is direct and simple, whereas the IRR approach is indirect and lengthy.

Even without knowing the complications of the solution of the internal rate of return for multiperiod cash flow profiles, which are discussed in Chapter 4, it is sufficient to note that IRR is a useful merit measure as long as the limit of its usefulness is recognized. However, the IRR decision criterion for profit optimization requires a roundabout computational procedure for incremental analysis. Thus, one can choose to use IRR as a merit measure without choosing the IRR decision criterion for profit optimization in selecting the best among a group of mutually exclusive proposals.

The defenders of IRR as a merit measure often defend IRR as a decision criterion for profit optimization. Their arguments usually do not tell the whole story. For example, a frequent argument is that when various decision criteria are properly applied, they lead to the same conclusion and therefore no preference need be given to any of them. On closer examination, one may find that the IRR decision criterion has not been properly treated to ensure a correct solution because the IRR decision rules provided by them are not robust or ironclad for all possible cases of cash flow profiles. Another argument does not even concede that the NFV and IRR decision criteria will lead to the same conclusion.

Knowing only the simple illustrations in this chapter, you may ask yourself whether you can just use the NFV decision criterion and ignore the roundabout procedure associated with the IRR decision criterion for selecting the best project from a group of mutually exclusive proposals. If you are interested in the IRR as a merit measure, can you compute it for the best project after that project is selected on the basis of the more direct NFV decision criterion? If you can answer these questions to your own satisfaction as you move on to more complicated cases in subsequent chapters, you have gone a long way in deciphering the confusion that has surrounded the controversies in the engineering economic literature.

2.10 THE ROLE OF FINANCIAL ANALYSIS IN CAPITAL INVESTMENT

Although we are primarily interested in economic analysis for capital investment, it is important to point out the role of financial analysis in influencing capital project selection. As a practical matter, financing a capital project through borrowing is often a necessity, and the choice of a financing package may increase or decrease the value of a capital project.

A financial manager or advisor to a capital investment project is expected not only to raise money for the project but also to make the best deal in the process. There are numerous borrowing opportunities in the capital markets, which include many types of financial instruments such as money market funds, bonds, and notes. Even under the assumption of a perfect capital market, there are still other borrowing opportunities outside the capital market. For example, the on-going market rate for loans to college students may be 10% per year, but government subsidized student loans may charge 6% per year. Similarly, the government may offer lower subsidized interest rates for loans to developers of innovative energy sources. In each of these cases, the borrowing opportunity is specially tied to a specific project, i.e., a student must attend college as a prerequisite to qualify for a government subsidized student loan, and an energy company must be engaged in the development of innovative energy sources to qualify for the government subsidized loans. In this section, we consider only the profit or net benefit of borrowing, which is treated independently from that of investment.

In a borrowing situation, the MARR represents the *maximum acceptable rate of borrowing* to the borrower (or the minimum attractive rate of return to the lender). Since a person is free to lend or borrow at the MARR in a perfect capital market, the MARR for borrowing and that for lending can be denoted by the same value i^*. If a person wants to make use of a proposed opportunity to borrow a sum of P_0 and promises to repay a sum of F_1 a year later, the net future value (NFV) is given by

$$\text{NFV} = P_0(1 + i^*) - F_1 \tag{2.12}$$

where $P_0(1 + i^*)$ is the amount that the borrower would have to pay at the end of one year by borrowing in a perfect capital market at the MARR. Hence, the NFV is a *direct* measure of the *size* of profit or net benefit at the end of one year by taking advantage of the proposed borrowing opportunity instead of borrowing at the MARR.

On the other hand, the internal rate of borrowing i' for 1 year from borrowing P_0, which requires a repayment of F_1 at the end of the year, is defined as

$$i' = \frac{F_1 - P_0}{P_0} \tag{2.13}$$

Since Eq. (2.13) for computing at the rate of borrowing i' is identical in appearance to Eq. (2.5) for computing the rate of return i', the term i' is often referred to as the internal rate of return (IRR) for both situations. If so, it should be clearly understood that in the investment case, IRR refers to the rate of return *to the investor* from the proposed investment opportunity, whereas in the borrowing case, IRR refers to the rate of return *to the lender* (or the internal rate of borrowing for the borrower) from the proposed borrowing opportunity. The IRR for the borrowing opportunity is an *indirect* measure of profit that is independent of the MARR.

The decision criteria for accepting or rejecting a proposed borrowing opportunity are analogous to those for evaluating the economic feasibility of a proposed investment project. Using the NFV as the merit measure, the decision rule can be stated as follows: Accept the proposed borrowing opportunity if NFV ≥ 0; reject it otherwise. On the other hand, using the IRR as the merit measure, the decision rule for the one-period borrowing becomes: "Accept the proposed borrowing opportunity if IRR $\leq$ MARR; reject it otherwise."

Of course, we may also encounter situations in which mutually exclusive borrowing opportunities are available. For example, the sponsoring agency of a proposed capital project in a developing country may receive three borrowing packages of different sizes and repayment terms including interest rates from three competing international financial institutions. It is sufficient to point out that for such situations, the NFV decision criterion remains direct and straightforward, while the IRR decision criterion still requires incremental analysis of the IRR. Hence, the issues of choosing an appropriate merit measure and a set of decision rules to carry out the financial analysis of a group of mutually exclusive borrowing opportunities are essentially the same as those in the economic analysis of investment opportunities. Since our interest in financial analysis is confined to the contribution of the profit or net benefit from borrowing to the value of a capital investment project, we shall not treat the criteria for financing decisions in detail.

Example 2.9

Suppose that the Department of Housing and Urban Development (HUD) offers a federally provided loan fund of $10,000 for the parking lot improvement project in Example 2.5 with the stipulation that the city must repay $10,600 at the end of year one. Noting that the MARR of the city is 10% per annum, is it worthwhile to accept this borrowing opportunity?

The net benefit of the borrowing opportunity at the end of the year is represented by the net future value, which can be obtained from Eq. (2.12):

$$\text{NFV} = (10{,}000)(1 + 0.10) - 10{,}600 = \$400$$

Thus, the city will gain an amount of $400 at the end of the year by borrowing from HUD instead of borrowing at the MARR. Furthermore, even if the city has $10,000, it can deposit it in the capital market to collect interest at the MARR while borrowing from HUD for the parking lot improvement project and still reap the net benefit of $400 at the end of the year from borrowing in addition to the net benefit resulting from the investment. Hence, the NFV merit measure indicates that the borrowing opportunity is worthwhile.

The IRR of the borrowing opportunity offered by HUD can be obtained from Eq. (2.13):

$$i' = \frac{10{,}600 - 10{,}000}{10{,}000} = 6\%$$

Since the borrowing rate of 6% from HUD is lower than the MARR of 10%, the IRR merit measure also indicates that the borrowing opportunity is worthwhile.

2.11 SUMMARY AND STUDY GUIDE

In this chapter, we have defined the problem of capital investment and established the objectives and criteria for investment decisions. Since the problem is inherently complex, simplifying assumptions have been introduced in problem formulation. Some of these assumptions can be removed as additional concepts are delineated and more sophisticated methods of analysis are introduced in subsequent chapters.

To emphasize the basic concepts, we have specifically assumed the following conditions:

1. A rate of time preference exists at which an individual is indifferent between consumption and investment.
2. Individual savings are channeled into productive uses through the financial markets, and the equilibrium of supply and demand is reached when the amount of capital supplied and the amount demanded are equal.
3. In a perfectly competitive market, an investor can borrow or lend freely at the market rate, which may be regarded as the minimum attractive rate of return for an investor.
4. The cash flows over the planning horizon can be accurately estimated from market prices.
5. There are no price level changes, i.e., no inflation or deflation.
6. The cash flows either are nontaxable in the case of a public agency, or represent the after-tax values in the case of private corporations.
7. Risk and uncertainty can be ignored.
8. The foremost objective in economic evaluation is to maximize the profit.
9. An investment decision criterion is based on the objective of profit optimization, whatever merit measure and the associated decision rules may be.
10. Financial analysis may influence the selection of capital projects since borrowing is often necessary in undertaking a capital project.

Using simple examples, we have considered the use of the net future value of discounted cash flows for an investment as the most direct measure of its profit, although the mechanics of computing the net future value for an investment or borrowing profile represented by cash payments and receipts for more than one time period are not discussed until Chapter 4. It has been pointed out that the internal rate of return to an investor is an indirect measure of profit that generally requires careful interpretation for multiperiod investment or borrowing cash flow profiles; a detailed explanation is deferred until Chapter 4. In any case, any decision criterion that consists of a merit measure and a set of decision rules is invariably tied to the minimum attractive rate of return one way or the other.

In the economic evaluation of investment proposals, we must differentiate independent projects from the mutually exclusive projects. We have examined the criteria for accepting independent projects and for selecting the best project among the mutually exclusive proposals. Although we have not discussed capital rationing in detail, it has been pointed out that the conditions of capital rationing must be dealt with according to the available information.

It cannot be overemphasized that the economic evaluation of investment proposals is based on certain generally accepted assumptions. Once the assumptions have been agreed upon, the objective profit maximization provides a consistent approach for making the most appropriate choice. As changes in these assumptions become warranted, they can be systematically incorporated in the decision-making process. Thus, the capital investment decision may be viewed as the logical result of a consensus process, in which the critical evaluation of acceptable assumptions is at least as important as the methods of economic analysis. Financial analysis is also important in determining the contribution of the profit from financing to the value of a capital project whenever appropriate. Consequently, the results and conclusions of any analysis based on these assumptions must be subjected to review by considering the objectives of the analysis.

REFERENCES

2.1 Bierman, H., Jr., and S. Smidt, *The Capital Budget Decision,* 6th ed. New York: Macmillan, 1984.

2.2 McKean, R. N., *Efficiency in Government through Systems Analysis.* New York: Wiley, 1958.

PROBLEMS

P2.1 Salisbury Corporation plans to invest $36,000 for a "quick fix" of its equipment with the expectation of reaping a profit of $40,000 a year later as a result of this investment. The corporation requires a minimum attractive rate of return of 12% per annum. Determine whether it is worthwhile to make this investment on the basis of both the NFV and IRR decision criteria.

P2.2 Vinceton County plans to spend $25,000 to improve its conservatory with the expectation that the receipts from visitors will increase by an amount of $32,000 at the end of one year as a result of this improvement. The county requires a minimum attractive rate of return of 8% per annum. Determine whether it is worthwhile to make this investment on the basis of both the NFV and IRR decision criteria.

P2.3 Zappan Company is considering a television advertising program for an expenditure of $3,000 that is expected to boost the sales revenue by $5,000 at the end of the year. The company uses a minimum attractive rate of return of 15% per year. Determine if this plan is worthwhile on the basis of the IRR decision criterion.

P2.4 Sandiana Amusement Park is considering a proposal of investing $15,000 now to improve its facility with an expectation to receive a profit of $20,000 from this improvement. The minimum attractive rate of return is 20% per year. Determine if this proposal is worthwhile on the basis of the IRR decision criterion.

P2.5 A manufacturer of a new line of heavy equipment has made projections on the annual production costs and sales revenues at various levels of production. These costs and revenues are expressed in future values shown below. Determine the level of production (number of units produced) that will maximize the profit. Show computation.

Number of units produced	Production costs in $ million	Sales revenues in $ million
1,000	2	6
2,000	4	9
3,000	6	12
4,000	12	15
5,000	20	18

P2.6 A parcel of urban land may have four possible different uses. The costs and benefits associated with development for such uses (expressed in future values in millions of dollars) are given below. Which alternative should be selected? Show computation.

Alternative use	Benefits	Costs
1. Parking lot	4	1
2. Children's playground	10	5
3. Public park with band shelter	14	7
4. Public library	16	12

P2.7 Zappan Company is considering two advertising plans to improve its profits for the coming year. The first calls for an expenditure of $1,000 in newspaper advertisements now and is expected to increase the sales revenues by $1,500 at the end of 1 year. The second calls for an expenditure of $3,000 in television advertisements now and is expected to boost the sales revenues by $5,000 at the end of the year. The company uses a minimum attractive rate of return of 15% per year for such expenditures. Determine which plan should be used (a) if there is no budget constraint, and (b) if the advertising budget is limited to $2,000. Use the NFV decision criterion.

P2.8 Repeat Problem 2.7 using the IRR decision criterion to select the best plan.

P2.9 Consider two mutually exclusive proposals for a one-period investment with cash flow profile shown.

Time	Proposal 1	Proposal 2
0	−$70,000	−60,000
1 year	+$80,000	+69,000

For a minimum attractive rate of return of 9% per year, select the best proposal on the basis of both the NFV and IRR decision criteria.

P2.10 Sandiana Amusement Park is considering two mutually exclusive proposals in improving its facilities for the next season. The cash flow profiles for these alternatives are given below. The minimum attractive rate of return per year is 8%. Select the best proposal on the basis of both the NFV and IRR decision criteria.

Time	Proposal 1	Proposal 2	Proposal 3	Proposal 4
0	−$8,000	−$10,000	−$12,000	−$14,000
1 year	+$9,520	+$11,700	+$14,160	+$16,240

P2.11 Consider three mutually exclusive proposals for a one-period investment with cash flow profiles shown:

Time	Proposal 1	Proposal 2	Proposal 3
0	−$12,000	−$16,000	−$17,500
1 year	+$14,400	+$19,040	+$20,650

For a minimum attractive rate of return of 10% rank the merits of these proposals on the basis of the NFV decision criterion.

P2.12 Repeat Problem 2.11 using the IRR decision criterion.

P2.13 Suppose that you are eligible to obtain a government subsidized student loan of $5,000 at 6% annual interest rate for a period of one year. If you do not receive the subsidized loan, you will have to borrow the same amount from a commercial bank at 10% annual interest rate. What is the net benefit of borrowing from the government subsidized loan as measured by the net future value at the end of one year?

P2.14 To encourage the development of renewable energy resources, a federal program offers a subsidized loan of up to $80,000 at 5% annual interest rate for a period of one year to the developers of windmills which are approved as promising. A developer intends to apply for $80,000 in subsidized loan instead of borrowing this amount from a commercial bank which will charge 11% annual interest rate. If she succeeds in obtaining the subsidized loan, find the net future value of this borrowing opportunity at the end of one year.

CHAPTER

3 Compound Interest Formulas and Applications

3.1 TIME VALUE OF MONEY

Before discussing the essential features of engineering economic analysis, it is necessary to acquire a basic understanding of the time value of money. This concept is most easily understood when it is presented in the context of transactions in the financial world.

The same amount of money spent or received at different times has different values because opportunities are available to invest the money in various enterprises to produce a return over a period of time. For this reason, financial institutions are willing to pay interest on deposits because they can lend the money to the investors. Based on the specified interest rate, deposits or investments will accumulate interest over time. As a result, the future value of a present amount of money will be larger than the existing amount because of the accumulated interest; conversely, the present value of a future amount of money to be received some time later would be smaller than the indicated amount after making a discount for interest that could have been accumulated if the money were available at present. Hence, the interest rate plays a significant role in determining the time value of money.

If an amount of money is deposited in a bank, interest accrues at regular time intervals. Each time interval represents an *interest period* at the end of which the earned interest on the original amount will be calculated according to a specified interest rate. The interest accrued in a single interest period is referred to as *simple interest*. If the earned interest is not withdrawn at the end of an interest period and is automatically redeposited with the original sum in the next interest period, the interest thus accrued is referred to as *compound interest*.

In business transactions, the interest period may be a month, a quarter, or a year. For example, the interest charge for the purchase of a household appliance on credit may be compounded monthly, while the interest accrued from a savings account in a bank may be compounded quarterly. Unless otherwise stated, practically all current transactions are based on compound interest; however, the length of the interest period for compounding as well as the interest rate per period must be specified for individual transactions. When the length of the interest period is finite, the compounding operation is referred to as *discrete compounding*. As will be explained later, when the length of the interest period is infinitesimally small, the compounding operation is called *continuous compounding*.

Thus, the *interest rate* may be interpreted as the rate at which money increases in value from present to future. Conversely, the *discount rate* refers to the rate by which the value of money is discounted from future to present. This basic concept of the time value of money is illustrated by simple examples.

Example 3.1

Jefferson University has recently received a bequest of $1 million to establish a trust for providing annual scholarships in perpetuity. The trust fund is deposited in a bank that pays 7% interest per annum, and only the annual interest will be spent for the designated purpose. What is the annual amount that is available for scholarships?

Since the interest will be withdrawn at the end of the interest period, the simple interest per annum is

$$(\$1{,}000{,}000)(0.07) = \$70{,}000$$

Note that at the end of each year, the trust fund remains intact after the interest is withdrawn from the bank. Thus, $70,000 is available annually for scholarships in perpetuity.

Example 3.2

A sum of $1,000 is invested in a 2-year savings certificate that pays 8% interest per year compounded annually. What is the total amount to be received at the end of 2 years?

The principal and interest at the end of each year for the 2 years are as follows:

End of year 1	$1{,}000 + (1{,}000)(0.08) = 1{,}080.00$
End of year 2	$1.080 + (1{,}080)(0.08) = 1{,}166.40$

Hence, the total amount to be received at the end of two years is $1,166.40.

Example 3.3

A manufacturer expects to receive $20,200 one month after the shipment of goods to a retailer. The manufacturer needs the cash and has arranged with a bank for a loan of $20,000 upon the shipment of goods on the condition that the bank will collect all of the \$20,200 from the retailer a month later. What is the monthly interest rate charged by the bank?

Since there is only one interest period, the interest rate per month is

$$\frac{20{,}200 - 20{,}000}{20{,}000} = 0.01 = 1\%$$

This is also the discount rate by which the future sum of \$20,200 is discounted to a present value of \$20,000.

3.2 EQUIVALENCE OF CASH FLOWS

Since an amount of money unspent or received may accrue interest over time, it has different value at different time if the interest accrued is included. It is important to set up a time scale with regular intervals as interest periods. To simplify the computation of interest, it is assumed that receipts and disbursements are made only at the dividing points of the interest periods. For a transaction whose cash flows cover n interest periods, the dividing points are denoted by $t = 0, 1, 2, \ldots, n$, with $t = 0$ representing the present time, and can be depicted by equidistant points on a horizontal line with $t = 0$ as the reference point.

Consider the simple case of a single sum of money at a specified reference point such as $t = 0$. Then, it is possible to find the value of an equivalent single sum at n interest periods later, provided that the interest rate per period is known. It does not matter whether the single sum is an amount to be received or to be paid. For a single sum P at present, its *future value* F^* after n interest periods will be greater than P; conversely, for a single sum F at n interest periods in the future, its *present value* P^* will be less than F. For example, if a single sum P is deposited in the bank at present, it would be *compounded* to a future value of F^* after n interest periods; conversely, if a single sum F is to be paid at n interest periods later, it would be *discounted* to a present value of P^*. These situations may be depicted respectively in parts (a) and (b) of Fig. 3.1, in which the solid arrows represent the given quantities and the dotted arrows represent the desired quantities. The present value P and the future value F^* in part (a) of Fig. 3.1, or the future value F and the present value P^* in part (b) of Fig. 3.1, are said to be equivalent with respect to the time value of money.

It is often convenient to describe a cash flow profile as representing either lending or borrowing. For the case of the single sum just cited, we may speak of depositing a present sum P in the bank with the expectation of receiving a future sum F from the bank after n interest periods. Conversely, we may conceive the idea of borrowing a present sum P in anticipation of paying back a future sum F. The lending and borrowing situations are depicted respectively in parts (a) and (b) of Fig. 3.2. Generally, receipts, which represent positive cash flows, and disbursements, which represent negative cash flows, are denoted schematically by upward and downward arrows, respectively, in a diagram.

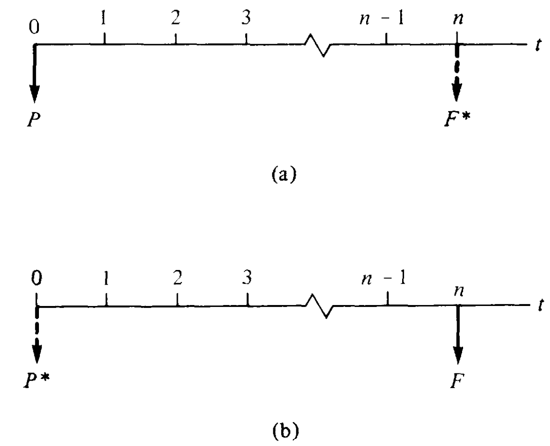

Figure 3.1 Equivalent present value and future value.

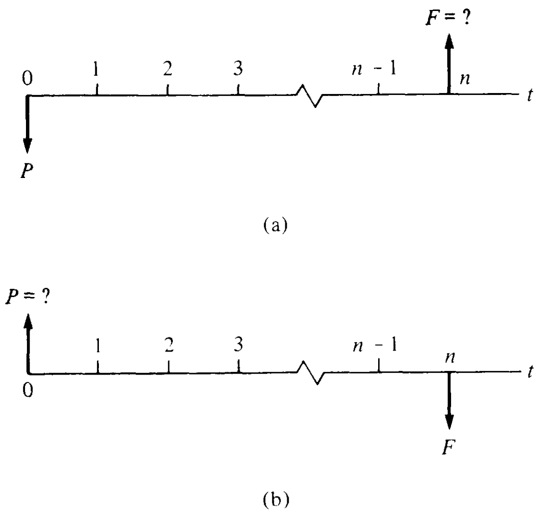

Figure 3.2 Relationship between present sum and future sum.

Example 3.4

Consolidated Financial Service Company advertises an interest rate of 12% per year for personal loans, while Northgate Finance Incorporated advertises an interest rate of 3% per quarter of a year. If you want to borrow \$1,000 now and pay back the principal and interest one year later, find the equivalent future sum that you owe when using each of these two services.

The interest period used by Consolidated Financial Service Company is a year. Since you will borrow a sum of \$1,000 for one interest period, the equivalent future sum after one year is given by

$$(\$1{,}000)(1 + 0.12) = \$1{,}120$$

The interest period used by Northgate Finance Incorporated is a quarter of a year. Since there are four quarters in a year, the equivalent future sum after one year is obtained by compounding the interests progressively over four periods. Thus,

$$(\$1{,}000)(1 + 0.03) = \$1{,}030.00$$
$$(\$1{,}030)(1 + 0.03) = \$1{,}060.90$$
$$(\$1{,}060.90)(1 + 0.03) = \$1{,}092.73$$
$$(\$1{,}092.73)(1 + 0.03) = \$1{,}125.51$$

3.3 RELATIONSHIP BETWEEN SINGLE SUMS

Consider the simplest case of a single sum of money that is being invested now for a future return. Let

i = percentage interest rate per interest period, expressed as decimals in computation
n = number of interest periods
P = a present sum of money
F = a future sum of money at the end of n interest periods starting from the present

What should be the sum of money F to be received at the end of n interest periods if a single sum P being invested now will receive interest that is compounded at the end of each interest period?

The interest accrued in any interest period is the product of the amount at the beginning of an interest period and the interest rate per period. Then, the amount of money at the end of each interest period is equal to the sum of the amount of money at the beginning of that period and the interest accrued in that period. Hence, we can determine the amount of money accumulated progressively at the end of each interest period through compounding as shown in Table 3.1. Note that the amount of money at the end of each period is obtained by adding the amount at the beginning

TABLE 3.1 COMPOUND INTEREST FOR A SINGLE SUM

Period	Amount at beginning of period	Interest for the period	Amount at end of period
1	P	iP	$P(1 + i)$
2	$P(1 + i)$	$iP(1 + i)$	$P(1 + i)^2$
3	$P(1 + i)^2$	$iP(1 + i)^2$	$P(1 + i)^3$
....	....	....	....
n	$P(1 + i)^{n-1}$	$iP(1 + i)^{n-1}$	$P(1 + i)^n$

of the period and the interest for the period, e.g., $P + iP = P(1 + i)$. Thus, the future value F of a single present sum P with interest rate i compounded over n interest periods is given by

$$F = P(1 + i)^n \tag{3.1}$$

Conversely, we may wish to find a single sum of money P which at present is equivalent to a single sum to be received at the end of n interest periods for a discount rate i per period. For computational purposes, we shall use the terms discount rate and interest rate interchangeably. In this case, because the amount of money at the beginning of each period is computed backward by subtracting the interest for the period from the amount at the end of that period, the net effect is to discount or reduce the future sum F to its present value P. By transposing P and F in Eq. (3.1),

$$P = \frac{F}{(1 + i)^n} = F(1 + i)^{-n} \tag{3.2}$$

In Eq. (3.1), the factor $(1 + i)^n$ has the effect of increasing the magnitude of the present sum P through compounding and is referred to as the *compound amount factor;* whereas in Eq. (3.2), the factor $(1 + i)^{-n}$ has the effect of discounting the magnitude of the future sum F and is referred to as the *discount amount factor*.

Example 3.5

Gretchen Boyd borrows \$1,000 for 4 years and agrees to pay 5% interest per year compounded annually. What is the total amount that she will repay the debt at the end of 4 years?

Since $P = \$1{,}000$, $i = 5\%$, and $n = 4$, we get from Eq. (3.1)

$$F = (1{,}000)(1 + 0.05)^4 = \$1{,}215.51$$

Example 3.6

Andrew Burke wants to put aside a sum of money in the bank now so that he can have \$1,000 available 2 years from now. If the bank pays 1.5% interest per quarter (every three months) compounded quarterly, what is the amount P that he should deposit now?

Since $F = \$1{,}000$, $i = 1.5\%$, $n = (2)(4) = 8$, we get from Eq. (3.2)

$$P = (1{,}000)(1 + 0.015)^{-8} = \$887.71$$

3.4 DECOMPOSITION AND SUPERPOSITION OF CASH FLOWS

In dealing with a series of cash flows, we can treat each sum separately in determining its equivalence and then obtain the combined effects of all sums on the final result. This operation is referred to as the *decomposition* and *superposition* of cash flows. Although the numerical computation is often tedious, it can be greatly facilitated by using automated computational aids. As will be explained later, compound interest tables are also available as aids for numerical computation.

In general, we attempt to *decompose* the cash flows into a number of single sums whose values are converted to equivalent amounts at the *same point in time* and then to *superimpose* the individual equivalent amounts to obtain the composite results. This principle of decomposition and superposition can be applied to obtain the present or future value of any series of cash flows, provided the interest rate per period and the number of interest periods associated with receipts or disbursements are known.

Let us consider a series of disbursements $P_0, P_1, \ldots, P_n$ at time $t = 0, 1, \ldots, n$ as shown in part (a) of Fig. 3.3. We can treat each of these disbursements P_t separately in determining its equivalent future value f_t at time $t = n$ as shown in part (b) of Fig. 3.3. Then, according to Eq. (3.1),

$$f_0 = P_0(1 + i)^n$$

$$f_1 = P_1(1 + i)^{n-1}$$

$$\cdots\cdots\cdots\cdots$$

$$f_t = P_t(1 + i)^{n-t}$$

$$\cdots\cdots\cdots\cdots$$

$$f_n = P_n$$

By summing these values, the equivalent future value F at $t = n$ resulting from the total effects of $P_0, P_1, \ldots, P_n$ is given by

$$F = f_0 + f_1 + f_2 + \cdots + f_n$$

or

$$F = \sum_{t=0}^{n} P_t(1 + i)^{n-t} \tag{3.3}$$

Similarly, for a series of receipts $F_0, F_1, \ldots, F_n$ at $t = 0, 1, \ldots, n$, as shown in Fig. 3.4, we can treat each of these receipts F_t separately in determining its equivalent present value p_t at time $t = 0$ by Eq. (3.2) as follows:

$$p_0 = F_0$$

$$p_1 = F_1(1 + i)^{-1}$$

$$\cdots\cdots\cdots\cdots$$

$$p_t = F_t(1 + i)^{-t}$$

$$\cdots\cdots\cdots\cdots$$

$$p_n = F_n(1 + i)^{-n}$$

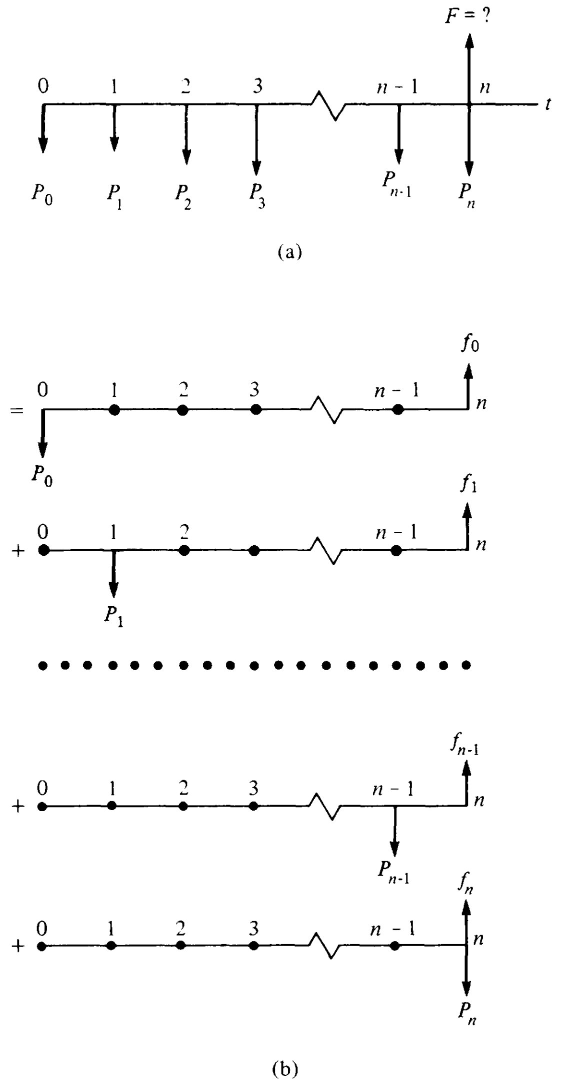

Figure 3.3 Equivalent future value of a series of cash flows.

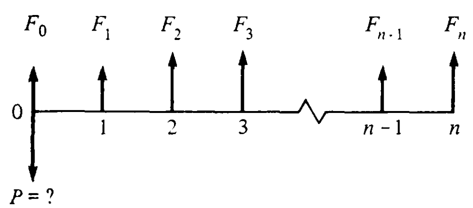

Figure 3.4 Equivalent present value of a series of cash flows.

By summing these values, the equivalent present value P at $t = n$ resulting from the total effects of $F_0, F_1, \ldots, F_n$ is given by

$$P = p_0 + p_1 + p_2 + \cdots + p_n$$

or

$$P = \sum_{t=0}^{n} F_t(1 + i)^{-t} \tag{3.4}$$

For the general case of a series of mixed positive and negative cash flows, the same principle may be applied. Let $A_0, A_1, A_2, \ldots, A_n$ be the cash flows at $t = 0, 1, 2, \ldots, n$ which are designated as positive for receipts and negative for disbursements. Then, the net future value (NFV) of the series at $t = n$ is defined as

$$\text{NFV} = \sum_{t=0}^{n} A_t(1 + i)^{n-t} \tag{3.5}$$

Similarly, the net present value (NPV) of the series at $t = 0$ is defined as

$$\text{NPV} = \sum_{t=0}^{n} A_t(1 + i)^{-t} \tag{3.6}$$

The derivations of Eqs. (3.5) and (3.6) are analogous to those of Eqs. (3.3) and (3.4). However, since A_t (for $t = 0, 1, 2, \ldots, n$) may be positive, negative, or zero, the summation must be carried out algebraically with due regard to the signs.

The cash flows $A_0, A_1, A_2, \ldots, A_n$ are schematically shown in part (a) of Fig. 3.5, while the NFV and the NPV are shown respectively in parts (b) and (c) of

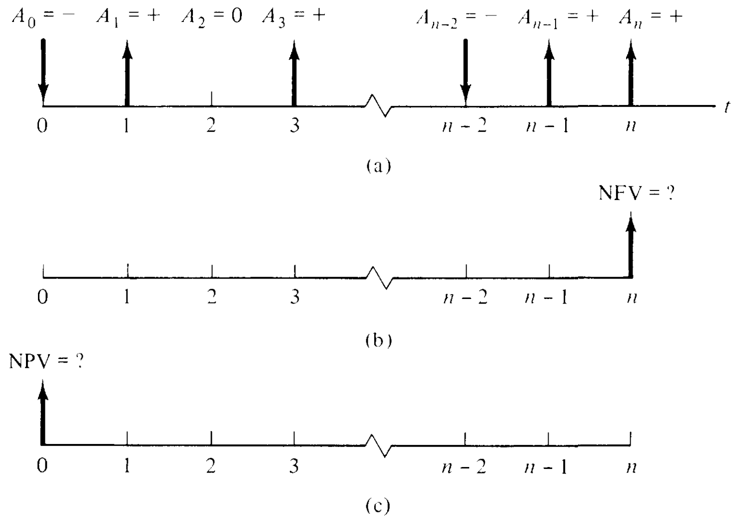

Figure 3.5 Net present value and net future value.

Fig. 3.5. The NFV is the resulting sum at $t = n$ if all positive cash flows are to accrue interest at a rate of i per period and all negative cash flows are obligated to pay interest also at a rate of i per period until the end of period n. The NPV is the equivalent sum of the NFV if the latter is discounted to the present, i.e., NPV = NFV$(1 + i)^{-n}$ or NFV = NPV$(1 + i)^n$.

Example 3.7

Rhoda Gould is planning a trip to Europe 3 years from next January. She intends to deposit her savings of $1,000 in a bank at the end of this year and expects to deposit another $500 at the end of next year. If the bank pays 5% interest per year compounded annually, how much money can she expect to be available to her at the time of her departure?

The cash flow for this problem is depicted in Fig. 3.6 in which $t = 0$ refers to next January. For P_0 = $1,000 and P_1 = $500, the amount available at $t = 3$ is

$$F = P_0(1 + i)^3 + P_1(1 + i)^2$$
$$= (1{,}000)(1 + 0.05)^3 + (500)(1 + 0.05)^2$$
$$= 1{,}157.63 + 551.25 = \$1{,}708.88$$

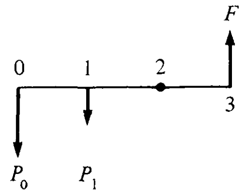

Figure 3.6 Cash flows for Example 3.7.

Example 3.8

Ron Alexander arranged to buy a used car from a friend through a deferred payment plan. He is allowed to possess the car immediately, but he must pay $500 two months from now and another $500 three months from now, which include interest charges of 1% per month compounded monthly. What is the equivalent of these payments if he can afford to make a single payment now?

The cash flow for this problem is depicted by $F_2 = F_3$ = $500 at $t = 2$ and $t = 3$, respectively. The equivalent single present sum is given by

$$P = F_2(1 + i)^{-2} + F_3(1 + i)^{-3}$$
$$= (500)(1 + 0.01)^{-2} + (500)(1 + 0.01)^{-3}$$
$$= 490.15 + 485.30 = \$975.45$$

3.5 UNIFORM SERIES OF CASH FLOWS

In many situations, the cash flows consist of a uniform series of disbursements or receipts continuing for a number of interest periods. For example, we may deposit an amount U at the end of each interest period in order to accumulate a single sum F at the end of n interest periods, as shown in Fig. 3.7, or we may invest a single sum P now with the expectation of receiving an amount U at the end of each interest period continuing for n interest periods, as indicated in Fig. 3.8. In each case, the disbursements or receipts U in the uniform series *are assumed to be made at the ends of interest periods.*[1] The assumption that disbursements and receipts are made only at the ends of interest periods is adopted to simplify the derivations of compound interest formulas for uniform series and is generally accepted in conventional practice. The point $t = 0$ representing the present time may be regarded as the end of a fictitious period 0.

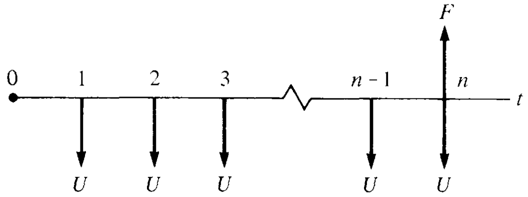

Figure 3.7 Future value of a uniform series.

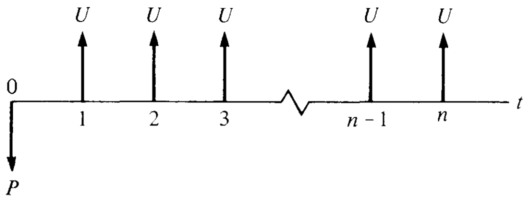

Figure 3.8 Present value of a uniform series.

Referring to Fig. 3.7, the future value F of the uniform series may be obtained by first converting each of the U values at $t = 1, 2, \ldots, n$ to a future value at $t = n$ and then superimposing these values to obtain F. Thus, for U at $t = 1$, its future value at $t = n$ is $U(1 + i)^{n-1}$, and the process of finding the future value equivalence of each U is continued until the last U at $t = n$. Then,

$$F = U(1 + i)^{n-1} + U(1 + i)^{n-2} + \cdots + U(1 + i) + U$$

This equation represents a geometric progression whose sum F can be easily obtained in a simple form. Factoring out U and multiplying both sides of the equation

[1] The notation U is used to denote the amount at the end of each interest period for a *uniform* series with U at period $t = 1, 2, \ldots, n$. This is a departure from the traditional use of the notation A for this purpose because in this book the notation A is used to denote net *annual* cash flow.

by $(1 + i)$, we get in two successive steps

$$F = U[(1 + i)^{n-1} + (1 + i)^{n-2} + \cdots + (1 + i) + 1]$$

$$(1 + i)F = U[(1 + i)^{n} + (1 + i)^{n-1} + \cdots + (1 + i)^2 + (1 + i)]$$

Subtracting the first from the second equation above,

$$iF = U[(1 + i)^n - 1]$$

Consequently,

$$F = U\left[\frac{(1 + i)^n - 1}{i}\right] \tag{3.7}$$

or, expressing U in terms of F,

$$U = F\left[\frac{i}{(1 + i)^n - 1}\right] \tag{3.8}$$

Similarly, referring to Fig. 3.8, the present value P of the uniform series may be obtained by superposition. Thus, for U at $t = 1$, its present value is $U(1 + i)^{-1}$; for U at $t = 2$, its present value is $U(1 + i)^{-2}$, and the process of finding the present value equivalence of U is continued until the present value of U at $t = n$ is found to be $U(1 + i)^{-n}$. Then,

$$P = U(1 + i)^{-1} + U(1 + i)^{-2} + \cdots + U(1 + i)^{-(n-1)} + U(1 + i)^{-n}$$

This equation also represents a geometric progression whose sum P can be easily obtained in a simple form. Factoring out U and multiplying the equation by $(1 + i)$, we get in two successive steps

$$P = U[(1 + i)^{-1} + (1 + i)^{-2} + \cdots + (1 + i)^{-(n-1)} + (1 + i)^{-n}]$$

$$(1 + i)P = U[1 + (1 + i)^{-1} + \cdots + (1 + i)^{-(n-2)} + (1 + i)^{-(n-1)}]$$

Subtracting the first from the second equation above yields

$$iP = U[1 - (1 + i)^{-n}]$$

Consequently, by carrying out the algebraic operations, we get

$$P = U\left[\frac{1 - (1 + i)^{-n}}{i}\right] = U\left[\frac{(1 + i)^n - 1}{i(1 + i)^n}\right] \tag{3.9}$$

or, by expressing U in terms of P, we find

$$U = P\left[\frac{i(1 + i)^n}{(1 + i)^n - 1}\right] \tag{3.10}$$

It should be emphasized that Eqs. (3.7) through (3.10) are based on a uniform series of amount U in each period at $t = 1, 2, \ldots, n$. If an amount U is also given for $t = 0$, this amount is not a part of the uniform series beginning from $t = 0$. On the other hand, if an amount U is given for $t = 1, 2, \ldots, n$ except for an intermediate period between $t = 1$ and $t = n$, the series does not constitute a uniform series.

The uniform series in Fig. 3.7 is typical of the situation where we try to build up a future fund by depositing uniform payments over a number of interest periods. Hence, the factor $[(1 + i)^n - 1]/i$ in Eq. (3.7) is referred to as the *compound uniform series factor*, while its reciprocal $i/[(1 + i)^n - 1]$ in Eq. (3.8) is called the *sinking fund factor*. On the other hand, the uniform series in Fig. 3.8 is typical of the situation where we attempt to recover an initial capital investment by receiving uniform repayments over a number of interest periods. Hence, the factor $[(1 + i)^n - 1]/i(1 + i)^n$ in Eq. (3.9) is referred to as the *discount uniform series factor*, while its reciprocal $i(1 + i)^n/[(1 + i)^n - 1]$ in Eq. (3.10) is called the *capital recovery factor*. These factors are applicable only to uniform series involving end-of-period payments continuing for n interest periods.

The relationships between various factors can be seen from Eqs. (3.1), (3.2), and (3.7) through (3.10). For example, the factors in Eqs. (3.8) and (3.10) are the respective reciprocals of those in Eqs. (3.7) and (3.9). Also, the factors in Eqs. (3.8) and (3.10) are related as follows:

$$\frac{i}{(1 + i)^n - 1} + i = \frac{i(1 + i)^n}{(1 + i)^n - 1} \tag{3.11}$$

Consequently, we can always derive appropriate relationships from these equations.

Example 3.9

An excavation contractor decided 7 years ago to put aside an amount U annually to build up a sinking fund to replace a piece of equipment by the end of 7 years. He encountered a run of financial difficulties and was not able to make deposits at the end of the third and fourth years, as shown in Fig. 3.9. Before deciding how much money should be added to the fund at the end of the seventh year in order to purchase the equipment, he wants to find out the value at this time equivalent to all previous payments that have accrued interests at a rate i compounded annually.

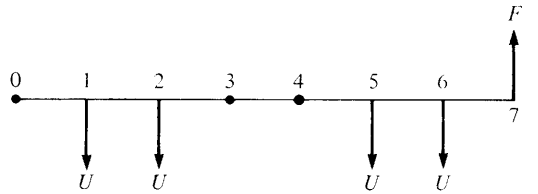

Figure 3.9 Cash flows for Example 3.9.

Since the payments in Fig. 3.9 do not constitute a uniform series, we can treat each payment U as a single sum and compute the value F at $t = 7$ by superposition. Thus,

$$F = U(1 + i)^6 + U(1 + i)^5 + U(1 + i)^2 + U(1 + i)$$

Alternatively, we may regard the payments as a uniform series for 7 years, except that payments at $t = 3$, $t = 4$, and $t = 7$ are to be deleted as single sums. Then, by applying Eq. (3.6) at $t = 7$ for the uniform series, we obtain the future value of the cash flow as

$$F = U\left[\frac{(1 + i)^7 - 1}{i}\right] - U(1 + i)^4 - U(1 + i)^3 - U$$

The results of both approaches are equivalent.

Example 3.10

John Benjamin wishes to deposit a sum of money in the bank on the tenth birthday of his daughter Mary for the purpose of financing her college education. She is to receive an amount U on her seventeenth, eighteenth, nineteenth, and twentieth birthdays, as shown in Fig. 3.10. If the bank pays interest at a rate i compounded annually, determine the single sum that he should deposit in the bank when his daughter reaches ten years old.

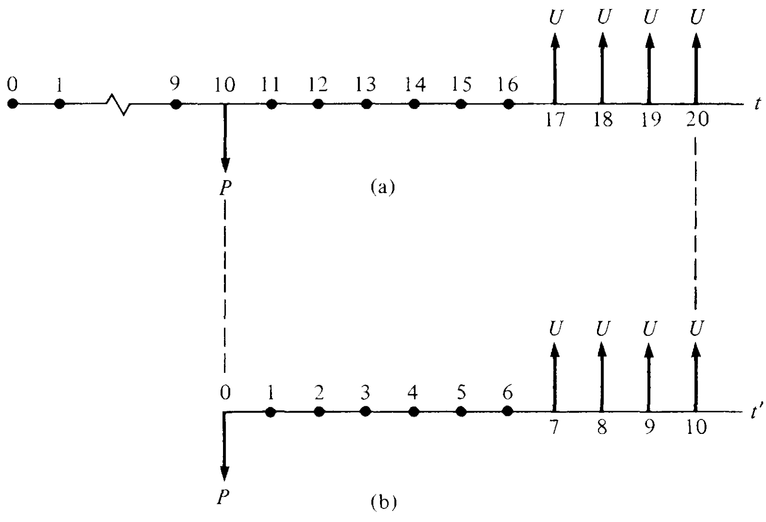

Figure 3.10 Cash flows for Example 3.10.

For the problem in Fig. 3.10, we can theoretically choose the day of Mary's birth as the origin of the time axis and her age as the chronological time scale. In that case, we will make reference to $t = 0$ for all payments. Thus, the sum of money to be deposited at $t = 10$ is related to other payments as follows:

$$P(1 + i)^{-10} = U[(1 + i)^{-17} + (1 + i)^{-18} + (1 + i)^{-19} + (1 + i)^{-20}]$$

or, multiplying through by $(1 + i)^{10}$,

$$P = U[(1 + i)^{-7} + (1 + i)^{-8} + (1 + i)^{-9} + (1 + i)^{-10}]$$

However, the computation can be simplified if we select a new time scale t' such that $t' = t - 10$. In other words, the origin $t' = 0$ is located at $t = 10$, and other points in time for payments become $t' = 7, 8, 9$, and 10. In this case, the sum of money P at $t' = 0$ is the present value which is identical to the expression for P in the last equation.

Alternatively, we can treat the problem as the difference between two uniform series: the first with uniform payments U from $t = 11$ to $t = 20$ (i.e., $t' = 1$ through 10) and the second with uniform payments U from $t = 11$ to $t = 16$ (i.e., $t' = 1$ through 6). The sum P is given by applying Eq. (3.9) to the two uniform series:

$$P = U\left[\frac{(1+i)^{10} - 1}{i(1+i)^{10}}\right] - U\left[\frac{(1+i)^{6} - 1}{i(1+i)^{6}}\right] = U\left[\frac{(1+i)^{4} - 1}{i(1+i)^{10}}\right]$$

It should be noted that this result is quite different from that of

$$P_6 = U\left[\frac{(1+i)^{4} - 1}{i(1+i)^{4}}\right]$$

where P_6 is a single sum at $t' = 6$, which is equivalent to the uniform series of payments U from $t' = 7$ through 10. However, it is not the answer to the question because we are interested in the sum P at $t' = 0$. Carrying out the conversion to $t' = 0$, we get

$$P = P_6(1+i)^{-6} = U\left[\frac{(1+i)^{4} - 1}{i(1+i)^{10}}\right]$$

This result is the same as that previously obtained by considering the difference of two uniform series.

3.6 FUNCTIONAL NOTATION AND COMPOUND INTEREST TABLES

The use of the compound interest formulas derived in this chapter is often tedious. Therefore, a functional notation is introduced to simplify our reference to such formulas. The numerical computation can easily be carried out with the aid of an electronic computer. The availability of automated computational aids has greatly reduced the efforts of carrying out the required numerical calculations. However, compound interest tables listing various factors in these formulas for a wide range of combinations of interest rates and interest periods are readily available. Appendix A shows a set of discrete compound interest tables which are used for the numerical examples in this text for pedagogic reasons. You can readily trace the steps of calculation by first listing the functional notation in an equation and then finding the corresponding value in the appendix to carry out the numerical solution. You are encouraged to use automated computational aids after you have mastered the basic principles underlying the calculations.

The various factors for the compound interest formulas in Eqs. (3.1), (3.2), and (3.7) through (3.10) are expressed in the functional notation as follows[2]:

(a) Compound amount factor

$$\frac{F}{P} = (1 + i)^n = (F \mid P, i, n) \tag{3.12}$$

(b) Discount amount factor

$$\frac{P}{F} = \frac{1}{(1 + i)^n} = (P \mid F, i, n) \tag{3.13}$$

(c) Compound uniform series factor

$$\frac{F}{U} = \frac{(1 + i)^n - 1}{i} = (F \mid U, i, n) \tag{3.14}$$

(d) Sinking fund factor

$$\frac{U}{F} = \frac{i}{(1 + i)^n - 1} = (U \mid F, i, n) \tag{3.15}$$

(e) Discount uniform series factor

$$\frac{P}{U} = \frac{(1 + i)^n - 1}{i(1 + i)^n} = (P \mid U, i, n) \tag{3.16}$$

(f) Capital recovery factor

$$\frac{U}{P} = \frac{i(1 + i)^n}{(1 + i)^n - 1} = (U \mid P, i, n) \tag{3.17}$$

The meaning of the functional notation can be explained as follows: In (a), the notation $(F \mid P, i, n)$ reads "to find F, given $P = 1$, for the specified i and n." In other words, if the present sum P is \$1, the compound amount factor $(F \mid P, i, n)$ yields the dollar value of the future sum F for the specified values of i and n. This value F is listed for each i value in a separate table and for each n value in a column under the heading $(F \mid P, i, n)$. Thus, for any given value of P that is different from \$1, we can obtain F by multiplying this factor by that value of P. Similarly, in (b), the notation $(P \mid F, i, n)$ reads "to find P, given $F = 1$, for the specified i and n." If the future sum F is \$1, this discount amount factor $(P \mid F, i, n)$ yields the present sum P for the specified values of i and n. In (c), the notation $(F \mid U, i, n)$ reads "to find F, given $U = 1$, for the specified i and n." Thus, for a uniform series with

[2]The term *compound amount factor* is traditionally referred to as *single payment compound amount factor*, the term *discount amount factor* referred to as *single payment present worth factor*, the term *compound uniform series factor* referred to as *uniform series compound amount factor*, and the term *discount uniform series factor* referred to as *uniform series present worth factor*. The terminology used in this book has simplified the designation of these factors.

U of \$1 per period for $t = 1, 2, \ldots, n$, the compound uniform series factor $(F \mid U, i, n)$ yields the dollar value of the future sum F for the specified values of i and n. This explanation of the functional notation applies to the compound interest factors in the remaining formulas. The use of these tables for numerical computation will be further explained in the examples.

To illustrate the application of the functional notation we find the present sum equivalent to a future sum of \$10,000 for $i = 6\%$ and $n = 7$. From Table A.7 for $i = 6\%$, we note that for $n = 7$, $(P \mid F, i, n) = 0.6551$. This discount amount factor indicates that given $F = 1$, for the specified $i = 6\%$ and $n = 7$, the present value P is 0.6651. Since F is 10,000 instead of 1, $P = (10{,}000)\,(0.6651) = 6{,}651$. The symbol $P \mid F$ denotes the value of P on the condition of $F = 1$, and is different from P/F, which means P divided by F.

Another example is to find the future sum equivalent to a present sum of 10,000 for $i = 6\%$ and $n = 100$; we use Table A.7 for $i = 6\%$ and read down from the first column to $n = 100$ and find $(F \mid P, i, n) = 339.3019$. This compound amount factor indicates that given $P = 1$, for the specified $i = 6\%$ and $n = 100$ the future value P is 339.3019. Since P is 10,000 instead of 1, $F = (10{,}000)\,(339.3019) = 3{,}393{,}019$.

Example 3.11

A home builder estimates that by adding better insulation, installed at an initial cost of \$1,000, to an average three-bedroom house the owner can save \$100 of heating fuel cost per year over the next 40 years. Compute the present value P of the uniform series with $U = \$100$ continuing over $n = 40$ years if the annual interest rate i is (a) 8% and (b) 10%.

(a) For $i = 8\%$, we can formulate the problem in terms of the functional notation as follows:

$$P = U(P \mid U, i, n) = (100)(P \mid U, 8\%, 40)$$

In the compound interest tables in Appendix A, we look for the page with interest rate $i = 8\%$. On that page, we search for the column heading $(P \mid U, i, n)$ and read downward for a number on a line corresponding to $n = 40$. The number 11.9246 is the discount uniform series factor $(P \mid U, 8\%, 40)$. Hence,

$$P = (100)(11.9246) = \$1{,}192.46$$

(b) For $i = 10\%$, we can find $(P \mid U, 10\%, 40) = 9.7791$ from another page with interest rate $i = 10\%$. Thus,

$$P = (100)(P \mid U, 10\%, 40) = (100)(9.7791) = \$977.91$$

Example 3.12

The Dutch merchant Peter Minuit bought the island of Manhattan in New York from the Manhattos Indians in 1626 for \$24. Assuming that he could have deposited \$24 in a bank in Holland that would pay compound interest at an annual interest rate of 6%, what would the value of his investment be in 1976?

This problem, as depicted in Fig. 3.11, can be formulated in functional notation and solved by using computational aids as follows:

$$F = P(F \mid P, i, n) = (24)(F \mid P, 6\%, 350) = \$17.2688 \times 10^9$$

In the table for $i = 6\%$ in Appendix A, the value of n goes up to 100, and we cannot find the compound amount factor $(F \mid P, 6\%, 350)$. However, from Fig. 3.11, we note that we can find the future value F_{100} at $t = 100$. Using F_{100} as present value, we can find the future value F_{200} at $t = 200$, etc. Hence,

$$F_{100} = P(F \mid P, 6\%, 100)$$
$$F_{200} = F_{100}(F \mid P, 6\%, 100)$$
$$F_{300} = F_{200}(F \mid P, 6\%, 100)$$
$$F_{350} = F_{300}(F \mid P, 6\%, 50)$$

From the table for $i = 6\%$ in Appendix A, $(F \mid P, 6\%, 100) = 339.3021$ and $(F \mid P, 6\%, 50) = 18.4202$. Hence, for $F = F_{350}$,

$$F = P(F \mid P, 6\%, 100)^3(F \mid P, 6\%, 50)$$
$$= (24)(339.3021)^3(18.4202)$$
$$= \$17.2689 \times 10^9$$

The significant figures have been rounded off in the computation.

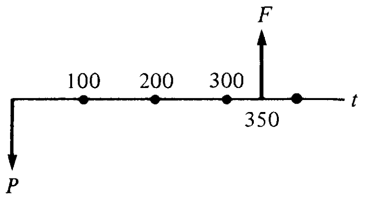

Figure 3.11 Cash flows for Example 3.12.

Example 3.13

In Example 3.9, let $U = \$1{,}000$ and $i = 8\%$, and determine the amount F at the end of 7 years.

If we treat each payment as a single sum, the problem can be expressed in functional notation as follows:

$$F = (1{,}000)(F \mid P, 8\%, 6) + (1{,}000)(F \mid P, 8\%, 5) + (1{,}000)(F \mid P, 8\%, 2) + (1{,}000)(F \mid P, 8\%, 1)$$

The compound amount factors can be found from the table for $i = 8\%$ in Appendix A. Then,

$$F = (1{,}000)(1.5869 + 1.4693 + 1.1664 + 1.080)$$
$$= (1{,}000)(5.3026) = \$5{,}302.60$$

Alternatively, if we regard the payments as a uniform series for 7 years and subtract those not included in the problem, the solution becomes

$$F = (1{,}000)(F \mid U, 8\%, 7) - (1{,}000)(F \mid P, 8\%, 4) - (1{,}000)(F \mid P, 8\%, 3) - 1{,}000$$
$$= (1{,}000)(8.9228 - 1.3605 - 1.2597 - 1)$$
$$= (1{,}000)(5.3026) = \$5{,}302.60$$

Example 3.14

In Example 3.10, let $U = \$4{,}000$ and $i = 6\%$, and determine the amount P to be deposited on Mary's tenth birthday.

If we treat each payment as a single sum, the problem can be expressed in functional notation as follows:

$$P = (4{,}000)[(P \mid F, 6\%, 7) + (P \mid F, 6\%, 8) + (P \mid F, 6\%, 9) + (P \mid F, 6\%, 10)]$$
$$= (4{,}000)(0.6651 + 0.6274 + 0.5919 + 0.5584)$$
$$= (4{,}000)(2.4428) = \$9{,}771.20$$

Alternatively, we can treat the problem as the difference between two uniform series: the first continuing for 10 years from the time of making the deposit and the second continuing for 6 years from the same time. Thus,

$$P = (4{,}000)(P \mid U, 6\%, 10) - (4{,}000)(P \mid U, 6\%, 6)$$
$$= (4{,}000)(7.3601 - 4.9173)$$
$$= (4{,}000)(2.4428) = \$9{,}771.20$$

If we treat the payments as a uniform series from the seventh through tenth years, the solution becomes

$$P_6 = (4{,}000)(P \mid U, 6\%, 4)$$

and

$$P = P_6(P \mid F, 6\%, 6)$$
$$= (4{,}000)(P \mid U, 6\%, 4)(P \mid F, 6\%, 6)$$
$$= (4{,}000)(3.4651)(0.7050) = \$9{,}771.58$$

Example 3.15

A power company is considering the installation of antipollution equipment in an old coal-based generating plant that is expected to be used for another 10 years. If antipollution equipment is not installed immediately, more expensive low-sulfur coal must be used in order to meet the new air-quality standards, and even more expensive fuel must be used 5 years from now as the air-quality standards become more stringent. It is estimated that by installing the antipollution equipment, the saving of fuel cost will be U_1 = $15,000 per year in the first 5 years and U_2 = $20,000 in the second 5 years. Determine the present value of the annual savings in the next 10 years if the annual interest rate is 10%.

The cash flow for this problem is depicted in Fig. 3.12. We can treat it as the difference between two uniform series: the first with $U = U_2 = 20{,}000$ continuing from $t = 1$ through $t = 10$, and the second with $U = U_2 - U_1 = 5{,}000$ from $t = 1$ through $t = 5$. Thus,

$$\begin{aligned} P &= (20{,}000)(P \mid U, 10\%, 10) - (5{,}000)(P \mid U, 10\%, 5) \\ &= (20{,}000)(6.1446) - (5{,}000)(3.7908) \\ &= 122{,}892 - 18{,}954 = \$103{,}938 \end{aligned}$$

If we try to treat the problem as the superposition of two uniform series not having the same starting point in time, the computation of the present value will be less direct. For example, we may consider the first series with $U = U_1 = 15{,}000$ continuing from $t = 1$ through $t = 10$, and the second with $U = U_2 - U_1 = 5{,}000$ from $t = 6$ through $t = 10$. Then

$$\begin{aligned} P &= (15{,}000)(P \mid U, 10\%, 10) + (5{,}000)(P \mid U, 10\%, 5)(P \mid F, 10\%, 5) \\ &= (15{,}000)(6.1445) + (5{,}000)(3.7908)(0.6209) \\ &= 92{,}175 + 11{,}769 = \$103{,}937 \end{aligned}$$

We can also treat the problem as the superposition of two other uniform series: the first with $U_1 = 15{,}000$ continuing from $t = 1$ through $t = 5$, and the second with $U_2 = 20{,}000$ continuing from $t = 6$ through $t = 10$. Then

$$\begin{aligned} P &= (15{,}000)(P \mid U, 10\%, 5) + (20{,}000)(P \mid U, 10\%, 5)(P \mid F, 10\%, 5) \\ &= (15{,}000)(3.7908) + (20{,}000)(3.7908)(0.6209) \\ &= 56{,}862 + 47{,}074 = \$103{,}936 \end{aligned}$$

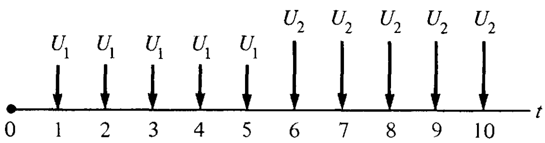

Figure 3.12 Cash flows for Example 3.15.

3.7 CASH FLOWS IN LINEAR GRADIENTS

In some situations, the cash flow profile may follow the form of a linearly increasing or decreasing gradient as the disbursements or receipts change linearly over time. For example, the maintenance cost of equipment may increase annually according to a linear gradient, or the savings in the operating cost of a physical plant after remodeling may decrease annually according to a linear gradient. Consequently, it is convenient to develop special formulas for finding the equivalence of cash flows in linear gradients.

Let us consider the case of a cash flow in the form of a linear gradient with increasing increments, as shown in Fig. 3.13. The receipts at $t_1, t_2, \ldots, t_n$ are, respectively, L_1; $L_2 = L_1 + G$; $L_3 = L_1 + 2G, \ldots, L_{n-1} = L_1 + (n-2)G$; and $L_n = L_1 + (n-1)G$, where L_1 is the constant portion of receipts in each period while G is a constant increment such that $L_1, L_2, \ldots, L_n$ is an arithmetic progression. Consequently, the linear gradient of part (a) in Fig. 3.13 may be decomposed into two components as represented by parts (b) and (c) of the figure. Part (b) is a uniform series whose present value P_1 can be obtained from Eq. (3.9). Part (c)

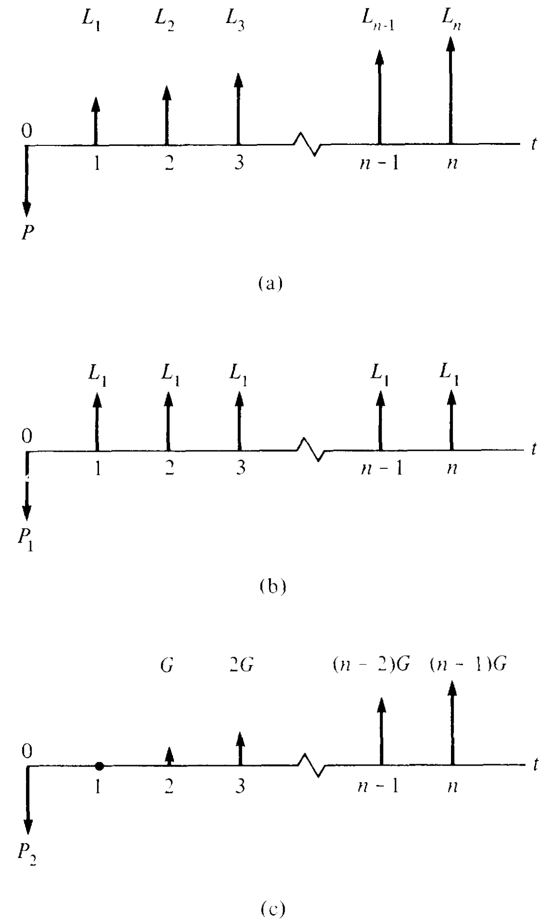

Figure 3.13 Cash flows with a linear gradient.

shows a basic linear gradient whose present value P_2 can be determined as later shown in Eq. (3.18). Then the present value of the linear gradient is $P = P_1 + P_2$.

A basic linear gradient is a special case of the linear gradient having an initial point of zero value at $t = 1$ and a constant increment of G at each succeeding point in time until the value reaches $(n - 1)G$ at $t = n$ as shown in part (a) of Fig. 3.14. Note the definition of the basic linear gradient in which the values $G, 2G, \ldots, (n - 1)G$ are located at $n = 2, 3, \ldots, n$, respectively. There is no value at $t = 1$ even though the starting point of the gradient is $t = 1$. The present value P of the basic linear gradient is given by

$$P = \frac{G}{(1+i)^2} + \frac{2G}{(1+i)^3} + \cdots + \frac{(n-2)G}{(1+i)^{n-1}} + \frac{(n-1)G}{(1+i)^n}$$

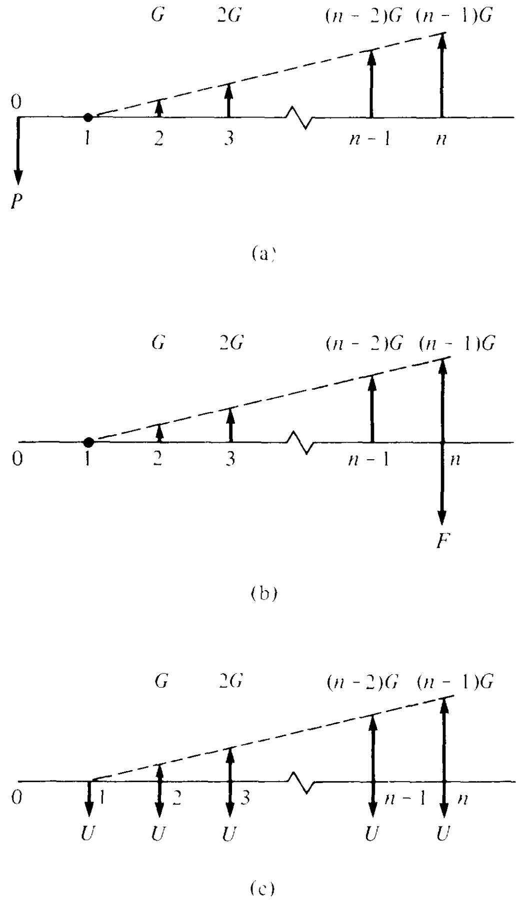

Figure 3.14 Equivalent sum of basic linear gradient.

Factoring out G and multiplying both sides of the equation by $(1 + i)$, we get in two successive steps

$$P = G\left[\frac{1}{(1+i)^2} + \frac{2}{(1+i)^3} + \cdots + \frac{n-2}{(1+i)^{n-1}} + \frac{n-1}{(1+i)^n}\right]$$

$$(1+i)P = G\left[\frac{1}{(1+i)} + \frac{2}{(1+i)^2} + \cdots + \frac{n-2}{(1+i)^{n-2}} + \frac{n-1}{(1+i)^{n-1}}\right]$$

Subtracting the first from the second equation above yields

$$iP = G\left[\frac{1}{(1+i)} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}} - \frac{n-1}{(1+i)^n}\right]$$

or

$$P = \frac{G}{i}\left[\frac{1}{(1+i)} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right] - \frac{nG}{i(1+i)^n}$$

The expression in the brackets on the right side of the above equation is the present value of a uniform series with $U = 1$, and its sum can be obtained by Eq. (3.9). Hence,

$$\begin{aligned} P &= \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] - \frac{nG}{i(1+i)^n} \\ &= G\left[\frac{(1+i)^n - 1 - ni}{i^2(1+i)^n}\right] \end{aligned} \tag{3.18}$$

Again, introducing the functional notation, the *discount gradient factor* is defined as

$$\frac{P}{G} = \frac{(1+i)^n - 1 - ni}{i^2(1+i)^n} = (P \mid G, i, n) \tag{3.19}$$

We can also derive directly the future value F. However, this is unnecessary since we can make use of the relationship in Eq. (3.1). Hence, the equivalent F in part (b) of Fig. 3.14 is obtained as

$$F = P(1+i)^n = G\left[\frac{(1+i)^n - 1 - ni}{i^2}\right]$$

and the *compound gradient factor* is defined as

$$\frac{F}{G} = \frac{(1+i)^n - 1 - ni}{i^2} = (F \mid G, i, n) \tag{3.20}$$

Similarly, from Eq. (3.10), the equivalent uniform series U in part (c) of Fig. 3.14 is obtained as

$$U = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = G\left[\frac{(1+i)^n - 1 - ni}{i(1+i)^n - i}\right]$$

and the *uniform series gradient factor* is defined as

$$\frac{U}{G} = \frac{(1 + i)^n - 1 - ni}{i(1 + i)^n - i} = (U \mid G, i, n) \tag{3.21}$$

The reciprocals of these relationships, G/P, G/F, and G/U, can easily be obtained if desired.

In the case of a cash flow in the form of a basic linear gradient with constantly decreasing increments, the formulas represented by Eqs. (3.19), (3.20), and (3.21) are still applicable except that G will be negative instead of positive. Thus, the cash flow equivalence of such a basic linear gradient is also negative.

The expressions in functional notation for various compound interest factors in Eqs. (3.19), (3.20), and (3.21) can also be tabulated in numerical computation for convenience. However, only one factor $(P \mid G, i, n)$ is listed in Appendix A, and the functional notation for this factor reads "to find P, given $G = 1$, for the specified i and n." The relationships between other compound interest factors and this factor can be obtained as follows:

1. Discount gradient factor

$$\frac{P}{G} = (P \mid G, i, n)$$

2. Compound gradient factor

$$\frac{F}{G} = (F \mid G, i, n) = (F \mid P, i, n)(P \mid G, i, n)$$

3. Uniform series gradient factor

$$\frac{U}{G} = (U \mid G, i, n) = (U \mid P, i, n)(P \mid G, i, n)$$

Example 3.16

The maintenance cost for a new school bus is expected to be \$1,500 in the first year and \$300 more for each additional year (i.e., \$1,800 in the second year, and \$2,100 in the third year, etc.) until the bus is disposed of at the end of 10 years. If the annual compound interest rate is 6%, determine the equivalent present value of the maintenance costs.

The problem can be decomposed into a uniform series with $U = \$1,500$ and a basic linear gradient with $G = \$300$. Thus,

$$\begin{aligned} P &= (1{,}500)(P \mid U, 6\%, 10) + (300)(P \mid G, 6\%, 10) \\ &= (1{,}500)(7.3601) + (300)(29.6023) \\ &= 11{,}040 + 8{,}881 = \$19{,}921 \end{aligned}$$

Example 3.17

A piece of solid waste disposal equipment was purchased for the purpose of reducing operating costs. In the first year of operation, the savings were \$38,000 with \$6,000 less savings in each succeeding year (i.e., \$32,000 in the second year, \$26,000 in the third year) until the equipment was retired after 5 years of service. If the annual interest rate was 12%, determine a single equivalent sum of the savings at the time the equipment was retired.

The problem can be decomposed into a uniform series with $U =$ \$38,000 and a basic linear gradient with $G =$ \$6,000. Hence,

$$\begin{aligned} F &= (38{,}000)(F \mid U, 12\%, 5) - (6{,}000)(P \mid G, 12\%, 5)(F \mid P, 12\%, 5) \\ &= (38{,}000)(6.3528) - (6{,}000)(6.3970)(1.7623) \\ &= 241{,}406 - 67{,}641 = 173{,}765 \end{aligned}$$

Example 3.18

A new office duplicating machine is expected to generate a stream of annual benefits in dollars for the next 9 years, as shown in part (a) of Fig. 3.15. If the annual discount rate is 8%, determine the present value of the cash flows.

The cash flows can be decomposed into two streams as shown in parts (b) and (c) of Fig. 3.15. That is, $U_1 = 600$ for $t = 1, 2, 3,$ and 4; $U_2 = 200$

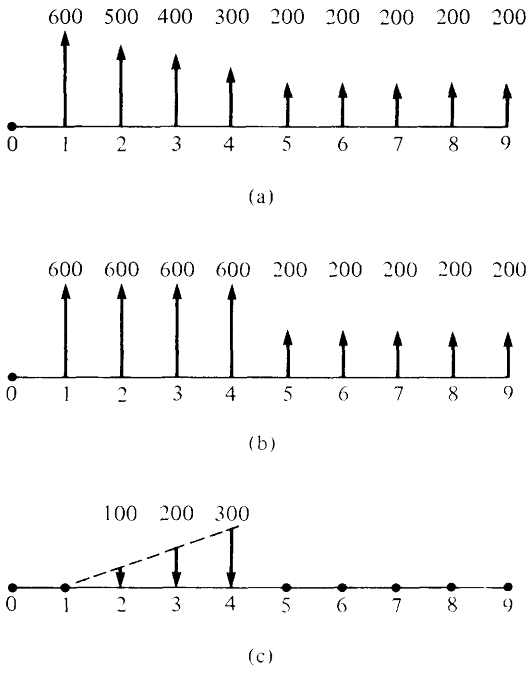

Figure 3.15 Cash flows for Example 3.18.

for $t = 5, 6, 7, 8$, and 9; and $G = -100$ for $n = 4$. Then, the present value is given by

$$P = (600)(P \mid U, 8\%, 4) + (200)(P \mid U, 8\%, 5)(P \mid F, 8\%, 4) - (100)(P \mid G, 8\%, 4)$$

Alternatively, it can be obtained by

$$P = (200)(P \mid U, 8\%, 9) + (400)(P \mid U, 8\%, 4) - (100)(P \mid G, 8\%, 4)$$

In either case, we obtain $P = \$2,109$.

3.8 COMPUTATION OF UNKNOWN INTEREST RATE

In each of the compound interest formulas derived in this chapter, only one unknown is to be determined from the other quantities given. So for, we have not dealt with the problem of computing the unknown interest rate i while all other quantities in the formula are given.

Let us consider first the simplest case involving a single present sum P (at $t = 0$) and a single future sum F (at $t = n$) as represented by Eq. (3.12) or (3.13), i.e.,

$$\frac{F}{P} = (1 + i)^n \quad \text{or} \quad \frac{P}{F} = \frac{1}{(1 + i)^n}$$

Solving for the interest rate i, we obtain

$$i = \sqrt[n]{\frac{F}{P}} - 1 \tag{3.22}$$

The interest rate i is conventionally defined as a positive quantity. If $F/P > 1$, Eq. (3.22) yields a unique positive real number for i. The interest rate i at which a single sum P will be accumulated to a single sum F over n interest periods is referred to as the *rate of return*.

It is important to note that Eq. (3.22) represents a special case in which the value of i can be obtained by taking the nth root of F/P. For other compound interest formulas, which involve either a series of receipts or a series of disbursements, the solution of i is not nearly as simple. For example, from Eqs. (3.14) and (3.16), we obtain, respectively,

$$\left(\frac{F}{U}\right)i = (1 + i)^n - 1 \tag{3.23}$$

and

$$\left(\frac{P}{U}\right)i(1 + i)^n = (1 + i)^n - 1 \tag{3.24}$$

The solution of i from either Eq. (3.23) or Eq. (3.24) requires the solution of an algebraic equation involving the nth power of i. Thus, the computation of the unknown interest rates generally can be quite tedious without automated computational aids.

For the group of compound interest formulas in Eqs. (3.12) through (3.17) corresponding to the six sets of cash flows in Fig. 3.16, a positive interest rate i exists when $F > P$, $F > nU$, and $P < nU$. For a specified value of n, we can find the numerical values of the compound interest factors for a range of values of i from the discrete compound interest tables in Appendix A, and plot such factors versus the interest rate i in a graph as shown schematically in Fig. 3.17. It can be seen that the compound interest factors represented by Eqs. (3.12) through (3.17) are monotonically increasing or decreasing functions of the interest rate i. Consequently, the computation of the unknown interest rate from any of these formulas is reduced to seeking the particular value of i which satisfies the corresponding formula.

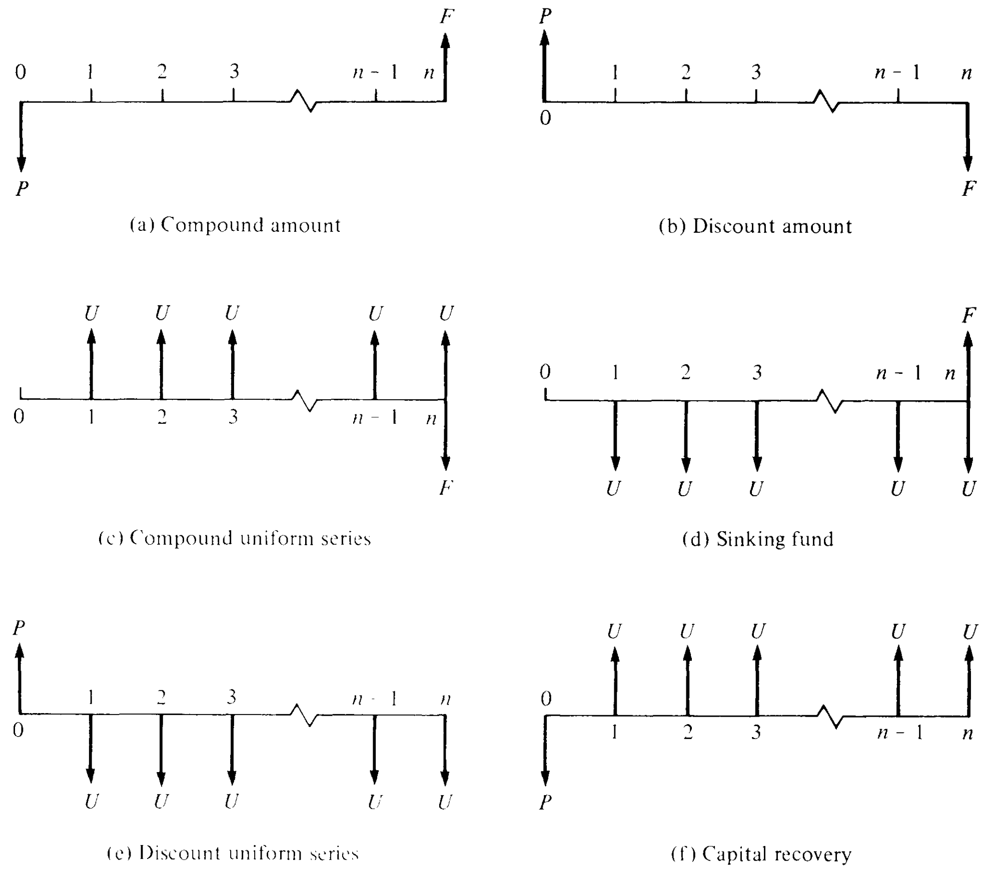

Figure 3.16 Typical cash flows of common interest.

Using the discrete compound interest tables in Appendix A, it is possible to compute the approximate value of i by linear interpolation of values for a factor corresponding to the formula under consideration. The accuracy of the interpolation depends on the interval of i values used. This method of interpolation for computing the unknown interest rate i can best be illustrated by examples. A more general approach in the formulation and solution of the unknown interest rate from a series of cash flows representing an investment will be discussed in Chapter 4.

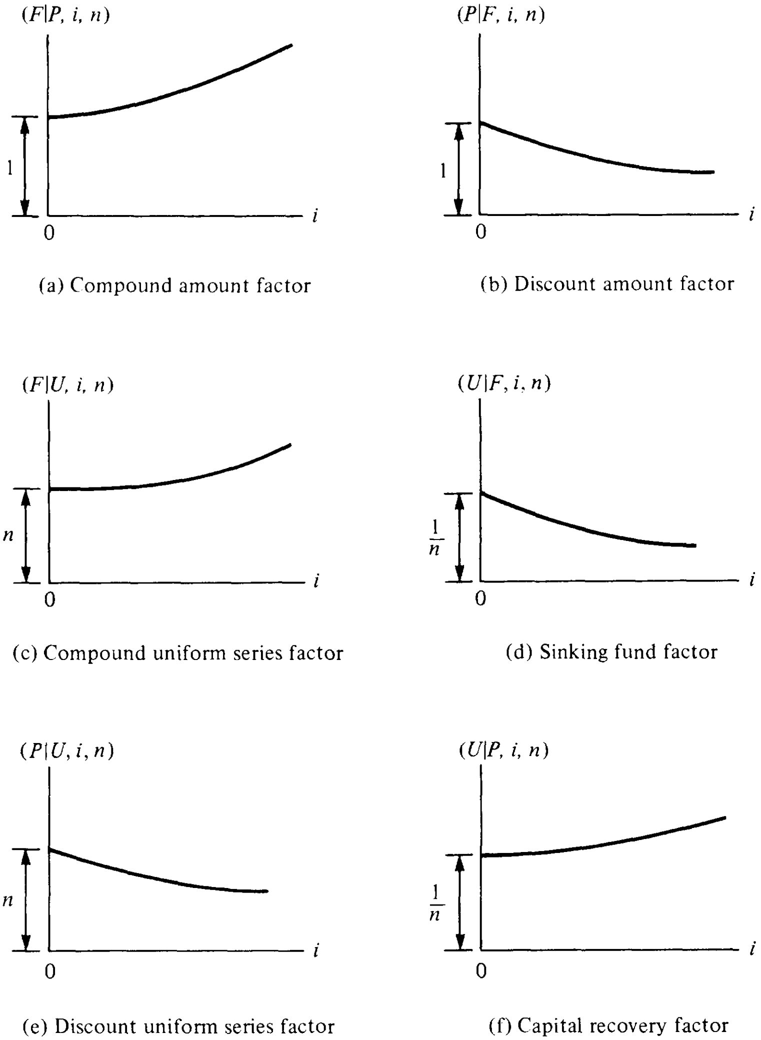

Figure 3.17 Compound interest factors.

Example 3.19

Tony MacDonald received a U.S. savings bond that was purchased as a gift for him by his grandfather at a discount price of \$75 six years ago. This savings bond is now redeemable at its face value of \$100. What is the annual compound interest for the bond?

For $P = 75$, $F = 100$, and $n = 6$, we can compute i according to Eq. (3.12) as follows:

$$i = \sqrt[6]{\frac{100}{75}} - 1 = \sqrt[6]{1.3333} - 1$$

$$= 1.049 - 1 = 0.049 = 4.9\%$$

Alternatively, we can use interpolation from the values of the $(F \mid P, i, n)$ factor in Appendix A. Since $F/P = 1.3333$, we try to find values of i in various tables in Appendix A that lead to F/P near 1.3333 for $n = 6$. For example,

Trial i	$(F \mid P, i, 6)$
3%	1.1941
4%	1.2653
5%	1.3401

Using linear interpolation between 4% and 5%, and letting $x\%$ be the increment from 4%, as indicated in Fig. 3.18, we obtain from the relationship of similar triangles,

$$\frac{x}{5-4} = \frac{1.3333 - 1.2653}{1.3401 - 1.2653} \quad \text{or} \quad x = \frac{0.0680}{0.0748} = 0.91$$

Hence,

$$i = 4 + x = 4 + 0.91 = 4.91\%$$

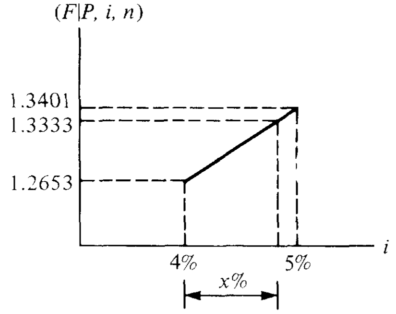

Figure 3.18 Interpolation of interest rates for Example 3.19.

Example 3.20

A bank solicits customers who will deposit \$1,000 at the end of each year for 20 years with the promise of a return of \$35,000 at the end of 20 years. What is the annual compound interest rate that the bank is offering?

The equivalence of the uniform payments $U = \$1{,}000$ is known to be a return $F = \$35{,}000$ for $n = 20$. For $F/U = 35$, we try to find values of i from various tables in Appendix A that lead to F/U near 35 for $n = 20$. For example,

Trial i	$(F \mid U, i, 20)$
4%	29.7781
5%	33.0659
6%	36.7856

Using linear interpolation between 5% and 6%, and letting x% be the increment from 5%, as shown in Fig. 3.19, we get

$$\frac{x}{6-5} = \frac{35.0 - 33.0659}{36.7856 - 33.0659} \quad \text{or} \quad x = \frac{1.934}{3.7197} = 0.52$$

Hence,

$$i = 5 + x = 5 + 0.52 = 5.52\%$$

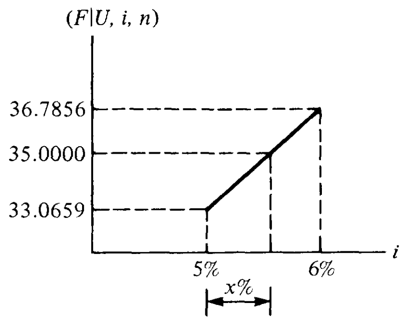

Figure 3.19 Interpolation of interest rates for Example 3.20.

Example 3.21

An insurance company offers an annuity policy to retired persons that requires an immediate payment of \$40,000 in return for a guaranteed annual income of \$2,500 for the next 30 years. What is the annual compound interest rate that the insurance company offers to pay in this policy?

In this problem, the present sum paid is $P = \$40{,}000$ and the annual receipt $U = \$2{,}500$ in a uniform series continuing for $n = 30$ years. For $P/U = 16$, we try to find values of i from various tables in Appendix A that lead to P/U near 16. For example,

Trial i	$(P \mid U, i, 30)$
3%	19.6004
4%	17.2920
5%	15.3725

Using linear interpolation between 4% and 5%, and letting $x\%$ be the increment from 4%, as indicated in Fig. 3.20, we get

$$\frac{x}{5-4} = \frac{16.0 - 17.292}{15.3725 - 17.292} \quad \text{or} \quad x = \frac{-1.292}{-1.9195} = 0.67$$

Hence,

$$i = 4 + x = 4 + 0.67 = 4.67\%$$

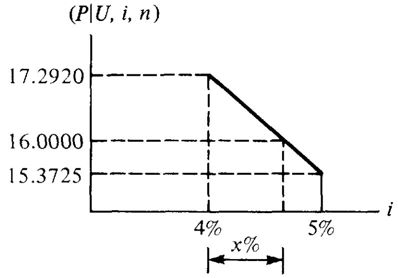

Figure 3.20 Interpolation of interest rates for Example 3.21.

3.9 ANNUAL PERCENTAGE RATE

In previous sections, we have emphasized compound interest accrued at an interest rate per period for a number of interest periods. In engineering economic analysis, a year is usually used as an interest period since investments in engineering projects are of long duration and a calendar year is a convenient period for accounting and tax computation. However, financial institutions that provide opportunities for making deposits and loans often offer more than one interest period per year in compounding the interest. Consequently, we introduce the terms *annual percentage rate* (APR) and *effective annual interest rate* to describe such situations more precisely.

If a financial institution offers more than one interest period per year in compounding the interest, it may advertise an *annual percentage rate,* which is the annual interest rate *neglecting the effect of any compounding during the year*. However, it pays its depositors on the basis of an *effective annual interest rate, considering the effect of all compounding during the year*. For example, an annual percentage rate of 8% compounded semiannually means that the interest rate per period is $8\%/2 = 4\%$ for each of the 6-month periods during the year. Then the interest for a sum of $1 accrued at the end of the year is

$$\left(1 + \frac{0.08}{2}\right)^2 - 1 = 0.0816$$

Thus, the effective annual interest rate is 8.16%. Similarly, an annual percentage rate of 8% compounded quarterly means that the interest rate per period is $8\%/4 = 2\%$ for each of the 3-month periods during the year. Hence, the effective annual interest rate is

$$\left(1 + \frac{0.08}{4}\right)^4 - 1 = 0.0824 = 8.24\%$$

Let r be the annual percentage rate per year and s be the number of interest periods per year. Then, the interest rate per interest period is r/s. The effective interest rate i_e is defined as the annual interest per dollar expressed in percentage which is accrued at an annual percentage rate r compounded over s interest periods during the year. Thus,

$$i_e = \left(1 + \frac{r}{s}\right)^s - 1 \tag{3.25}$$

For the special case where $s = 1$, i.e., one interest period per year with $r/s = r$, Eq. (3.25) reduces to $i_e = r$. That is, the effective annual interest rate is identical to the annual percentage rate.

If a sum P is deposited to accrue interest for n years, the future sum at the end of n years is

$$F = P(1 + i_e)^n$$

Noting the value of $(1 + i_e)$ obtained from Eq. (3.25), we get

$$F = P\left(1 + \frac{r}{s}\right)^{sn} \tag{3.26}$$

Some financial institutions offer a large number of interest periods per year, such as $s = 365$ for daily compounding. As the number of interest periods s becomes very large, the interest rate per interest period r/s becomes very small. If s approaches infinity and r/s approaches zero as a limit, the limiting condition is equivalent to continuous compounding. Thus,

$$F = \lim_{s \to \infty} P\left(1 + \frac{r}{s}\right)^{sn} \tag{3.27}$$

The limit in Eq. (3.27) can be evaluated by first noting the limit of the following expression through binomial expansion, which results in an infinite series:

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$= 2.71828 \cdots = e$$

Let $r/s = 1/x$; then $s = rx$ and $sn = rxn$. Furthermore, $s \to \infty$ as $x \to \infty$. Then, Eq. (3.27) can be expressed in the form

$$F = \lim_{x \to \infty} \left[P\left(1 + \frac{1}{x}\right)^{rxn}\right]$$

$$= P\left[\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x\right]^{rn} = Pe^{rn}$$

Hence, for continuous compounding, we have

$$F = Pe^{rn} \tag{3.28}$$

and

$$P = Fe^{-rn} \tag{3.29}$$

The factors e^{rn} and e^{-rn} in Eqs. (3.28) and (3.29) are listed in the table of basic factors for continuous compounding in Appendix B.

For $n = 1$, Eq. (3.28) yields the sum of principal and interest at the end of the first year. Then, the *effective annual interest rate* for continuous compounding is obtained from $(F - P)/P$ as follows:

$$i_e = (Pe^r - P)\frac{1}{P} = e^r - 1 \tag{3.30}$$

Consequently, we can always compute the effective annual interest rate if we know the annual percentage rate and the number of interest periods per year, including an infinite number of periods corresponding to continuous compounding.

Example 3.22

Mr. and Mrs. Cosgrave have a home mortgage that requires a monthly payment of $600 due on the first day of each month. There is a grace period without penalty until the 15th of the month. If the payment is made on or after the 16th of the month, the required payment is $618. The Cosgraves regard the increase in payment from $600 to $618 as a subtle form of interest charge for a two-week period. According to this view, what is the effective annual interest?

For the two-week period between the 1st and the 15th of the month, the interest rate is

$$\frac{618 - 600}{600} = \frac{18}{600} = 0.03 = 3\%$$

Since there are 52 weeks or 26 two-week periods in a year, the effective annual interest is

$$\begin{aligned} i_e &= (1 + 0.03)^{26} - 1 = (F \mid P, 3\%, 26) - 1 \\ &= 2.1566 - 1 = 1.1566 = 155.66\% \end{aligned}$$

Example 3.23

Peoples State Bank advertises an annual percentage rate of 6% compounded semiannually, and Consolidated Savings and Loans Association advertises an annual percentage rate of 6% compounded continuously. What are their effective annual interest rates?

For Peoples State Bank, $r = 6\%$ and $s = 2$. Then, the effective annual interest rate is obtained from Eq. (3.26) as follows:

$$i_e = \left(1 + \frac{0.06}{2}\right)^2 - 1 = 0.0609 = 6.09\%$$

For Consolidated Savings and Loans Association $r = 6\%$ and $n = 1$, or $rn = 0.06$ for entry to Table B1 in Appendix B for continuous compounding. From Eq. (3.30),

$$i_e = e^{0.06} - 1 = 1.0618 - 1 = 0.0618 = 6.18\%$$

3.10 CONTINUOUS COMPOUNDING INTEREST FORMULAS

With the availability of automatic data processing, the use of continuous compounding in computing interest has become most common. For every given annual percentage rate, it is easy to compute the corresponding effective annual interest rate from continuous compounding by Eq. (3.30). However, in engineering economic analysis, the receipts and disbursements are often assumed to occur only at the end of the time periods $t = 0, 1, 2, \ldots, n$, where each time period is 1 year. Therefore, we can introduce a new functional notation and compound interest tables based on continuous compounding at the annual percentage rate r.

The compound interest factors resulting from continuous compounding of single sums of money can readily be derived from Eqs. (3.28) and (3.29). We shall define a new functional notation for these factors as follows:

$$\frac{F}{P} = e^{rn} = (F \mid P, r, n)_c \tag{3.31}$$

$$\frac{P}{F} = e^{-rn} = (P \mid F, r, n)_c \tag{3.32}$$

where r is the annual percentage rate and n is the number of years. The subscript c outside the parenthesis denotes continuous compounding.

For the uniform series shown in Fig. 3.7, we see that the future value F at $t = n$ due to continuous compounding is given by

$$F = Ue^{r(n-1)} + Ue^{r(n-2)} + \cdots + Ue^r + U$$

Factoring out U and multiplying both sides by e^r, we get in two successive steps

$$F = U[e^{r(n-1)} + e^{r(n-2)} + \cdots + e^r + 1]$$

$$e^rF = U[e^{rn} + e^{r(n-1)} + \cdots + e^{2r} + e^r]$$

Subtracting the first from the second equation above, we obtain

$$(e^r - 1)F = U(e^{rn} - 1)$$

or

$$F = U\left(\frac{e^{rn} - 1}{e^r - 1}\right) \tag{3.33}$$

Similarly, for the uniform series in Fig. 3.8, we see that the present value at $t = 0$ due to continuous compounding is given by

$$P = Ue^{-r} + Ue^{-2r} + \cdots + Ue^{-(n-1)r} + Ue^{-nr}$$

Factoring out U and multiplying both sides by e^r, we get in two successive steps

$$P = U[e^{-r} + e^{-2r} + \cdots + e^{-(n-1)r} + e^{-nr}]$$

$$e^rP = U[1 + e^{-r} + \cdots + e^{-(n-2)r} + e^{-(n-1)r}]$$

Subtracting the first from the second equation above yields

$$(e^r - 1)P = U(1 - e^{-nr})$$

or

$$P = U\left(\frac{1 - e^{-nr}}{e^r - 1}\right) \tag{3.34}$$

Using the new functional notation, the compound interest factors resulting from continuous compounding of a uniform series are obtained from Eqs. (3.33) and (3.34) as follows:

$$\frac{F}{U} = \frac{e^{rn} - 1}{e^r - 1} = (F \mid U, r, n)_c \tag{3.35}$$

$$\frac{U}{F} = \frac{e^r - 1}{e^{rn} - 1} = (U \mid F, r, n)_c \tag{3.36}$$

$$\frac{P}{U} = \frac{1 - e^{-nr}}{e^r - 1} = (P \mid U, r, n)_c \tag{3.37}$$

$$\frac{U}{P} = \frac{e^r - 1}{1 - e^{-nr}} = (U \mid P, r, n)_c \tag{3.38}$$

For the basic gradient in part (a) of Fig. 3.14, we use superposition to obtain

$$P = G[e^{-2r} + 2e^{-3r} + \cdots + (n - 2)e^{-(n-1)r} + (n - 1)e^{-nr}]$$

Multiplying by e^r gives

$$e^rP = G[e^{-r} + 2e^{-2r} + \cdots + (n - 2)e^{-(n-2)r} + (n - 1)e^{-(n-1)r}]$$

Subtracting from the last equation the previous one yields

$$(e^r - 1)P = G[e^{-r} + e^{-2r} + \cdots + e^{-(n-1)r} + e^{-nr} - ne^{-nr}]$$

Noting the relationship in Eq. (3.34), we get

$$(e^r - 1)P = G\left(\frac{1 - e^{-nr}}{e^r - 1} - ne^{-nr}\right)$$

or

$$\frac{P}{G} = \frac{1}{(e^r - 1)}\left(\frac{1 - e^{-rn}}{e^r - 1} - ne^{-rn}\right) \tag{3.39}$$

Hence

$$\frac{F}{G} = \left(\frac{F}{P}\right)\left(\frac{P}{G}\right) = \frac{e^{rn}}{(e^r - 1)}\left(\frac{1 - e^{-rn}}{e^r - 1} - ne^{-rn}\right) \tag{3.40}$$

and

$$\frac{U}{G} = \left(\frac{U}{P}\right)\left(\frac{P}{G}\right) = \frac{1}{1 - e^{-nr}}\left(\frac{1 - e^{-rn}}{e^r - 1} - ne^{-rn}\right) \tag{3.41}$$

With the new functional notation, the above three equations are denoted by

$$\frac{P}{G} = (P \mid G, r, n)_c \tag{3.42}$$

$$\frac{F}{G} = (F \mid G, r, n)_c \tag{3.43}$$

$$\frac{U}{G} = (U \mid G, r, n)_c \tag{3.44}$$

It should be noted that Eqs. (3.33) through (3.41) involve the quotient of the differences of two numbers with the same order of magnitude. Consequently, many significant figures must be carried out to obtain accurate values for various factors in the continuous compound interest tables that are given in Appendix C.

Example 3.24

If the annual percentage rate is 10% compounded continuously for Example 3.15 while all other quantities remain unchanged, determine the present value of the annual savings in the next 10 years.

Using the same approach but a different set of values for $r = 10\%$ from the continuous compound interest tables in Appendix C, we get

$$\begin{aligned} P &= (20{,}000)(P \mid U, 10\%, 10)_c - (5{,}000)(P \mid U, 10\%, 5)_c \\ &= (20{,}000)(6.0104) - (5{,}000)(3.7412) \\ &= 120{,}208 - 18{,}706 = 101{,}502 \end{aligned}$$

Note that because of a higher effective annual interest rate due to continuous compounding, the present value obtained in this problem is smaller than that in Example 3.15.

3.11 SUMMARY AND STUDY GUIDE

In this chapter, we have examined the concept of the time value of money and the equivalence of cash flows. We have discussed the principle of decomposition and superposition of cash flows which is the foundation for computing the net present value or the net future value of a cash flow profile.

Discrete compound interest formulas have been derived for converting present sums, future sums, uniform series, and linear gradients to specified points in time. For ease of reference, a functional notation has been introduced. You are encouraged to conceptualize the problems in the functional notation before carrying out the numerical computation. Although it is possible to derive additional formulas for computing the present value or future value of cash flows in nonlinear gradients and other special cases, such formulas are rather cumbersome in application and have not been included. While the topic of the rates of return will be discussed more thoroughly in Chapter 4, the computation of unknown interest rates for simple cash flow profiles can be obtained by interpolation of numerical values from compound interest tables.

We have also covered the concepts of annual percentage rate and effective annual interest rate, leading to the idea of continuous compounding. Continuous compound interest formulas were then derived. However, we have considered only discrete cash flows, i.e., cash flows received or disbursed at the end of finite interest periods, even though the interest for such cash flows may be compounded discretely or continuously. We have not considered cash flows that are received or disbursed continuously over time since such coverage will divert the primary focus of our attention on the equivalence of cash flows.

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PROBLEMS

P3.1 In considering a 5-year budget for remodeling a testing laboratory, it is expected that $10,000 each will be spent at the end of the first and third years, but no money will be spent at the end of the second and fourth years. How much money can be spent at the end of the fifth year if an initial sum of $25,000 for this purpose is deposited in a bank that pays an interest of 6% compounded annually?

P3.2 The Public Works Department of a city plans to repave some of the city streets in its 6-year budget. The plan calls for the expenditures of $20,000 at the end of the first year, $10,000 each at the end of the second and third year, and $12,000 each at the end of the fourth, fifth, and sixth year. If the annual discount rate is 6%, what is the present value of the expenditures?

P3.3 Eight years ago, Mr. Williams decided to set aside $2,000 at the end of each year in the bank in anticipation of his need to buy a new truck in the future. He was able to do so in all 8 years except at the end of the third and fifth years when he ran into financial difficulties. Having just deposited $2,000 at the end of the eighth year in a bank where the money has accrued interest at 6% per annum, he found a new truck priced at $15,000. Has he sufficient money in the bank to purchase it?

P3.4 An issue of 6-year U.S. savings bonds has a face value of $100 at maturity. If the annual interest rate is 6% per year, how much should a bond be sold for when it is issued?

P3.5 A company invests $5,000 in a new machine that is expected to produce a return of $1,000 per year for the next 10 years. At a 10% annual interest rate, is the investment worthwhile?

P3.6 A solar heating system for residential buildings is advertised for $3,000 cash. The company marketing the system offers a deferred payment plan so that the first payment will be made at the end of 2 years and subsequent payments of the same amount will be paid every 6 months until the end of 5 years, i.e., seven equal payments. If the nominal interest rate is 4% per period of 6 months, what should be the amount of each payment so that the deferred payment plan will be equivalent to the $3,000 cash payment now?

P3.7 Mr. and Mrs. Payne would like to buy a $40,000 home with a down payment of $10,000 and a 20-year mortgage on the rest. If they are allowed to make uniform end-of-year payments at 8% annual interest, what will be the annual payment over a 20-year period?

P3.8 A contractor expects to receive $2,000 per year during the next 10 years, except the end of the second and the sixth year. What is the present value of these receipts at 9% annual interest?

P3.9 The annual maintenance cost on a piece of construction equipment is $200 for each of the first 3 years, with an increase of $50 more in each subsequent year (i.e., $250 for the fourth year, $300 for the fifth year, etc.) until the end of the seventh year, when the equipment will be disposed of. At 6% annual interest, what is the equivalent sum of the maintenance costs at the end of the seventh year?

P3.10 An engineer's starting annual salary is $16,000. For the next 10 years, he expects a $2,000 raise each year. What is the present value of his expected earnings in the next 10 years if the annual interest rate is 6%?

P3.11 Ms. McDonald bought a car for $6,000 and agreed to pay for it in 24 equal end-of-month payments at 1% monthly interest. Immediately after making the sixth payment, she wants to pay the balance due in a single sum. What should the final single payment be if there is no penalty for making the extra payment?

P3.12 Mr. Goodman has just reached his fifty-fifth birthday and wants to save $800 per year at 7% annual interest for 10 years until his sixty-fifth birthday. If he plans to spend this money in five equal amounts to be withdrawn on his sixty-sixth, sixty-seventh, sixty-eight, sixty-ninth, and seventieth birthdays, what will be the amount in each withdrawal?

P3.13 The annual maintenance cost of a nuclear power plant is expected to be $300,000 for the first year and to decline $30,000 each subsequent year until it reaches $150,000 in the sixth year, beyond which the annual maintenance cost will remain uniform at $150,000. If the useful life of the power plant is 40 years and the maintenance costs are end-of-year charges, what is the present value of these charges at a 10% annual interest rate?

P3.14 A small construction firm entered into a 10-year contract with a quarry for gravel supply with an agreement to pay $50,000 immediately plus $10,000 per year beginning with the end of the sixth year and ending with the end of the contract period. Because of substantial profits at the end of the fourth year, the firm requested that it be allowed to pay off the rest of the contract with a lump sum. Both the firm and the quarry agreed to an interest rate of 6% compounded annually. What should this lump sum be?

P3.15 A new telephone system is being installed in a business office with an expectation of reduction in staff. The annual savings will be $40,000 in the first year, with $5,000 less savings in each subsequent year (i.e., $35,000 savings in the second year, $30,000 in the third year) until the equipment is retired after 6 years of service. At a 6% annual interest rate, determine the equivalent sum of the savings at the time the equipment is retired.

P3.16 A person borrows $6,000 at 6% compounded annually and wishes to pay back the loan with annual payments over a 5-year period. It is agreed that the annual payments will decrease by $300 per year. That is, the payment in the second year is to be $300 less than that in the first year, the payment in the third year is to be $300 less than that in the second year, etc. What should be the payment at the end of the first year?

P3.17 Martha Donohue expects to receive an annual bonus from her employer and wishes to deposit it in an account that pays interest at a rate of 8% compounded annually. Her bonus in the first 4 years is to be $2,000 per year and will be increased at a rate of 6% per year from the fifth to the seventh year. That is, the increase in the fifth year is 6% over that of the fourth year, and the increase in the sixth year is 6% over that of the fifth year, etc. What will be the accumulated amount in the account after she deposits the last bonus at the end of the seventh year?

P3.18 Jean bought a car for $14,000 and agreed to pay for it in 30 equal payments at the end of each month at 1% monthly interest. Immediately after making the payment at the end of 12 months, she plans to go abroad for a year and wants to pay a single sum for the next 12 months in order to save interest charges as well as for her own convenience of not having to make payments during that period. Upon her return, she intends to resume payments regularly in the last 6 months. What should be this single sum to be paid at the end of 12 months if she is allowed to do so without penalty?

P3.19 Mr. and Mrs. Vainwright begin to save money for their son's college education when he reaches three years old. They plan to put aside 14 uniform annual amounts on his third (three years after birth) through 16th birthdays, thus providing $15,000, $14,000, $13,000, and $12,000 on his 17th, 18th, 19th, and 20th birthdays, respectively. If the interest rate is 8% compounded annually, what should be this annual amount?

P3.20 A machine that costs $80,000 is expected to last for 15 years. The annual maintenance cost is $1,000 per year in the first five years and $1,500 per year in the second five years. Then the annual maintenance cost goes up by an amount of $100 each year according to a linear gradient starting with $1,600 in the 11th year and ending with $2,000 in the 15th year. The used machine can be sold for $4,000 at the end of 15 years. The operating of this machine will generate a uniform series of income of $12,000 per year for 15 years. If the discount rate is 8%, find the net present value of the investment cash flow profile including all disbursements and receipts.

P3.21 If you buy a car by borrowing $12,000 now and make 36 monthly payments of $500 each including an immediate first payment, what is the monthly interest rate?

P3.22 Cecilia bought a refrigerator on credit. It costs $1,400 and is to be paid off at a monthly payment of $70 for the next 25 months beginning from the end of the first month. What is the monthly interest rate?

P3.23 If a bank gives you an education loan of $20,000 now and you are required to repay $3,000 annually for the next 10 years beginning from the end of the first year, what is the annual interest rate?

P3.24 A bank makes a loan to an entrepreneur in the form of a series of 9 annual payments of $1,000 each beginning from the end of the first year. At the end of 10 years, a single sum of $14,193 will be required to repay the entire loan and interest. What is the annual interest rate?

P3.25 An elderly couple won a Pennsylvania lottery that would pay them a series of 20 uniform annual payments of $250,000 each. Immediately after receiving the tenth payment, the couple entered into an agreement that they receive a single sum of $1.5 million at that time from a friend who in turn will receive the remaining 10 payments from the state according to the original schedule. What is the discount rate used by their friend in determining this single sum?

P3.26 An entrepreneur in a new engineering venture attempts to borrow $10,000 for 5 years with no repayment for 5 years. Babcock National Bank offers him the loan at an annual percentage rate of 12% compounded semiannually, and Kensington State Bank makes an offer at a nominal annual interest rate of 11% compounded continuously. What will be the repayment at the end of 5 years from each of these offers?

P3.27 The First National Bank offers 8% interest compounded quarterly for 8-year savings certificates. The Wilkin City Bank offers 7.5% interest compounded continuously for the same type of certificate. What are the values of the certificates from the two banks after 8 years if the initial value of each is $1,000?

P3.28 A Savings and Loan Association advertises an annual percentage rate of 7% compounded continuously for an 8-year savings certificate. (a) What is the effective annual interest rate? (b) What is the total interest at the end of 8 years for a certificate of $1,000 at the time of purchase?

P3.29 Jim Wilson deposited $1,000 in a savings account 10 years ago. For the first 5 years, the bank offered an annual percentage rate of 6% compounded semiannually, and for the last 5 years, it offered an annual percentage rate of 8% compounded quarterly. How much was in the account at the end of 10 years?

P3.30 A bank is offering an interest rate of 0.5% per month for its savings account and is considering a switch to continuous compounding at an annual percentage rate of 6.25%. What are the effective annual interest rates before and after the switching?

P3.31 A bill sent to a customer by the City Water Authority contains the following information: "Pay $36.00 on or before the due date of January 23, or $37.80 after the due date. The last date of acceptance for $37.80 without additional penalty is March 23." If you intend to pay either $36.00 on January 23 or $37.80 on March 23, what is the effective annual interest rate by choosing the late payment date?

P3.32 If the annual percentage rate is 6% compounded continuously for Problem P3.9 while all other quantities remain unchanged, determine the equivalent sum at the end of the seventh year.

P3.33 If the annual percentage rate is 10% compounded continuously for Problem P3.13 while all other quantities remain unchanged, determine the present value of the charges.

P3.34 If the annual percentage rate is 8% compounded continuously for Problem P3.17 while all other quantities remain unchanged, determine the equivalent sum of the savings at the end of the seventh year.

CHAPTER

4 Discounted Cash Flows and Rates of Return

4.1 INVESTMENT AND BORROWING CASH FLOW PROFILES

In Chapter 2, we introduced the net future value (NFV) and the internal rate of return (IRR) as merit measures for the one-period examples in capital investment analysis. After having discussed in Chapter 3 the compound interest formulas and operations for some special cases of multiperiod cash flow profiles, we are now ready to examine the merit measures associated with the general case of a multiperiod cash flow profile, which may consist of a variety of transactions in the discrete time periods over the planning horizon.

Consider an n-period cash flow profile $A_0, A_1, A_2, \ldots, A_n$ in which each cash flow A_t (for $t = 0, 1, 2, \ldots, n$) may be positive, negative, or zero. To be more specific, A_t represents the net cash flow in period t, which is the difference between income and expenditure or between benefit and cost in that period. Thus, a positive value of A_t represents a net receipt and a negative value represents a net disbursement from the transactions in that period.

Generally, a multiperiod cash flow profile represents either an investment opportunity or a borrowing opportunity. A conventional investment cash flow profile is characterized by one or more early periods of disbursements, followed by one or more early periods of receipts; conversely, a conventional borrowing cash flow profile is characterized by one or more early periods of receipts, followed by one or more later periods of disbursements. In each case, the cash flow profile can be unmistakenly identified either as an investment or a borrowing opportunity. A special case of a conventional investment often encountered is an outlay at $t = 0$ followed by a series of returns in subsequent periods, and that of a conventional borrowing is a loan at $t = 0$ followed by a series of repayments in subsequent periods.

In complex business ventures, an investment opportunity or a borrowing opportunity may be represented by unconventional cash flow profiles with intermittent periods of disbursements and receipts, such as those shown schematically in Figs. 4.1 and 4.2. An unconventional cash flow profile may be presented to a decision maker for evaluation as either an investment or a borrowing opportunity. For example, Fig. 4.1 may represent an investment opportunity in which additional outlays at intermediate periods are expected, but Fig. 4.2 may also represent an investment opportunity in which a payment is received at $t = 0$ as an enticement for participation and an outlay is needed at $t = n$ for the decommissioning or disposal of the physical asset. Conversely, Fig. 4.1 may represent a borrowing opportunity as much as Fig. 4.2 resulting from unconventional transactions. It is up to the decision maker to evaluate each one in the proper context, i.e., whether it is proposed as an investment or borrowing opportunity. The investment problem and the borrowing problem represent two sides of the same coin. In a corporation, the chief operating officer will look out for the most profitable opportunities for investment, while the chief financial officer will be responsible for finding the cheapest ways to raise funds for such projects.

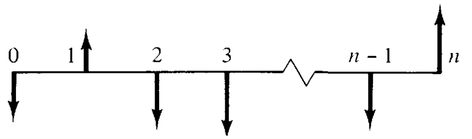


Figure 4.1 Unconventional cash profile with an initial disbursement.

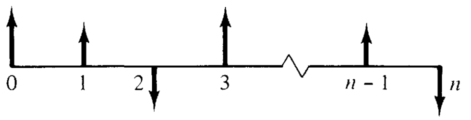


Figure 4.2 Unconventional cash profile with an initial receipt.

A decision criterion for accepting an independent proposal or for selecting the best among a group of mutually exclusive proposals consists of two elements: a merit measure and a set of decision rules. In this chapter, we introduce a number of merit measures associated with multiperiod discounted cash flow analysis. The reason for the existence of so many merit measures is that decision makers like to consider various factors such as profit size, percentage rate of return for the funds committed, and the overall percentage rate of return over the planning horizon. Each merit measure can be used as a decision criterion for accepting or rejecting an independent project, but only the direct merit measures can be used to rank mutually exclusive proposals in selecting the best among them. Some of these measures are tied directly to the minimum attractive rate of return (MARR) based on the assumption of perfect capital markets; others are independent of the MARR but will be compared to the MARR when the decision rules are invoked.

4.2 NET PRESENT VALUE AND NET FUTURE VALUE

In Section 3.4, we introduced the principles of decomposition and superposition of cash flows to find the combined effects of their equivalent amounts at the same point in time. These principles can be applied to obtain the present value or future value of any cash flow series, provided that the discount rate per period and the number of discrete discount periods associated with receipts or disbursements are known.

Consider an n-period cash flow series $A_0, A_1, A_2, \ldots, A_n$, each of which may be positive, negative, or zero. The present value of such a cash flow series, when the algebraic signs of the cash flows are properly taken into consideration, is called the *net present value* and is equal to

$$\text{NPV} = A_0 + A_1(1 + i)^{-1} + A_2(1 + i)^{-2} + \cdots + A_n(1 + i)^{-n} \tag{4.1}$$

in which all cash flows in future periods subsequent to period 0 are discounted to the present at period 0. Similarly, the future value of such a cash flow series, when the algebraic signs of the cash flows are properly taken into consideration, is called the *net future value* and is equal to

$$\text{NFV} = A_0(1 + i)^n + A_1(1 + i)^{n-1} + \cdots + A_n(1 + i) + A_n \tag{4.2}$$

in which all cash flows in periods prior to period n are compounded to the future at period n.

Using an abbreviated notation, the net present value of a general cash flow profile of n periods is given by

$$\text{NPV} = \sum_{t=0}^{n} A_t(1 + i)^{-t} = \sum_{t=0}^{n} A_t(P \mid F, i, t) \tag{4.3}$$

Similarly, the net future value is given by

$$\text{NFV} = \sum_{t=0}^{n} A_t(1 + i)^{n-t} = \sum_{t=0}^{n} A_t(F \mid P, i, n - t) \tag{4.4}$$

Furthermore, it can easily be shown that

$$\text{NPV} = \text{NFV}(1 + i)^{-n} \tag{4.5}$$

Conversely,

$$\text{NFV} = \text{NPV}(1 + i)^n \tag{4.6}$$

In the context of capital investment analysis, the NPV and NFV are numerical measures of the equivalence of a cash flow profile discounted or compounded at the minimum attractive rate of return (MARR) since an investor can lend or borrow freely at the MARR under the assumption of a perfect capital market. Let i^* be the MARR for an investment cash flow profile with n discount periods. By substituting

$i = i^*$ into Eq. (4.2), the net future value (NFV) evaluated at the MARR means that the cash flows in all periods prior to the terminal period $t = n$ are reinvested externally at a rate of MARR. For a conventional investment cash flow profile, NFV is a direct merit measure of the size of gain that an investor would have gotten at the terminal period by investing in a project as specified by this cash flow profile instead of investing in the foregone opportunity represented by the MARR. For a conventional borrowing opportunity, the MARR refers to the *maximum acceptable rate of borrowing* to the borrower (or the minimum attractive rate of return to the lender). Then, the NFV of a conventional borrowing cash flow profile is also a direct merit measure of the size of gain that a borrower would have gotten at the terminal period by taking this borrowing opportunity instead of borrowing at the MARR. For an unconventional cash flow profile, the minimum attractive rate of return (MARR) is applied to investment periods and the maximum acceptable rate of borrowing (also designated as MARR) is applied to borrowing periods of the profile. Hence, it does not matter whether the cash flow profile is conventional or unconventional since the same MARR is applied to both positive and negative cash flows. A positive NFV indicates a net gain from the investment and a negative NFV indicates a net loss.

Since the net present value (NPV) can be obtained by multiplying the NFV by a positive factor of $(1 + i)^{-n}$ to find the equivalent value at $t = 0$, as indicated in Eq. (4.5), it is also a direct measure of an investment cash flow profile evaluated at $i = i^*$. Whether it is obtained directly from Eq. (4.1) or indirectly from Eqs. (4.2) and (4.5), the NPV represents the size of gain as measured by its time value at $t = 0$ for a cash flow profile with n interest periods by investing in a project as specified by this cash flow profile instead of investing in the foregone opportunity represented by the MARR. The NPV can be a net gain or a net loss, as indicated by a positive or a negative sign, respectively.

Since both the NFV and NPV are direct merit measures, the decision criteria associated with them are simple and straightforward. For the NFV criterion, the decision rule for accepting or rejecting an investment or a borrowing proposal simply is: "Accept if NFV ≥ 0; reject otherwise." For the NPV criterion, it becomes: "Accept if NPV ≥ 0; reject otherwise."

Example 4.1

Hilda Lang has the opportunity to secure a loan with the receipts and repayments specified for months 0 to 6 in a six-month schedule as follows: $A_0 = +\$5{,}000$, $A_1 = +\$5{,}000$, $A_2 = A_3 = -\$1{,}800$, $A_4 = A_5 = -\$2{,}200$, and $A_6 = -\$2{,}500$. If the MARR of the borrower is 1% per month, find the net present value of this borrowing opportunity.

The net present value (NPV) of this cash flow profile can be obtained by using Eq. (4.3), and the computation is tabulated in Table 4.1.

The result indicates that Hilda is worse off by an amount of \$123.62 as expressed in the time value of money at $t = 0$ if this borrowing opportunity is used instead of borrowing at a MARR of 1% per month.

TABLE 4.1 COMPUTATION OF NET PRESENT VALUE

t	A_t	$(P \mid F, i, t)$	$A_t(P \mid F, i, t)$
0	+5,000	1.0000	+5,000.00
1	+5,000	0.9901	+4,950.50
2	−1,800	0.9803	−1,764.54
3	−1,800	0.9706	−1,747.08
4	−2,200	0.9610	−2,114.20
5	−2,200	0.9515	−2,093.30
6	−2,500	0.9420	−2,355.00
			NPV = −123.62

Example 4.2

Find the net present value of the following investment profile with mixed cash flows expressed in millions of dollars if the MARR is specified at 8% per period. $A_0 = +18$, $A_1 = +10$, $A_2 = -40$, $A_3 = -60$, $A_4 = +30$, and $A_5 = +50$.

The net present value (NPV) of the cash flow profile can be obtained by using Eq. (4.3), and the computation is tabulated in Table 4.2.

The result indicates that there is a net gain of $1.419 million expressed in the time value of money at $t = 0$ in comparison with investing or borrowing at a MARR of 8% per period.

TABLE 4.2 COMPUTATION OF NET PRESENT VALUE

t	A_t	$(P \mid F, i, t)$	$A_t(P \mid F, i, t)$
0	+18	1.0000	+18.000
1	+10	0.9259	+ 9.259
2	−40	0.8573	−34.292
3	−60	0.7938	−47.628
4	+30	0.7350	+22.050
5	+50	0.6806	+34.030
			NPV = +1.419

4.3 THE INTERNAL RATE OF RETURN

In the analysis of an n-period cash flow profile $A_0, A_1, A_2, \ldots, A_n$, *the internal rate of return* (IRR) is defined as the discount rate that will set the net present value or the net future value of the cash flow profile equal to zero. Hence, the IRR refers to the rate of return that is "internal" to the cash flow profile and is independent of the external foregone opportunity represented by the MARR. By setting NPV = 0 in Eq. (4.1), we obtain

$$A_0 + A_1(1+i)^{-1} + A_2(1+i)^{-2} + \cdots + A_n(1+i)^{-n} = 0 \qquad (4.7)$$

and by setting NFV = 0 in Eq. (4.2), we get

$$A_0(1 + i)^n + A_1(1 + i)^{n-1} + \cdots + A_{n-1}(1 + i) + A_n = 0 \tag{4.8}$$

Note that Eq. (4.8) can also be obtained if Eq. (4.7) is multiplied by a factor of $(1 + i)^n$. Thus, the IRR can be obtained by finding the value(s) of i from Eq. (4.7) or Eq. (4.8).

The numerical solution of Eq. (4.8) can be simplified by letting $x = 1 + i$. Then, we have

$$A_0x^n + A_1x^{n-1} + \cdots + A_{n-2}x^2 + A_{n-1}x + A_n = 0 \tag{4.9}$$

The algebraic function of the left side of this equation can be abbreviated as

$$f(x) = A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n \tag{4.10}$$

The function $f(x)$ is said to have a *change of sign* if the coefficients of two successive terms in $f(x)$ have opposite signs. Powers of x having zero coefficients are not counted. Then, according to the Descartes rule of signs, the number of positive real roots in Eq. (4.9) cannot exceed the number of changes in sign in $f(x)$.

It is possible that there is no positive real root for $f(x) = 0$, regardless of the number of sign changes. In general, if there is only one sign change in $f(x)$, there will be a positive real root for $f(x) = 0$. However, if there are two or more sign changes in $f(x)$, there may be 0, 1, 2, . . . , c positive real roots for $f(x) = 0$ where c is the actual number of sign changes.

Since $x = 1 + i$, we can obtain the value of $i = x - 1$ corresponding to every positive real root of $f(x) = 0$. That is,

$$i = x - 1 = \begin{cases} \text{Positive} & \text{for } x > 1 \\ 0 & \text{for } x = 1 \\ \text{Negative} & \text{for } x < 1 \end{cases} \tag{4.11}$$

Hence, for every positive root x, i can be either positive or zero because a negative value of i is not admissible as an internal rate of return. Consequently, the number of nonnegative values of the internal rate of return i for a cash flow profile of A_0, A_1, . . . , A_n also cannot exceed the number of sign changes in the sequence, not counting the periods with zero cash flow, if any, in the sequence.

The plotting of the NPV versus i curve (or the NFV versus i curve) is accomplished by assigning a range of values for i starting from $i = 0$ and by computing the NPV (or the NFV) corresponding to each i. Since the internal rate of return that sets the net *present* value of a series of cash flows equal to zero will also set the net *future* value of the same series equal to zero, we can obtain the internal rate of return for a series of cash flows from either NPV = 0 or NFV = 0.

In seeking the solution of the internal rate of return (IRR), it is important to distinguish a cash flow profile either as an investment opportunity or a borrowing opportunity in the interpretation of the IRR as a merit measure whether the cash flow profile is conventional or unconventional. The IRR resulting from a conventional investment opportunity represents the *internal rate of return to the investor,* while the IRR resulting from a conventional borrowing opportunity represents the *internal rate of borrowing* (or the *internal rate of return to the lender*), even though both are customarily referred to as the internal rate of return. For unconventional cash flow profiles, the IRR may represent the internal rate of return in some periods and the internal rate of borrowing in other periods. However, it is possible that for some unconventional cash flow profiles, the IRR in all periods represents the internal rate of return or the internal rate of borrowing only. If a unique value of IRR exists for an investment cash flow profile whether the profile is conventional or unconventional, the case is referred to as *pure investment*. Similarly, if a unique value of IRR exists for a borrowing cash flow profile whether the profile is conventional or unconventional, the case is referred to as *pure borrowing*.[1]

The NPV versus i curves for the pure investment and pure borrowing cases where a unique value i' of the IRR exists are shown in Figs. 4.3 and 4.4 respectively. The unique value of the IRR is a useful merit measure of the percentage of return for the funds committed in accordance with the cash flow profile. When the value of IRR is not unique, it usually means either that no value of IRR exists in the positive range of i as shown in Figs. 4.5 and 4.6, or that multiple values of IRR are present in the positive range of i as shown in Figs. 4.7 and 4.8 in which two values of IRR are designated as i' and i''.

[1] The terms pure investment and pure borrowing follow the definitions in Ref. 4.3.

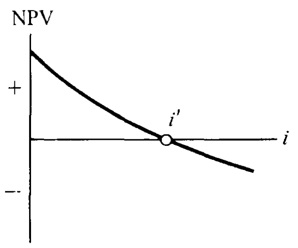


Figure 4.3 Unique IRR for pure investment.

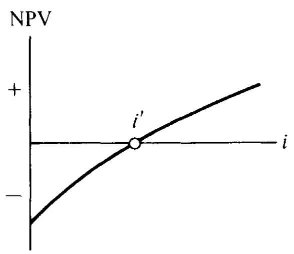


Figure 4.4 Unique IRR for pure borrowing.

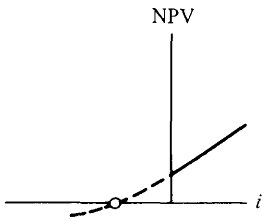


Figure 4.5 No positive IRR for positive NPV at $i = 0$.

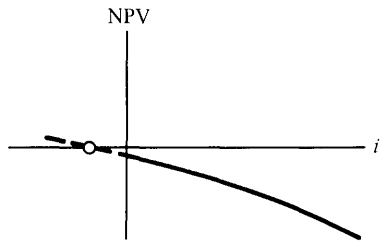


Figure 4.6 No positive IRR for negative NPV at $i = 0$.

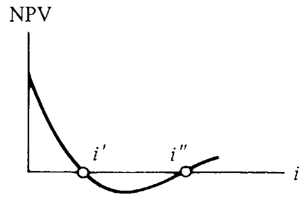


Figure 4.7 Multiple IRR for positive NPV at $i = 0$.

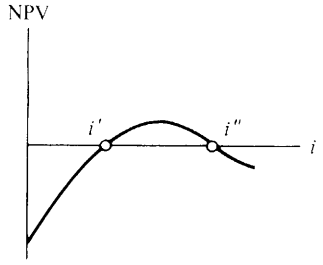


Figure 4.8 Multiple IRR for negative NPV at $i = 0$.

When a unique value of IRR exists, the set of decision rules for accepting or rejecting an investment or borrowing proposal is relatively simple. When no unique value of IRR exists, it does not mean that decision rules cannot be established, but such decision rules can be very complicated because they are related to the multiple values of IRR. (They will be discussed in detail in Chapter 8.) While the unique value of IRR is a merit measure if it exists, none of the multiple values of IRR can be used as a merit measure if they occur.

However, even if a unique value of IRR can be shown to exist, different rules are still needed to determine the acceptability of an investment opportunity or a borrowing opportunity because the term IRR refers to the internal rate of return in the former, but refers to the internal rate of borrowing in the latter. Under such circumstances, the set of decision rules consists of the following:

1. For a pure investment opportunity, accept if IRR $\geq$ MARR; reject otherwise.
2. For a pure borrowing opportunity, accept if IRR $\leq$ MARR; reject otherwise.

This set of decision rules associated with a unique value i' of IRR can be verified graphically by observing the positive or negative sign of NPV in the appropriate ranges of i relative to i' in Figs. 4.3 and 4.4, respectively.

Example 4.3

An investment proposal for a duration of 2 years indicates that a cost of $1,000 will be incurred in the beginning and a benefit of $600 will result at the end of each of the first and second years. Plot a NPV versus i graph for annual discount rates of 0%, 5%, 10%, 20%, and 25%. Also find the internal rate of return.

From the given cash flows over two interest periods, we get

$$\text{NPV} = -1{,}000 + 600(1 + i)^{-1} + 600(1 + i)^{-2}$$

The values of NPV for the range of given i are tabulated as follows:

i	0%	5%	10%	15%	20%	25%
NPV	200	116	41	−25	−83	−136

For NPV = 0, we get

$$-1{,}000(1 + i)^2 + 600(1 + i) + 600 = 0$$

Expanding and collecting terms, we have

$$-1{,}000i^2 - 1{,}400i + 200 = 0$$

Solving the quadratic equation, we obtain $i = 0.1307$ or -1.5307. Taking the positive root only, we have $i = 13.07\%$. The results of the net present values are plotted in Fig. 4.9.

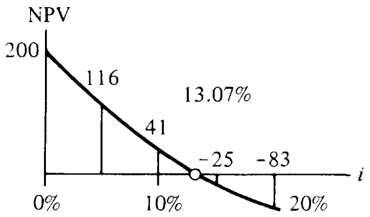


Figure 4.9 IRR for a two-period investment.

The positive real root of this equation can also be obtained by interpolation from interest tables. For this problem, for example, we can rewrite the equation in the form

$$\text{NPV} = -1{,}000 + 600(P \mid U, i, 2) = 0$$

or

$$(P \mid U, i, 2) = \frac{1{,}000}{600} = 1.6667$$

We try several values of i from the discrete compound interest tables in Appendix A that lead to $P \mid U$ near 1.6667 for $n = 2$. For example,

Trial i	$(P \mid U, i, 2)$
10%	1.7355
12%	1.6901
15%	1.6257

Using linear interpolation between 12% and 15%, we get

$$i = 12\% + (15\% - 12\%)\frac{1.6901 - 1.6667}{1.6901 - 1.6257} = 12\% + 1.09\% = 13.09\%$$

4.4 SOLVING FOR THE INTERNAL RATE OF RETURN

The solution of the internal rate of return from Eq. (4.9) is generally quite complicated when n is large. Fortunately, it can be carried out with automated computational aids such as computer-based spreadsheets or equation solvers. Even so, an approximate estimate of the value of IRR is often needed as an input to facilitate the automated search for the more accurate value. For problems involving multiple values of IRR, the knowledge of an appropriate estimate of each value will reduce the chance of missing some of the multiple values. Consequently, it is helpful to know the procedure for finding an approximate numerical solution of Eq. (4.9) with the aid of the discrete compound interest tables in Appendix A. As the interval for interpolation increases in size, the results obtained by interpolation will be less accurate, and it will be prudent to check the results by the available automated computational aids. The numerical solution of algebraic equations used in most computer programs is based on the Newton-Ralphson method. A simple procedure using a commercial computer-based spreadsheet is given in Appendix D.

In finding an approximate numerical solution of Eq. (4.9) by trial and error, we can try a range of values of x until we find all possible positive real roots of x that will lead to $f(x) = 0$. Let x_1 and x_2 be two successive values of x in a trial. If $f(x_1)$ and $f(x_2)$ are of opposite signs (one positive and the other negative), then a positive real root exists between x_1 and x_2, as shown in Fig. 4.10. Conversely, if $f(x_1)$ and $f(x_2)$ have the same sign, there may be none or an even number of positive real roots between x_1 and x_2, as shown in Fig. 4.11. Therefore, care must be taken so

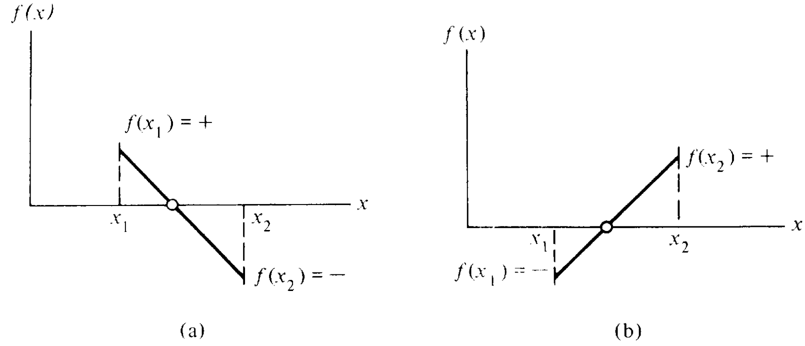


Figure 4.10 Finding the positive real root of a function by interpolation.

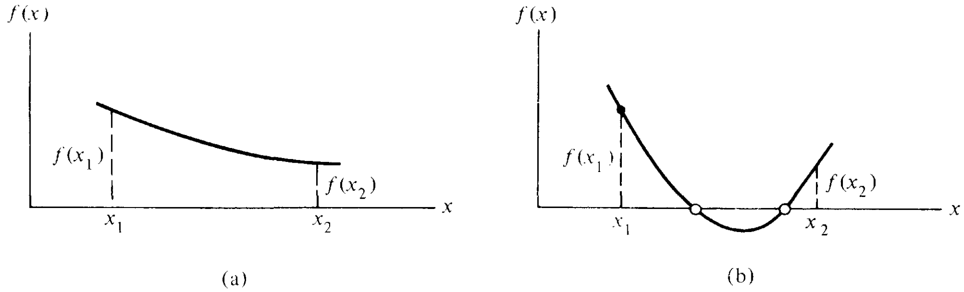


Figure 4.11 Multiple roots of a function in an interval.

that the unsearched interval between any two values of x will be sufficiently small if there are multiple sign changes in $f(x) = 0$. When the curve representing $f(x)$ is tangent to the horizontal axis x, the root at the tangent point is referred to as a repeated root because two real roots merge as one at that point. We can find the smallest positive real root if we start the trial solution from $x = 0$. When two or more positive real roots exist, each root may be obtained successively from the remaining equation after the smallest one has been found and factored out. Finally, the values of i corresponding to positive values of x found in Eq. (4.9) can be obtained from Eq. (4.11).

The approximate numerical solution of Eq. (4.9) may also be obtained by trial solution with the aid of the discrete compound interest tables in Appendix A. Again, it is important to observe the number of sign changes in the equation and to check for possible multiple positive roots when the conditions warrant. Thus, we try a range of values of i in Eq. (4.9) until we find all possible nonnegative value(s) of i that will lead to NPV = 0. It should be pointed out again that we can easily miss some positive values of i if we choose too large an interval of i in the search of the multiple values of i. The detailed procedure for obtaining nonnegative value(s) of i by trial solution in combination with the method of linear interpolation will be illustrated by examples.

Example 4.4

A cash flow profile with two interest periods consists of $A_0 = +1{,}000$, $A_1 = -2{,}500$, and $A_2 = +1{,}540$. Find the internal rate of return per interest period.

First, we find the net present value of the cash flow profile and set it equal to zero. Thus,

$$\text{NPV} = 1{,}000 - (2{,}500)(1 + i)^{-1} + (1{,}540)(1 + i)^{-2} = 0$$

Letting $x = 1 + i$ and multiplying through with x^2, we get

$$1{,}000x^2 - 2{,}500x + 1{,}540 = 0$$

For this simple quadratic equation, we recognize that

$$(10)(10x - 11)(10x - 14) = 0$$

Hence, $x = 1.1$ and $x = 1.4$. Then from $i = x - 1$, we get $i = 0.1 = 10\%$ and $i = 0.4 = 40\%$. A plot of NPV versus i is shown in Fig. 4.12 in which the internal rates of return are seen to be 10% and 40%.

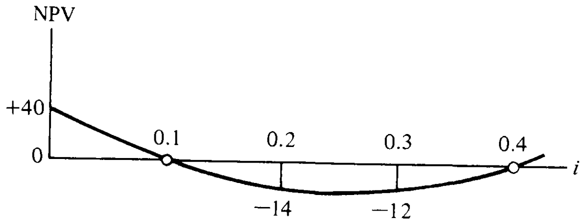


Figure 4.12 Two IRR for an unconventional cash flow profile.

Example 4.5

A cash flow profile with three interest periods consists of $A_0 = -1{,}000$, $A_t = +3{,}700$, $A_2 = -4{,}540$, and $A_3 = +1{,}848$. Verify that the internal rate(s) of return per interest period are 10%, 20%, and 40%, using the analytical approach.

We find the net present value of the cash flow profile and set it equal to zero. Thus,

$$\begin{aligned} \text{NPV} &= -1{,}000 + (3{,}700)(1 + i)^{-1} - (4{,}540)(1 + i)^{-2} + (1{,}848)(1 + i)^{-3} \\ &= 0 \end{aligned}$$

By letting $x = 1 + i$ and multiplying through by x^3 we get

$$-1{,}000x^3 + 3{,}700x^2 - 4{,}540x + 1{,}848 = 0$$

Generally, the analytical solution of a cubic equation is lengthy. However, for this problem, the results are already given, i.e., $i = 0.1$, 0.2, and 0.4. Consequently, we can verify the analytical solution by factoring, i.e.,

$$(10x - 11)(10x - 12)(10x - 14) = 0$$

which indeed leads to the given results of $i = 10\%$, 20%, and 40%.

Example 4.6

Find the numerical values of the internal rate(s) of return for the cash flow profile in Example 4.5, using the trial and error approach.

From the previous example, it is seen that for NPV = 0 we have

$$f(x) = -1{,}000x^3 + 3{,}700x^2 - 4{,}540x + 1{,}848 = 0$$

Since there are three sign changes in $f(x)$, there may be up to three positive real roots. Suppose that we try a set of values $x = 0$, $x = 1$, and $x = 1.3$. We find that

$$\begin{aligned} f(0) &= +1{,}848 \\ f(1) &= -1{,}000 + 3{,}700 - 4{,}540 + 1{,}848 = +8 \\ f(1.3) &= -2{,}197 + 6{,}253 - 5{,}902 + 1{,}848 = +2 \end{aligned}$$

We may infer that there is no positive real root between $x = 0$ and $x = 1$, or between $x = 1$ and $x = 1.3$. While the former conclusion turns out to be correct, the latter is not. There is no way to verify this fact other than to try smaller intervals of x between $x = 1$ and $x = 1.3$. Let us try a number of values between $x = 1$ and $x = 1.5$ and compute the corresponding values of $f(x)$ as follows:

$$\begin{aligned} x &= 1 \qquad & f(1) &= +8 \\ x &= 1.1 \qquad & f(1.1) &= 0 \end{aligned}$$

$$x = 1.15 \qquad f(1.15) = -1$$
$$x = 1.2 \qquad f(1.2) = 0$$
$$x = 1.3 \qquad f(1.3) = +2$$
$$x = 1.4 \qquad f(1.4) = 0$$
$$x = 1.5 \qquad f(1.5) = -12$$

From these results, we can conclude that there are three positive real roots, $x = 1.1$, $x = 1.2$, and $x = 1.4$ as shown in Fig. 4.13. The choice of $x = 1.15$ for trial is deliberate in order to verify that $f(x)$ is negative between $x = 1.1$ and $x = 1.2$, as indicated in the above equations. The corresponding values of the internal rate of return obtained from $i = x - 1$ are $i = 0.1 = 10\%$, $i = 0.2 = 20\%$, and $i = 0.4 = 40\%$.

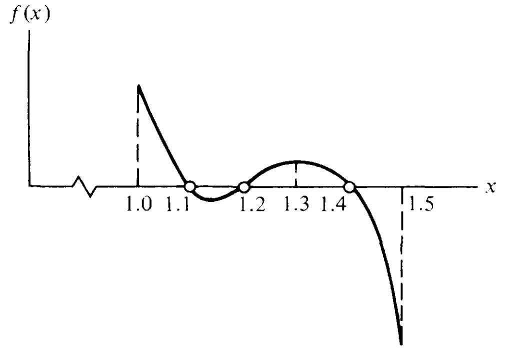


Figure 4.13 Three IRR for an unconventional cash flow profile.

Example 4.7

The Maxwell Manufacturing Company plans to invest \$77,000 in an energy-saving device with the expectation of receiving benefits for the next 5 years in the amounts of \$38,000, \$32,000, \$26,000, \$20,000, and \$14,000 at the end of first, second, third, fourth, and fifth years, respectively. What is the internal rate of return for investing in this device?

This series of cash flows has only one sign change since the cash flow at $t = 0$ is negative and all other cash flows at $t = 1, 2, \ldots, 5$ are positive. We expect only one positive real root of i, which can be obtained by using the method of linear interpolation. Since the benefits can be represented by a uniform series with $U = \$38{,}000$ and a basic linear gradient with $G = -\$6{,}000$ for $n = 5$, then the net present value of the cash flows in \$1,000 is

$$\text{NPV} = -77 + (38)(P \mid U, i, 5) - (6)(P \mid G, i, 5)$$

We can try various values of i until we find out which one sets the NPV equal to zero. We shall observe the change of the NPV as the value of i is increased until we note a change of NPV either from positive to negative, or vice versa.

The accuracy of the result of linear interpolation depends, of course, on the interval of i. For example, if we try $i = 20\%$, we can find the values of $(P \mid U, 20\%, 5)$ and $(P \mid G, 20\%, 5)$ from Appendix A and compute NPV from the above equation. The results of the computation for $i = 20\%$, 25%, and 30% are summarized as follows:

Trial i	$(P \mid U, i, 5)$	$(P \mid G, i, 5)$	NPV in \$1,000
20%	2.9906	4.9061	$-77 + 113.643 - 29.437 = +7.206$
25%	2.6893	4.2035	$-77 + 102.193 - 25.221 = -0.028$
30%	2.4356	3.6297	$-77 + 92.553 - 27.778 = -6.225$

Using linear interpolation of NPV between $i = 20\%$ and $i = 25\%$, and letting $x\%$ be the increment from 25% as shown in Fig. 4.14, we get

$$\frac{x}{25 - 20} = \frac{7.206}{7.206 + 0.028} \qquad x = (5)\frac{7.206}{7.234} = 4.98$$

Hence, the internal rate of return is

$$i = 20 + x = 20 + 4.98 = 24.98\% \text{ (close to 25\%)}$$

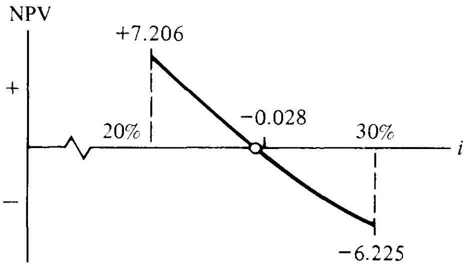


Figure 4.14 Example of finding IRR by interpolation.

In general, when the series of cash flows cannot be represented by a combination of the uniform series and the basic linear gradient, we can use Eq. (4.3) expressed in the functional notation:

$$\text{NPV} = \sum_{t=0}^{n} A_t(P \mid F, i, t)$$

For this particular problem, the NPV (in \$1,000) is

$$\text{NPV} = -77 + (38)(P \mid F, i, 1) + (32)(P \mid F, i, 2) + (26)(P \mid F, i, 3) + (20)(P \mid F, i, 4) + (14)(P \mid F, i, 5)$$

For each trial value of i, we can find the compound interest factors in the above equation from the tables in Appendix A. The resulting terms for each trial value of i are given in the vertical columns in Table 4.3, with the NPV in the bottom line.

TABLE 4.3 COMPUTATION OF INTERNAL RATE OF RETURN BY TRIAL

Year	Cash flow	$A_t(P \mid F, i, t)$				
t	A_t	0%	10%	20%	25%	30%
0	−77	−77	−77	−77	−77	−77
1	+38	+38	+34.546	+31.665	+30.400	+29.230
2	+32	+32	+26.445	+22.221	+20.480	+18.934
3	+26	+26	+19.534	+15.046	+13.312	+11.835
4	+20	+20	+13.660	+ 9.646	+ 8.192	+ 7.002
5	+14	+14	+ 8.693	+ 5.627	+ 4.588	+ 3.770
NPV		+53	+25.878	+ 7.205	− 0.028	− 6.229

It can be seen that the net present values for $i = 20\%$, 25%, and 30% are slightly different from those obtained previously due to truncation of digits in the tables. Nevertheless, the internal rate of return remains close to 25%.

Example 4.8

A government bond of $1,000 denomination redeemable at full value in 15 years with an annual interest rate of 4% payable at the end of each year was originally sold at its face value. Since the issuance of this bond 10 years ago, market interest rates have gone up steadily. After collecting the interest at the end of 10 years, this $1,000 bond can sell for only $700 in the market. If a holder of this bond wants to sell at this price, what is the internal rate of return for his investment? If the buyer keeps the bond to maturity, what is the internal rate of return for the buyer?

The annual interest from the bond is based on 4% of the face value of $1,000 and is $40 irrespective of the market price. If a holder of this bond keeps it until maturity, the cash flow profile of the investment will be as shown in part (a) of Fig. 4.15, and the internal rate of return for the investment will be 4%.

If the holder of this bond sells it at $700 after 10 years, the cash flow profile of the investment will be as shown in part (b) of Fig. 4.15, and the internal rate of return of the investment can be obtained as follows:

$$\text{NPV} = -1{,}000 + (40)(P \mid U, i, 10) + (700)(P \mid F, i, 10) = 0$$

We try the following values of i to obtain the compound interest factors in the tables in Appendix A:

i	$(P \mid U, i, 10)$	$(P \mid F, i, 10)$	NPV
1%	9.4713	0.9053	+12.56
2%	8.9826	0.8203	−66.49

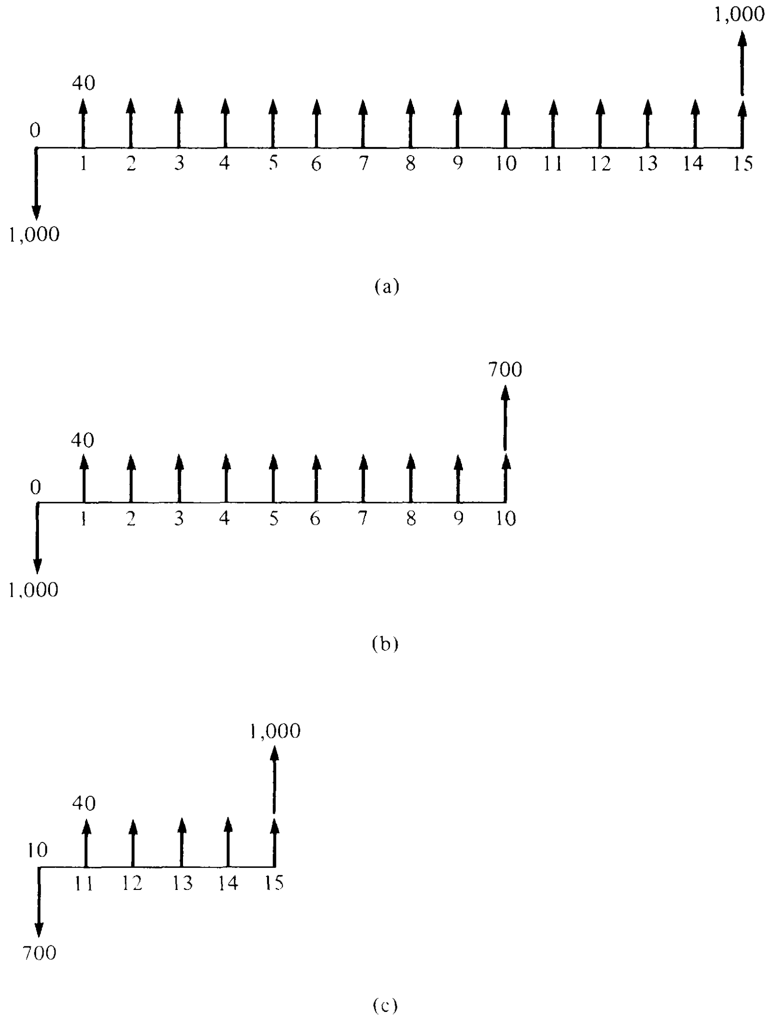


Figure 4.15 Example of computing IRR for coupon bonds.

Using linear interpolation between 1% and 2%, we find

$$i = 1\% + (2\% - 1\%)\frac{12.56}{12.56 + 66.49} = 1.16\%$$

For the buyer, the cash flow profile of the investment will be as shown in part (c) of Fig. 4.15 and the rate of return of the investment can be obtained as follows:

$$\text{NPV} = -700 + (40)(P \mid U, i, 5) + (1{,}000)(P \mid F, i, 5) = 0$$

We try the following values of i to obtain the compound interest factors in the tables in Appendix A:

i	$(P \mid U, i, 5)$	$(P \mid F, i, 5)$	NPV
10%	3.7908	0.6209	+71.53
12%	3.6048	0.5674	+11.59
15%	3.3522	0.4972	−68.71

Using linear interpolation between 12% and 15%, we find

$$i = 12\% + (15\% - 12\%)\frac{11.59}{11.59 + 68.71} = 12.43\%$$

Example 4.9

The Chapman Construction Company has been engaged by a foreign government to build an oil refinery. The cash flows (in millions of dollars) in the 5-year period of construction are +18, +10, −40, −60, +30, and +50, for $t = 0, 1, 2, \ldots, 5$, respectively. Find the internal rate of return from the construction of the refinery.

Since this series of cash flows has two sign changes, we expect to find at most two positive real roots of i. The net present value of the cash flow in millions of dollars is

$$\begin{aligned} \text{NPV} = {} & +18 + (10)(P \mid F, i, 1) - (40)(P \mid F, i, 2) - (60)(P \mid F, i, 3) \\ & + (30)(P \mid F, i, 4) + (50)(P \mid F, i, 5) \end{aligned}$$

Not knowing what the values of these roots may be, we try a range of values i = 0%, 10%, 20%, 30%, 40%, and 50%. The results of the computation are given in Table 4.4.

The results indicate that one positive real root of i lies between 10% and 20% and the other between 40% and 50%, as shown in Fig. 4.16.

TABLE 4.4 COMPUTATION OF INTERNAL RATE OF RETURN BY TRIAL

Year	Cash flow	$A_t(P \mid F, i, t)$					
t	A_t	0%	10%	20%	30%	40%	50%
0	+18	+18	+18	+18	+18	+18	+18
1	+10	+10	+ 9.091	+ 8.333	+ 7.692	+ 7.143	+ 6.667
2	−40	−40	−33.056	−27.776	−23.668	−20.408	−17.776
3	−60	−60	−45.078	−34.722	−27.312	−21.864	−17.778
4	+30	+30	+20.490	+14.469	+10.503	+ 7.809	+ 5.925
5	+50	+50	+31.045	+20.095	+13.465	+ 9.295	+ 6.585
NPV		+ 8	+ 0.492	− 1.601	− 1.320	− 0.025	+ 1.623

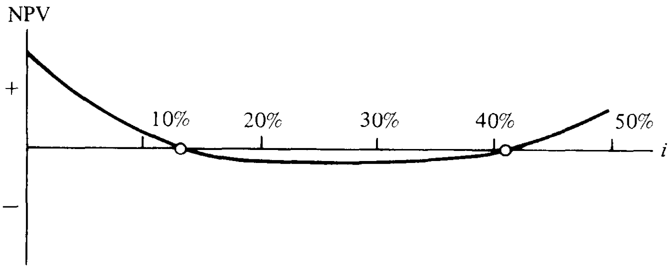


Figure 4.16 Example of two IRR for an investment project.

Using linear interpolation, we get

$$i = 10\% + (20\% - 10\%)\frac{0.492}{0.492 + 1.601} = 12.35\%$$

and

$$i = 40\% + (50\% - 40\%)\frac{0.025}{0.025 + 1.623} = 40.15\%$$

It should be noted that the multiple roots of i are found to be 11.30% and 40.16% by using an electronic spreadsheet. The first root departs significantly from that obtained by interpolation from the values in the compound interest tables because of the large interval for i used in the interpolation. The second root is virtually the same since it is very close to 40%, even though the interval for i is large. In any case, since each of these two roots of i satisfies Eq. (4.9), both are mathematically correct solutions of IRR. However, because each value of i represents the rate of return in some periods and the rate of borrowing in the remaining periods, neither one can be used as a merit measure for the internal rate of return or the internal rate of borrowing only.

4.5 INTERNAL RATE OF RETURN IN BORROWING

The internal rate of return for an investment may be defined alternatively as the rate of expected earnings on the unrecovered balances in various time periods such that the unrecovered balance at the end of the life of the investment project will be zero. Similarly, the internal rate of return for a borrowing opportunity may be defined as the interest rate on the unpaid balances in various time periods such that the unpaid balance at the end of the life of the borrowing arrangement will be zero. These two definitions are the two sides of the same coin, and the IRR in the borrowing situation actually refers to the internal rate of borrowing. We shall discuss the alternative definition of IRR in borrowing first since it is easier to understand the principal and interest payments in borrowing. The *principal* refers to the amount of money borrowed, and the principal repayment in a period refers to the portion in the scheduled payment above the interest accrued for the period.

For example, if you purchase a car with a sticker price of \$14,000 on credit and agree to make a monthly payment for a period of 30 months, then at a monthly interest rate of 1% arranged by the car dealer, you are to pay \$542 per month at the end of each month. For each payment you make, a portion is for the interest payment for the month and a portion is for the repayment of the principal, which reduces the unpaid balance. At the end of each month, your payment is the same, but the portion for interest payment and the portion for repayment of the principal vary because the interest will decrease and the principal repayment will increase as the unpaid balance at the beginning of the month becomes smaller. On the other hand, if you borrow an amount of \$1,000 from a friend for 6 months and agree on a monthly interest rate of 1% for compounding monthly, you will pay an amount of \$1,061.50 at the end of 6 months with no interim payments. At the end of each month, an interest will automatically be accrued as an increase in unpaid balance for that month.

In the borrowing situation, it is often important for accounting purposes to distinguish the portion of each payment that is interest and the portion that is left for reducing the principal amount borrowed, and it is also important to know the principal amount remaining to be paid after each payment. If no payment is made at a period, the interest accrued on the previously unpaid principal amount will be accumulated to produce a larger principal amount remaining to be paid. If additional borrowing takes place, not only is the interest accrued on the previously unpaid principal amount, but the newly borrowed amount will also be accumulated to the principal amount remaining to be paid.

To develop the algebraic relationships leading to the alternative definition of IRR for the borrowing opportunity, let us introduce the following notation:

i = the borrowing interest rate per period charged on the borrowing balance, which refers to the principal amount of loan remaining.

A_t = the cash flow incurred at the end of period t. A positive sign indicates a receipt and a negative sign indicates a payment.

R_t = the borrowing balance at the end of period t, after accounting for the required repayment of interest and principal for the period t. A positive sign for R_t indicates an unpaid balance; a negative sign indicates an overpayment of the principal.

I_t = the interest accrued to the borrowing balance in period t based on the interest rate i. Since i is positive, the sign of I_t follows that of R_{t-1} at the beginning of period t. A positive sign indicates the interest charged for an unpaid balance; a negative sign indicates interest earned from an overpayment of the principal.

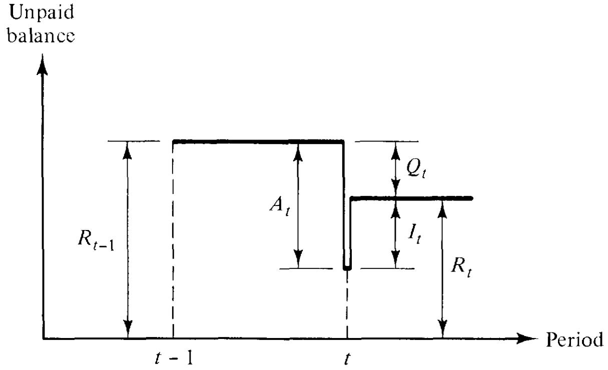
Q_t = the change in the borrowing balance for period t. A negative change indicates a net decrease in the borrowing balance; a positive change indicates a net increase in the borrowing balance.

In Fig. 4.17, the borrowing balance at the end of period $t - 1$ is designated by R_{t-1} and that at the end of period t is designated by R_t. In part (a) of this figure, the cash flow A_t incurred in period t consists of the interest I_t owed in period t and a repayment Q_t that leads to the reduction of the unpaid balance; in part (b) of the figure, the cash flow A_t represents new borrowing in addition to the interest I_t owed in period t, leading to an increase Q_t in the unpaid balance for the period. The signs of these quantities must be strictly observed as defined earlier so that all possible cases of repayment or new borrowing in a period will be covered. Then, the following algebraic relationships exist for $t = 1, 2, 3, \ldots, n$:

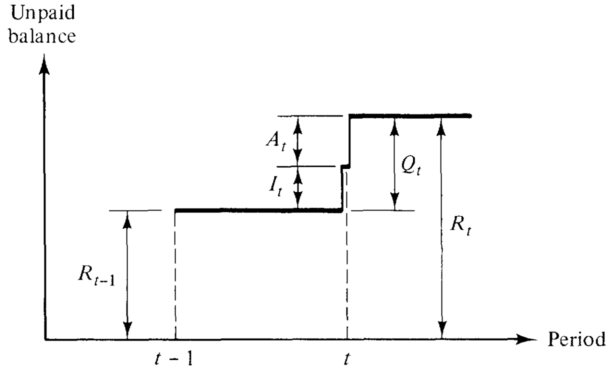
$$I_t = R_{t-1} i \tag{4.12}$$

$$Q_t = A_t + I_t \tag{4.13}$$

$$R_t = R_{t-1} + Q_t \tag{4.14}$$



(a) Repayment in period t



(b) Borrowing in period t

Figure 4.17 Unpaid balance of a loan for financing.

Substituting Eq. (4.12) into Eq. (4.13), we have

$$Q_t = A_t + R_{t-1}i \tag{4.15}$$

Furthermore, substituting Eq. (4.15) into Eq. (4.14), we get

$$R_t = A_t + R_{t-1}(1 + i) \tag{4.16}$$

Initially, $R_0 = A_0$ since the unrecovered balance at $t = 0$ is the amount received at that point in time. Hence, according to Eqs. (4.12) through (4.16), we have for $t = 1$

$$I_1 = R_0 i = A_0 i$$

$$Q_1 = A_1 + I_1$$

$$R_1 = R_0 + Q_1 = A_0 + Q_1$$

We can obtain similar quantities for subsequent periods $t = 2, 3, \ldots, n$. Note that Eq. (4.16) provides a recursive relationship for $t = 1, 2, \ldots, n$ as follows:

$$R_1 = A_1 + R_0(1 + i) = A_1 + A_0(1 + i)$$

$$R_2 = A_2 + R_1(1 + i) = A_2 + A_1(1 + i) + A_0(1 + i)^2$$

$$\cdots$$

$$R_t = A_t + R_{t-1}(1 + i) = A_t + A_{t-1}(1 + i) + \cdots + A_1(1 + i)^{t-1} + A_0(1 + i)^t$$

Thus, for $t = n$,

$$R_n = A_n + A_{n-1}(1 + i) + \cdots + A_1(1 + i)^{n-1} + A_0(1 + i)^n \tag{4.17}$$

Note that this value of R_n is the same as the net future value (NFV) of the cash flow profile $A_0, A_1, \ldots, A_n$ in Eq. (4.2). If i is the internal rate of return for this borrowing cash flow profile, then by the definition introduced in Section 4.3, R_n must equal zero. Hence, the alternative definition of IRR holds true.

If $i = i'$ is the IRR of a pure borrowing opportunity, then according to the alternative definition of IRR, the borrowing balances have the following characteristics:

1. $R_0 > 0$ since $R_0 = A_0$ must be a receipt
2. $R_t > 0$ at $i = i'$ for $t = 1$ to $t = n - 1$
3. $R_n = 0$ at $i = i'$

These are sufficient conditions to ensure the existence of a unique value of IRR. The NFV is negative at $i = 0$ for pure borrowing and rises to zero at $i = i'$. It is easy to verify the conditions by computing the borrowing balance period by period according to Eqs. (4.12) through (4.16) until the end of the borrowing.

For a given interest rate i, the amount of interest paid for each of the n periods over the entire time span of a loan is given by Eq. (4.12) as $I_1 = R_0 i$, $I_2 =$

$R_1 i, \ldots, I_n = R_{n-1} i$. Hence, the total amount of interest paid in n periods is given by

$$\sum_{t=1}^{n} I_t = i \sum_{t=0}^{n-1} R_t \tag{4.18}$$

The value of ΣI_t or ΣR_t in Eq. (4.18) indicates the extent of the use of money for a loan arrangement.

Since the repayment of a loan through a series of uniform payments over a number of periods is a common form of consumer credit for purchase of durable goods, such as automobiles and houses, a simplified formula may be developed for this special case. By letting $A_0 = P$ and $A_1 = A_2 = \cdots = A_n = -U$, Eq. (4.17) becomes

$$R_n = -U[1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-1}] + P(1 + i)^n$$

Hence,

$$R_n = -U\left[\frac{(1 + i)^n - 1}{i}\right] + P(1 + i)^n$$

At the end of period t, we have

$$R_t = -U\left[\frac{(1 + i)^t - 1}{i}\right] + P(1 + i)^t$$

In functional notation,

$$R_t = -U(F \mid U, i, t) + P(F \mid P, i, t) \tag{4.19}$$

This is the principal amount remaining to be paid at the end of period t for a borrowed amount P with a uniform series of repayment of U. It is the same as the remaining balance at the beginning of period $(t + 1)$, i.e., using t as the beginning point until the end of period n for $(n - t)$ periods.

$$R_t = U(P \mid U, i, n - t) \tag{4.20}$$

Example 4.10

Gerald Heins borrows \$200 from a finance company at 10% per annum with the understanding that he will pay back \$115.24 each at the end of the first and second year. Determine the principal amount and interest in each payment, and the principal amount remaining to be paid at the end of each year.

Since the interest rate is known to be 10%, we can verify that NPV = 0 and NFV = 0. Thus,

$$\text{NPV} = 200 - (115.24)(P \mid U, 10\%, 2)$$
$$= 200 - (115.24)(1.7355) = 0$$

and

$$\text{NFV} = (200)(F \mid P, 10\%, 2) - (115.24)(F \mid U, 10\%, 2)$$
$$= (200)(1.2100) - (115.24)(2.1000) = 0$$

Using the general relationships in Eqs. (4.12) through (4.16), we get the following:

At $t = 0$

$$R_0 = A_0 = 200$$

At $t = 1$

$$A_1 = -115.24$$
$$I_1 = R_0 i = (200)(10\%) = 20$$
$$Q_1 = A_1 + I_1 = -115.24 + 20 = -95.24$$
$$R_1 = R_0 + Q_t = 200 - 95.24 = 104.76$$

At $t = 2$

$$A_2 = -115.24$$
$$I_2 = R_1 i = (104.76)(10\%) = 10.48$$
$$Q_2 = A_2 + I_2 = -115.24 + 10.48 = -104.76$$
$$R_2 = R_1 + Q_2 = 104.76 - 104.76 = 0$$

For this particular problem involving a uniform series, we could have obtained R_1 and R_2 directly from Eq. (4.19). Thus,

$$\begin{aligned} R_1 &= (-115.24)(F \mid U, 10\%, 1) + (200)(F \mid P, 10\%, 1) \\ &= (-115.24)(1.0000) + (200)(1.1000) = 104.76 \\ R_2 &= (-115.24)(F \mid U, 10\%, 2) + (200)(F \mid P, 10\%, 2) \\ &= (-115.24)(2.1000) + (200)(1.2100) = 0 \end{aligned}$$

Thus, at the end of the first year, \$20 is for the interest and \$95.24 is for the principal amount, making a total of \$115.24. At the end of the second year, \$10.48 is for the interest and \$104.76 is for the principal amount, again making a total of \$115.24.

Example 4.11

Determine the year-by-year interest and principal amount for each of the loan arrangements represented by the following cash flow profiles. The borrowing interest rate is known to be 8% per annum for all cases.

(a) A loan of \$10,000 to be repaid with a single sum of \$14,693 at the end of 5 years
(b) A loan of \$10,000 with a series of repayments of \$2,505 per year for the next 5 years

(c) A series of loans of $2,505 each year beginning at the end of the first year for 5 years, to be repaid with a single sum of $14,693 at the end of 5 years

These three cases are identified respectively as (a) discount amount problem, (b) discount uniform series problem, and (c) compound uniform series problem. The computation can be carried out by using Eqs. (4.12) through (4.16), and the results are tabulated in Table 4.5.

The extent of the use of money for each of the loan arrangements is represented by part (a), (b), or (c) in Fig. 4.18. The area under each step function

TABLE 4.5 COMPUTATION OF YEAR-BY-YEAR UNRECOVERED BALANCE

(a) Discount Amount Problem

t	A_t	I_t	Q_t	R_t
0	+10,000	0	0	+10,000
1	0	+ 800	+ 800	+10,800
2	0	+ 864	+ 864	+11,664
3	0	+ 933	+ 933	+12,597
4	0	+1,008	+ 1,008	+13,605
5	−14,693	+1,088	−13,605	0

(b) Discount Uniform Series Problem

t	A_t	I_t	Q_t	R_t
0	+10,000	0	0	+10,000
1	− 2,505	+800	−1,705	+ 8,295
2	− 2,505	+664	−1,841	+ 6,454
3	− 2,505	+516	−1,989	+ 4,465
4	− 2,505	+357	−2,148	+ 2,317
5	− 2,505	+185	−2,320	− 3

(c) Compound Uniform Series Problem

t	A_t	I_t	Q_t	R_t
0	0	0	0	0
1	+ 2,505	0	0	+ 2,505
2	+ 2,505	+200	+ 2,705	+ 5,210
3	+ 2,505	+417	+ 2,922	+ 8,132
4	+ 2,505	+651	+ 3,156	+11,288
5	+ 2,505 } −14,693 }	+903	−11,285	+ 3

(expressed in the unit of dollar-years) is a measure of the term ΣR_t in Eq. (4.18).

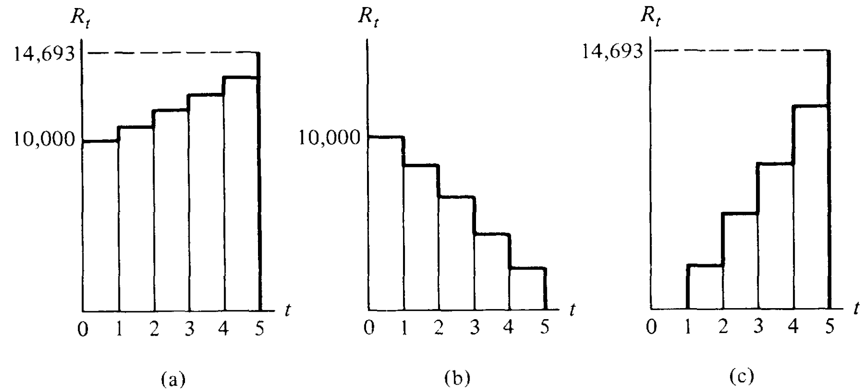


Figure 4.18 Funds supplied at various time periods by a loan.

4.6 INTERNAL RATE OF RETURN FROM INVESTMENT

In an investment situation, a receipt at the end of a period may consist of a portion that is the expected earnings and a portion that represents an increase or decrease in the project balance, which refers to the funds that remain invested after the receipt. If there is no receipt at the end of a period, the expected earnings for that period will accrue to the project balance; furthermore, a disbursement at the end of the period may include new funding as well as the plowback of the expected earnings for the period to produce an increase in the project balance.

For example, if you pay $1,000 to purchase a ten-year coupon bond that pays an interest at a rate of 3% per 6 month at its face value of $1,000, you will receive an interest of $30 at 6-month periods for 10 years, at the end of which you will also get back the principal of $1,000. Thus, the receipt at the end of each period represents only interest of $30 except in the last period, at which time the receipt includes both the interest of $30 and a decrease in project balance of $1,000. The internal rate of return from investing in this bond is 3% per 6-month period, and the amount that remains invested in each period is $1,000 except in the last period when it becomes zero. On the other hand, if you purchase a zero-coupon bond of $1,000 that promises semiannual interest of 3% compounded semiannually, you will receive $1,806.10 total at the end of 10 years, but nothing in the interim periods. In that case, the internal rate of return for investing in this bond is also 3% per 6-month period, but the amount that remains invested in each period is the sum of the amount accumulated up to the previous period and the expected earnings for the period.

An alternative definition of IRR for an investment project is introduced to examine the period-by-period unrecovered project balance until the end of the project life. To develop the algebraic relationships leading to the alternative definition, let us introduce the following notation:

i = the rate of return per period earned on the funds that remain invested in the project.

A_t = the cash flow at the end of period t. A positive sign indicates a receipt and a negative sign indicates a disbursement.

R_t = the project balance at the end of period t, after accounting for the expected earnings accrued and the cash flow at the end of period t. A negative sign for R_t indicates an unrecovered balance which remains invested; a positive sign indicates a surplus project balance.

I_t = the expected earnings accrued to the project in period t based on the rate of return i. Since i is positive, the sign of I_t follows that of R_{t-1} at the beginning of period t. A negative sign indicates the expected earnings charged against the unrecovered balance; a positive sign indicates the expected earnings received from the surplus project balance.

Q_t = the change in project balance for period t. A positive change indicates an increase in the project balance; a negative change indicates a decrease in the project balance.

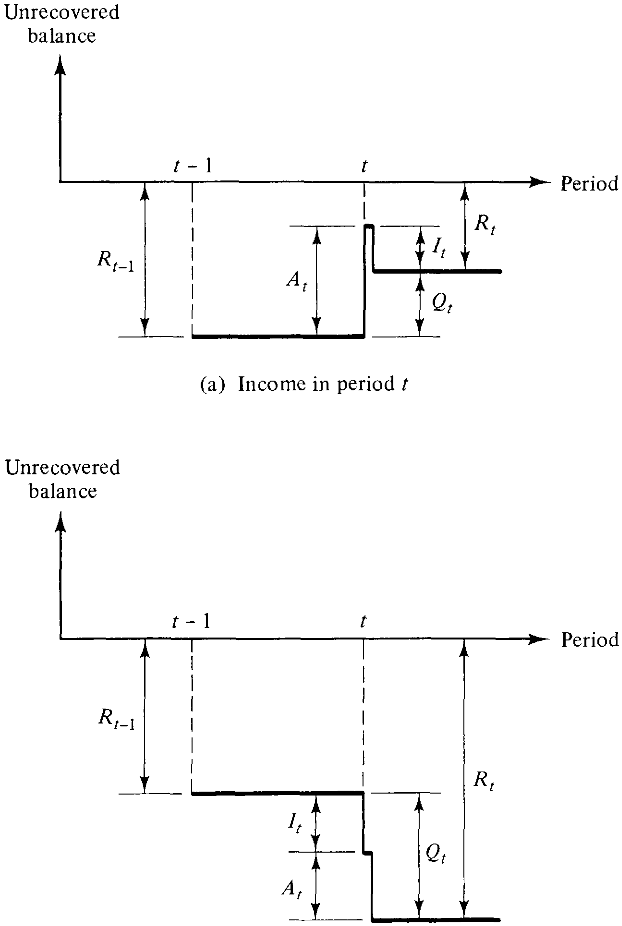
Thus, regardless of the number of sign changes in an investment cash flow, the following algebraic relationships in Eqs. (4.12) to (4.16) are also applicable for $t = 1, 2, 3, \ldots, n$, as illustrated schematically in Fig. 4.19. If i is the internal rate of return of an investment, the value of R_n in Eq. (4.17) is the same as the NFV in Eq. (4.2). Then, by the definition introduced in Section 4.3, R_n must equal 0. Hence, the internal rate of return of an investment project may alternatively be defined as the percentage rate per period earned on the unrecovered balance such that the unrecovered balance at the end of the planning horizon will be zero.

If $i = i'$ is the IRR of a pure investment project, then according to the alternative definition of the IRR, the project balances have the following characteristics:

1. $R_0 < 0$, since $R_0 = A_0$ must be a disbursement
2. $R_t < 0$ at $i = i'$ for $t = 1$ to $t = n - 1$
3. $R_n = 0$ at $i = i'$

These are sufficient conditions to ensure the existence of a unique value of IRR. The NFV is positive at $i = 0$ for pure investment and diminishes to zero at $i = i'$.

It is easy to verify that for a conventional investment cash flow profile with only one sign change, these conditions will always be satisfied since such a profile will always yield a single value of IRR. An unconventional investment project with two or more sign changes will yield a unique value of IRR only if all these conditions are satisfied. In that case, the unique IRR is a merit measure of the percentage rate per period of the expected earnings from the project balance. If an unconventional investment project does not satisfy all of these conditions, multiple values of



(b) Outlay in period t

Figure 4.19 Unrecovered balance of an investment.

IRR may exist but none of them can be used as a merit measure because each IRR represents the internal rate of return for some periods and internal rate of borrowing for the remaining periods of the cash flow profile.

For a specified rate of return i, the amount of required return for each of the n periods in the planning horizon is given by Eq. (4.12) as $I_1 = R_0 i$, $R_2 = R_1 i$, . . . , $I_n = R_{n-1} i$. The total amount of expected earnings in n periods is given by Eq. (4.18). Hence, the value of ΣR_t in Eq. (4.18) indicates the extent of commitment of funds that remain invested in a project.

Example 4.12

Verify that for the investment in Example 4.7 the unrecovered balance at the end of 5 years is zero if $i = 25\%$ is the internal rate of return.

The year-by-year unrecovered balance has been computed by using Eqs. (4.12) through (4.14), and the results in thousands of dollars are tabulated in Table 4.6. A few steps of computation are given to illustrate the results in the tabulation. For example, at $t = 0$, $R_0 = A_0 = -77$. Then, at $t = 1$,

$$I_1 = R_0 i = (-77)(25\%) = -19.250$$

$$Q_1 = A_1 + I_1 = +38 - 19.250 = +18.750$$

$$R_1 = R_0 + Q_1 = -77 + 18.750 = -58.250$$

At $t = 2$,

$$I_2 = R_1 i = (-58.250)(25\%) = -14.653$$

$$Q_2 = A_2 + I_2 = +32 - 14.653 = +17.437$$

$$R_2 = R_1 + Q_2 = -58.250 + 17.437 = -40.813$$

and so on. Because $i = 25\%$ has been obtained by linear interpretation, it introduces small numerical errors in the values of R_t. Consequently, at $t = 5$, $R_5 = -0.087$ instead of zero.

TABLE 4.6 YEAR-BY-YEAR UNRECOVERED BALANCE FOR EXAMPLE 4.12

t	A_t	I_t	Q_t	R_t
0	−77	—	—	−77
1	+38	−19.250	+18.750	−58.250
2	+32	−14.563	+17.437	−40.813
3	+26	−10.203	+15.797	−25.016
4	+20	− 6.254	+13.746	−11.270
5	+14	− 2.818	+11.183	− 0.087
	Σ = +53			

Since the cash flow profile representing the investment has only one sign change, a unique value of IRR is expected. It can be seen that all conditions required for the existence of a unique value of IRR are fulfilled.

Example 4.13

The unrecovered balances of an unconventional investment project have been computed for a discount rate of 10% as shown in Table 4.7. Verify that the unique value of IRR for this investment is 10%.

First, $R_0 = A_0 = -20{,}000$ is negative. Second, R_t resulting from $i =$ 10% is negative for $t = 1$ to $t = 5$. Third, $R_6 = 0$ at $i = 10\%$. Since all conditions required for the existence of a unique IRR are satisfied, $i = 10\%$ is the IRR representing the merit measure of the investment cash flow profile. As a

TABLE 4.7 YEAR-BY-YEAR UNRECOVERED BALANCE FOR EXAMPLE 4.13

t	A_t	I_t	Q_t	R_t
0	−20,000	0	0	−20,000
1	+12,000	−2,000	+10,000	−10,000
2	−19,000	−1,000	−20,000	−30,000
3	+23,000	−3,000	+20,000	−10,000
4	0	−1,000	−1,000	−11,000
5	−15,900	−1,100	−17,000	−28,000
6	+30,800	−2,800	+28,000	0
Σ =	+10,900			

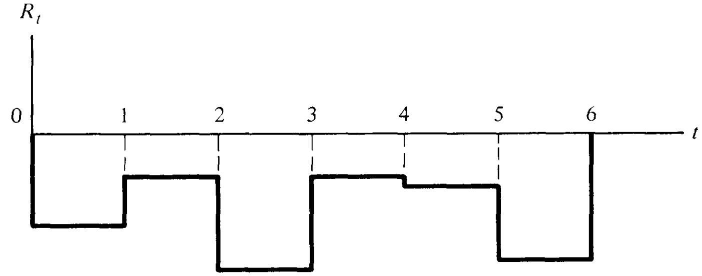


Figure 4.20 Funds committed at various time periods to a project.

check, NFV = +10,900 at i = 0 as indicated by the sum of A_t from t = 0 to t = 6, and that is typical of pure investment.

The extent of commitment of funds that remain invested in this project in various periods is represented by a graph of R_t versus t shown in Fig. 4.20.

Example 4.14

Verify that for the unconventional investment in Example 4.9 the unrecovered balance at the end of 5 years is zero if (a) i = 11.30% or (b) i = 40.16%, each of which satisfies the definition of the internal rate of return.

The unconventional cash flow profile consists of two sign changes. For each of the two internal rates of return i = 11.30% and i = 40.16%, the year-to-year unrecovered balances are computed as shown in Table 4.8. In both cases, R_t at t = 5 is approximately zero. Because of sign changes of R_t from t = 0 to t = 4 in each case, the IRR represents the internal rate of return in some periods and the internal rate of borrowing in the remaining periods. Since there are two values of IRR, neither one of them should be used as a merit measure. Note also that neither case satisfies the conditions for the existence of a unique value of IRR.

TABLE 4.8 YEAR-BY-YEAR UNRECOVERED BALANCE FOR EXAMPLE 4.14

t	A_t	I_t	Q_t	R_t
		Internal Rate of Return = 11.30%		
0	+18	0	0	+18
1	+10	+2.034	+12.034	+30.034
2	−40	+3.394	−36.606	− 6.572
3	−60	−0.743	−60.743	−67.315
4	+30	−7.607	+22.393	−44.922
5	+50	−5.076	+44.924	− 0.002
		Internal Rate of Return = 40.16%		
0	+18	0	0	+18
1	+10	+7.229	+17.229	+35.229
2	−40	+14.148	−25.852	+ 9.377
3	−60	+3.766	−56.234	−46.857
4	+30	−18.818	−11.182	−35.675
5	+50	−14.327	+35.673	− 0.002

4.7 REINVESTMENT AND FINANCING POLICIES

We have examined the conditions under which a unique value of IRR exists, and the computational procedure for the solution of the internal rate of return. If a unique value of IRR exists, whether the cash flow profile is conventional or unconventional, the IRR is a useful merit measure since it shows the percentage rate of profit for the funds *committed* in various periods for an investment project, or the percentage rate of interest for borrowed funds *used* in various periods for a borrowing opportunity. However, the IRR does not take into account the reinvestment of the interim receipts prior to the end of the project, or the financing of the interim disbursements after the initial point of the project. The discount rate used for reinvestment and for financing is sometimes referred to as the *external rate of return* (ERR), which is generally different from the internal rate of return.

Of all investment situations, the simple case of investing a single sum at the beginning and receiving a single sum at the end of the project life is the only case that cannot possibly be affected by an external rate of return. In that case, the cash flow accumulated at the end of each period will be totally committed for *internal reinvestment* at the IRR. Similarly, the simple case of borrowing a single sum at the beginning and repaying a single sum at the end of the project life is the only case of borrowing that cannot possibly be affected by an external rate of return. In this case, the accumulation of the principal and interest at the end of each period will be totally used for *internal borrowing* by compounding the accumulated sum in each period at the IRR. For all other situations, an external rate of return is either implicitly or explicitly specified for reinvestment of interim receipts and/or for financing of interim disbursements.

Without any explicit statement to the contrary, the external rate of return for reinvestment and for financing is assumed to be equal to the minimum attractive rate of return (MARR) in a perfect capital market. Such reinvestment and financing policies will be consistent with the definitions of the net future value and net present value in Section 4.2 and can be easily explained for conventional cash flow profiles. For an investment opportunity, the NFV will give a direct measure of the size of gain at the end of the project life by investing in a capital project as dictated by the cash flow profile such that receipts from the interim periods prior to the end are reinvested at the minimum attractive rate of return (MARR). Similarly, for a borrowing opportunity, the NFV will give a direct measure of the size of gain at the end of the project life by borrowing from a proposed borrowing opportunity as specified by the cash flow profile such that repayments required in the interim periods prior to the end are financed at the maximum acceptable rate of borrowing (also designated as MARR). The NPV of an investment or a borrowing cash flow profile will likewise give a direct measure of the size of gain with time value at the beginning of the time span. These policies are borne out by the mathematical relationships if $i = i^*$ for the MARR is substituted into Eqs. (4.1) and (4.2).

For an unconventional cash flow profile in which two or more changes of sign occur, Eqs. (4.1) and (4.2) are still applicable since the external rate of return is independent of the existence or nonexistence of a unique value of IRR. By using the MARR as the external rate of return for reinvestment and financing, we can provide a consistent basis for the analysis. When no IRR exists for such cash flow profiles, as for the cases shown in Figs. 4.5 and 4.6, the applicability of NPV or NFV is not in question since the NPV or NFV will not change sign as the discount rate i increases. When multiple values of IRR occur, as for the cases shown in Figs. 4.7 and 4.8, it may appear illogical at first glance that an investment project that is acceptable at one discount rate (for $i^* > i''$) may not be acceptable at some range of lower discount rates (for $i' < i^* < i''$). However, there is nothing illogical or inconsistent in such a phenomenon when the external rate of return is assumed equal to the MARR for both reinvestment and financing. In the case of unconventional investment profiles, an increase in the MARR for reinvestment and financing may produce the net effect of increasing the NPV and NFV, thus allowing such values to change from negative to positive. Similarly, in the case of unconventional borrowing profiles, an increase in the MARR may produce the opposite effect. Hence, the NPV and NFV are also applicable as merit measures for both investment and borrowing projects with multiple values of IRR.

Example 4.15

The cash flow profiles of two investment projects are given as follows: (a) Invest an amount of \$10,000 now and receive a return of \$14,693 only at the end of 5 years, and (b) invest an amount of \$10,000 now and receive a yearly amount of \$2,505 for the next 5 years. Find the future value of the receipts from the investment project at the end of 5 years under the following different circumstances:

1. The external rate of return per year is identical to the internal rate of return of the project.
2. The minimum attractive rate of return (MARR) per year for reinvestment is 6%.

For both investment proposals, the internal rate of return is found to be 8%. However, for case (a), the future value of the receipt at the end of 5 years is simply $F = \$14,693$ for all circumstances since there is no money for reinvestment or consumption until the end of 5 years.

For case (b), each of the situations presents a different future value F of the *receipts* at the end of 5 years:

1. For ERR = IRR = 8%,

$$F = (2,505)(F \mid U, 8\%, 5) = (2,505)(5.8666) = 14,696$$

2. For ERR = MARR = 6%,

$$F = (2,505)(F \mid U, 6\%, 5) = (2,505)(5.6371) = 14,121$$

Example 4.16

Consider an investment opportunity for a construction company engaged in a foreign project that has a cash flow profile in million dollars, as shown in Example 4.9. The multiple values of IRR for the unconventional cash flow profile are found to be 11.30% and 40.16%. If the MARR of this company is 8%, find the unrecovered balances for periods $t = 1$ to $t = 5$. Also determine whether this opportunity should be accepted on the basis of the NFV decision criterion.

For this investment proposal, the disbursements in period 2 are financed in part by the receipts in periods 0 and 1, which would earn interests at the MARR until period 2, as indicated by the computation in Table 4.9. For the remaining periods, the disbursements are invested until maturity with a net future value $R_5 = +2.080$. In other words, by financing unrecovered balances and reinvesting surplus project balances from interim periods at the MARR,

TABLE 4.9 UNRECOVERED BALANCES OF AN UNCONVENTIONAL INVESTMENT OPPORTUNITY

t	A_t	I_t	Q_t	R_t
0	+18	0	0	+18.000
1	+10	+1.440	+11.440	+29.440
2	−40	+2.355	−37.645	− 8.205
3	−60	−0.656	−60.656	−68.861
4	+30	−5.509	+24.491	−44.370
5	+50	−3.550	+46.450	+ 2.080

NFV = +2.080 in period 5 is the profit relative to the foregone opportunity based on the MARR only. Since this investment cash flow profile has multiple values of IRR, the IRR decision criterion cannot be applied. However, the NFV decision criterion is still applicable. Since NFV is positive, the investment proposal should be accepted.

Example 4.17

An unconventional borrowing opportunity in Table 4.10 has the following cash flows: $A_0 = -18$, $A_1 = -10$, $A_2 = +40$, $A_3 = +60$, $A_4 = -30$, and $A_5 = -50$. It is offered to the contractor in Example 4.9 by a bank, which will take over the receipts and the disbursement obligations of the contractor for this foreign project. Then, the values of the IRR for this series remain 11.30% and 40.16%. Should the contractor accept this offer if the MARR for both lending and borrowing is 15%?

The project balances for lending and borrowing at MARR = 15% are shown in Table 4.10, which indicates $R_5 = +1.990$. Even though the borrowing cash flow profile has multiple values of IRR, this financing opportunity is attractive according to the NFV decision criterion since NFV = +1.990. Hence, the contractor should accept this borrowing opportunity offered by the bank.

TABLE 4.10 UNRECOVERED BALANCES FOR AN UNCONVENTIONAL BORROWING PROPOSAL

t	A_t	I_t	Q_t	R_t
0	−18	0	0	−18.000
1	−10	−2.700	−12.700	−30.700
2	+40	−4.605	+35.395	+4.695
3	+60	+0.704	+60.704	+65.399
4	−30	+9.810	−20.190	+45.209
5	−50	+6.781	−43.219	+1.990

4.8 OVERALL RATE OF RETURN

The internal rate of return is intuitively appealing, if a unique value exists, since it resembles the interest rate that most people are familiar with. Furthermore, it is independent of the minimum attractive rate of return (MARR), which is tied to market expectations and the judgment of the decision maker. Consequently, the internal rate of return has long been accepted as a useful merit measure as long as it is assured that a unique value can be found. Although the computational procedure is complex, it can be relatively easily overcome by the use of automated computational aids.

However, the attraction of the IRR is also its weakness because by itself the IRR does not tell the whole story about net gain or loss of a project at the end of the project life even when a unique value of IRR exists. Consequently, a merit measure that expresses the percentage rate of the gain or loss over the project life is highly desirable. This measure must take into consideration the reinvestment and financing policies in treating the interim receipts and disbursements of a cash flow profile which will contribute to the overall gain or loss. A merit measure that can meet such requirements is referred to as the *overall rate of return* (ORR).

Consider again the example of purchasing a ten-year coupon bond at its face value of \$1,000 which pays 3% or \$30 interest at 6-month periods for 10 years. Suppose that the interest payments at interim periods are reinvested in other opportunities at 4% per 6-month period. You will expect to receive a sum of $F = (30)(F \mid U, 4\%, 20) = \$893.34$ in addition to the original investment of \$1,000 at the end of 10 years. That is, the amount (\$1,000) that remains invested in each period will grow at an internal rate of return of 3%, while the interest payment (\$30) reinvested in each per period will grow at an external rate of return of 4%. Thus, the overall rate of return per 6-month period from investing in this bond over the 10-year period is higher than the internal rate of return of the bond. Similarly, it is often noted in advertisements for mutual funds that the annual rates of return over a 10-year period include gains from reinvestment of dividends received in the interim periods, and such rates represent the overall rates of return for the original investment.

Like the internal rate of return, the ORR resulting from an investment opportunity represents the *overall rate of return to the investor,* while the ORR resulting from a borrowing opportunity represents the *overall rate of borrowing* (or the *overall rate of return to the lender*). Although the concept of overall rate of return is applicable to unconventional cash flows as well as conventional cash flows, in this chapter we shall confine our discussion to its application to the conventional cash flows only. For a conventional investment cash flow profile, the objective is to establish an *investment base* at the beginning ($t = 0$) to cover the necessary disbursements in interim periods, and to find a *resultant return* at the end ($t = n$) from the receipts in the profile. For a conventional borrowing cash flow profile, the objective is to establish a *borrowing base* at the beginning ($t = 0$) to account for subsequent receipts in interim periods, and to find a *resultant repayment* at the end ($t = n$) from the disbursements in the profile.

The *transition point* of a conventional cash flow profile A_t (for $t = 0, 1, 2, \ldots, n$) is defined as the time period such that the profile can be broken into two parts, each of which has only one sign change.[2] Thus, the transition point $t = e(0 \leq e < n)$ is simply the time period prior to the sign change of A_t so that A_0 through A_e will have one sign and A_{e+1} through A_n will have another. The sign of A_0 through A_e is negative if the profile represents an investment opportunity, and positive if it represents a borrowing opportunity. Then for a MARR equal to i^*, the equivalent present value of the cash flows before the sign change of A_t and the equiv-

[2] The concept of transition point was first introduced in Ref. 4.1, for application to both conventional and unconventional cash flow profiles.

alent future value of the cash flows after the sign change are obtained respectively as follows:

$$P_e = \sum_{t=0}^{e} A_t(1 + i^*)^{-t} \tag{4.21}$$

$$F_e = \sum_{t=e+1}^{n} A_t(1 + i^*)^{n-t} \tag{4.22}$$

Note that for an investment cash flow profile, the investment base is $-P_e$ and the resultant return is F_e, while for a borrowing cash flow profile, the borrowing base is P_e and the resultant repayment is $-F_e$.

The overall rate of return i^0 can be obtained from the following relationship:

$$P_e(1 + i^0)^n + F_e = 0 \tag{4.23}$$

Hence,

$$i^0 = \left(-\frac{F_e}{P_e}\right)^{1/n} - 1 \tag{4.24}$$

The ORR can also be obtained by interpolation from the discount amount factor in Appendix A by noting the following relation from Eq. (4.23):

$$(P \mid F, i^0, n) = -\frac{P_e}{F_e} \tag{4.25}$$

Consequently, the overall rate of return may also be used as an indirect merit measure whether the cash flow profile is conventional or unconventional. However, the decision rule for an investment opportunity is different from that for a borrowing opportunity because the ORR refers to the overall rate of return in the former, but refers to the overall rate of borrowing in the latter. The set of decision rules associated with the ORR decision criterion can be stated as follows:

1. For an investment opportunity, accept if ORR $\geq$ MARR; reject otherwise.
2. For a borrowing opportunity, accept if ORR $\leq$ MARR; reject otherwise.

The ORR decision criterion for accepting or rejecting a project refers to the percentage rate of profit and thus is size independent.

Example 4.18

An organization is making an investment plan over the next 5 years. The minimum attractive rate of return for its investments is specified to be 6%. The two investment projects in Example 4.15 are found to have an internal rate of return of 8%. The early year returns from each project are expected to be reinvested at the minimum attractive rate of return specified by the organization. What is the profit potential of each project for the organization as measured by the overall rate of return over 5 years?

For case (a), the overall rate of return is the same as the internal rate of return, which is found to be 8%.

For case (b), we can find a resultant return F_e at time $t = n$ for the receipts reinvested at 6% from Eq. (4.22). Since the transition point $e = 0$, we have

$$F_e = (2{,}505)(F \mid U, 6\%, 5) = (2{,}505)(5.6371) = 14{,}121$$

For $-P_e = 10{,}000$ at $t = 0$, the overall rate of return i^0 may be obtained from Eq. (4.25). Thus,

$$(P \mid F, i^0, 5) = \frac{10{,}000}{14{,}121} = 0.7082$$

By the method of linear interpolation from the tables in Appendix A, i^0 is found to be 7.15%. Note that the overall rate of return to the investor is lower than the internal rate of return for the project because the reinvestment rate of 6% for the early year returns is less than IRR = 8%.

Example 4.19

An investment project requires a series of disbursements of $2,505 each year at the end of years 1, 2, 3, 4, and 5, with an expected return of $14,693 at the end of 5 years. Determine the overall rate of return for the 5-year period if the external rate of return is 6%.

Note that the transition is $e = 4$ since there is a net receipt at the end of year 5. If an investment base $-P_e$ is available at $t = 0$ and is invested at 6% in order to provide the amounts for later disbursements, we have from Eq. (4.21)

$$-P_e = (2{,}505)(P \mid U, 6\%, 4) = (2{,}505)(3.4651) = 8{,}680$$

Since $F_e = 14{,}693 - 2{,}505 = 12{,}188$ at $t = 5$, we have from Eq. (4.25)

$$(P \mid F, i^0, 5) = \frac{8{,}680}{12{,}188} = 0.7122$$

By the method of interpolation from the tables in Appendix A, i^0 is found to be 7.02%.

4.9 COMPARISON OF MERIT MEASURES

In this chapter, we have introduced several merit measures for multiperiod cash flow profiles that may represent investment or borrowing opportunities. The direct merit measures associated with profit maximization include the net present value (NPV) and net future value (NFV). The indirect measures include the internal rate of return (IRR) and the overall rate of return (ORR). While merit measures are invariably linked to the decision rules for accepting or rejecting independent proposals, this chapter emphasizes the precise definitions of these measures and the procedures for computing them.

Both the NPV and the NFV of an investment cash flow profile can be used as merit measures of the gain at a given point in time relative to what would be attainable by investing at the MARR. For a borrowing cash flow profile, they refer to the gain at the given time relative to what would have been gotten by borrowing at the MARR. The measures can be applied to an investment or a borrowing opportunity by invoking the same decision rule, whether the cash flow profile is conventional or unconventional. There are no exceptions under any circumstances, and the rules are simple to apply.

The IRR is an absolute measure of the percentage rate of profit for the funds committed to an investment project if a unique value exists. That is, the funds that remain invested at the end of each period will earn returns at the IRR. However, the funds released at the interim periods before the end of the planning horizon will be reinvested at the external rate of return (ERR), which is usually assumed to be the minimum attractive rate of return (MARR) rather than the IRR. Since the IRR has nothing to do with the MARR, it does not tell the whole story about the profit from the reinvestment of released funds up to the end of the project life. With some exceptions, no unique value of IRR can be established for unconventional cash flow profiles. Furthermore, the decision rule of accepting or rejecting an investment opportunity is different from that for a borrowing opportunity. Hence, a set of two different decision rules delineating these two types of opportunities is required. The IRR from investment refers to the internal rate of return to the investor while the IRR in borrowing refers to the internal rate of borrowing for the borrower or the internal rate of return to the lender.

The ORR provides a merit measure in percentage rate of profit over the project life, considering both the funds that remain committed to the project and the reinvestment of the released funds in the interim periods. While it is applicable to both conventional and unconventional cash flow profiles, only the computational procedure for the conventional case is discussed in this chapter. Furthermore, a set of two different decision rules is required for separate treatment of an investment opportunity and a borrowing opportunity. The ORR from investment refers to the overall rate of return to the investor, while the ORR in borrowing refers to the overall rate of borrowing for the borrower or the overall rate of return to the lender.

Different merit measures may be preferred for various reasons. Consequently, all those that are introduced in this chapter have been fully explored and illustrated with examples. Personal preferences notwithstanding, some measures are more restrictive than others. Specifically, the IRR is meaningful only if a unique value of IRR exists, and even when that condition is satisfied, the IRR does not tell the whole story about the net gain or loss of the cash flow profile over the planning horizon. The NPV and NFV do not have such limitations since they are direct merit measures. While the ORR appeals to those who prefer to deal with percentage rate of profit including external reinvestments and/or financing, it is size independent.

4.10 SUMMARY AND STUDY GUIDE

This chapter has covered the techniques of computing the net present value (NPV), the net future value (NFV), the internal rate of return (IRR), and the overall rate of return (ORR). It has also explored the meaning of these merit measures as well as their usefulness and limitations. The detailed explanations provide a road map through the maze of engineering economics literature on various aspects of discounted cash flow analysis.

The computation of the NPV and NFV is straightforward, and the mathematical concepts behind these merit measures are simple. If the MARR is defined as the minimum attractive rate of return for an investment opportunity and as the maximum acceptable rate of borrowing for a borrowing opportunity, it is applicable to investment or borrowing cash flow profiles without exception whether these profiles are conventional or unconventional.

The IRR may be defined as the discount rate that sets NPV = 0 or NFV = 0. It can alternatively be defined as the interest rate earned on the unrecovered project balance at each period such that the unrecovered amount equals zero at the end of the planning horizon. While multiple values of IRR are possible mathematically, they are meaningless as merit measures in the context of capital investment analysis. Consequently, an attempt has been made to define the conditions under which a unique value of IRR exists in unconventional cash flow profiles. Furthermore, the term IRR should be interpreted as the internal rate of return to the investor for an investment opportunity, but as the internal rate of borrowing (or the internal rate of return to the lender) for a borrowing opportunity.

The term ORR refers to the overall rate of return to the investor for an investment project but it means the overall rate of borrowing (or the overall rate of return to the lender) for a borrower. In this chapter the application of the ORR is confined to conventional cash flows only. Applications to unconventional cash flows will be treated in Sections 16.7 and 16.8 (Chapter 16).

As you move on to later chapters and learn about the decision criteria for implementing the objective of profit maximization in capital investment analysis, the significance of these merit measures will be unfolded. With a good understanding of the basic principles, you can judge for yourself the simplicity or complexity of various decision criteria associated with these merit measures.

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PROBLEMS

P4.1 Find the net present value of the following unconventional borrowing profile: $A_0 = +\$10,000$, $A_1 = -\$50,000$, $A_2 = +\$93,000$, $A_3 = -\$77,500$, and $A_4 = +\$24,024$. The MARR specified is 8% per period.

P4.2 Find the net future value at $t = 5$ for the following investment profile in thousands of dollars: $A_0 = -77$, $A_1 = +38$, $A_2 = +32$, $A_3 = +26$, $A_4 = +20$, and $A_5 = +14$. The MARR specified is 6% per period.

P4.3 Find the net present value of the following unconventional investment cash flow profile: $A_0 = -1,000$, $A_1 = +3,700$, $A_2 = -4,520$, and $A_3 = +1,820$. The specified MARR is 20% per period.

P4.4 Find the net future value of the following unconventional investment cash flow profile: $A_0 = -1,000$, $A_1 = +3,000$, and $A_2 = -2,160$. The specified MARR is 25% per period.

P4.5 Find the net present value of the following borrowing cash flow profile: $A_0 = A_1 = A_2 = A_3 = 2,000$, $A_4 = A_5 = A_6 = A_7 = A_8 = 1,500$, and $A_9 = -18,000$. The specified MARR is 2% per period.

P4.6 The cash flow profile of an operation for 2 years is as follows: $A_0 = -1,000$, $A_1 = +3,000$, and $A_2 = -2,160$. Find the annual internal rate(s) of return by solving the algebraic equation resulting from setting the net present value of the cash flow profile equal to zero.

P4.7 The cash flow profile of an investment project for 3 years shows that $A_0 = -1,000$, $A_1 = +3,700$, $A_2 = -4,520$, and $A_3 = +1,820$. Verify that the annual internal rates of return for this cash profile are $i = 0\%$, $i = 30\%$, and $i = 40\%$ by substituting each value of i into the equation NPV $= 0$.

P4.8 The cash flow profile of an investment project for 4 years shows that $A_0 = +10,000$, $A_1 = -50,000$, $A_2 = +93,500$, $A_3 = -77,500$, and $A_4 = +24,024$. Verify that the internal rates of return of this cash profile are $i = 10\%$, $i = 20\%$, $i = 30\%$, and $i = 40\%$ by substituting each value of i into the equation NPV $= 0$.

P4.9 Chester has obtained a loan from a bank in the form of a series of 4 monthly payments of $2,000 each beginning immediately, plus another series of 5 monthly payments each of $1,500 beginning from the end of the fourth month. At the end of the ninth month, he is required to pay $18,000 for the entire loan and interest. What is the internal rate of return for the bank?

P4.10 Ms. Brennen has secured a loan of $6,000 with an agreement to pay back $800 at the end of the first year, and $300 more at each subsequent year, i.e., $1,100 at the end of the second year, \$1,400 at the end of the third year, etc., for a period of 5 years. Determine the annual internal rate of return for the loan.

P4.11 A government office is installing a new telephone system at a cost of \$120,000 with an expectation of reducing the staff for answering the telephone. The annual savings will be \$40,000 the first year, with \$5,000 savings less in each subsequent year until the equipment is retired after 6 years, i.e., the annual savings will be \$35,000 the second year, \$30,000 the third year, etc. The equipment will have no salvage value at the end of 6 years. Find the annual internal rate of return of the investment.

P4.12 Mr. Eberhart invested \$10,000 in a venture from which he received no return in the first 5 years, and an amount of \$4,000 each at the end of the sixth through the end of the tenth year. What was the annual internal rate of return of the venture?

P4.13 A city issues a new 20-year bond for the construction of a sewage system. Each unit for sale has a face value of \$1,000 and pays 6% of its face value as annual interest at the end of the year. Ten years after Mr. Stone bought a unit at \$1,000, the market value of each unit was dropped to \$900. Mr. Stone decided to sell it to Ms. Sand at that price after collecting the interest at the end of 10 years. What is the internal rate of return for Ms. Sand if she keeps the bond for the remaining 10 years?

P4.14 Charles Jones invested \$1,000 five years ago in a 10-year coupon bond which pays interest at an annual percentage rate of 8% payable every six months, i.e., 4% interest rate for each 6-month period. After collecting the interest at the end of the fifth year, he sold the bond at the market price of \$700.

(a) Find the rate of return per 6-month period.

(b) Find the effective annual rate of return.

P4.15 Ann Drexel invested \$1,000 eight years ago in a 20-year coupon bond that pays 8% annual interest payable at the end of each year. After collecting the interest rate at the end of the eighth year, she sold the bond at the market price of \$800.

(a) Find the internal rate of return of the bond per year.

(b) If the early returns for interest were reinvested at an external rate of return of 7% per year, find the overall rate of return for the period of eight years.

P4.16 Mr. Jackson purchased a 30-year bond having a face value of \$1,000 and bearing interest of 5% payable annually for a price of \$960. Twenty years later, after collecting the annual interest for that year, he sold the bond at a price of \$720. What is the actual annual interest rate on Mr. Jackson's investment? If the new buyer intends to keep the bond until maturity at the end of the thirtieth year, what is the actual annual interest rate for the new buyer?

P4.17 A turnpike authority plans to finance a bridge project through the sale of bonds. It issues a 30-year bond with a total face value of \$5,000,000 bearing interest of 5% payable annually. However, at the bond market, the authority can sell the bonds for only \$4,800,000. What is the actual annual interest rate that the authority must pay for the funds it receives?

P4.18 Lisa Sampson borrows \$9,000 with an agreement to pay back \$3,367 per year in the next 3 years. Determine the actual interest rate for the loan. Find also the year-by-year principal and interest payments for the loan.

P4.19 Joe Brennen borrows \$5,500 with an agreement to pay back \$800 at the end of the first year, and \$300 more at each subsequent year, i.e., \$1,100 at the end of the second year, \$1,400 at the end of the third year, etc., for a period of five years. Determine the actual annual interest rate for the loan. Find also the year-by-year principal and interest payments for the loan.

P4.20 Determine the year-by-year unrecovered balance for each of the following two cash flow profiles, both of which have an internal rate of return of 10% per year.

(a) A loan of $20,000 with no return other than a single sum of $29,282 at the end of 4 years.

(b) A loan of $20,000 with a series of repayments of $6,310 per year for the next 4 years.

P4.21 Mr. and Mrs. Garcia begin to save money for the college education of their son, John, by putting aside $2,700 annually from John's seventh birthday (seven years after birth) until his 16th birthday, i.e., 10 uniform annual amounts. They expect that these amounts together with interest compounded annually will provide John with $12,000, $13,000, $14,000, and $15,000 on his 17th, 18th, 19th and 20th birthdays, respectively. What is the annual interest rate?

P4.22 An investment has an unconventional cash flow profile of 6 years (in thousand dollars) as follows: $A_0 = -40.0$, $A_1 = +33.6$, $A_2 = -29.1$, $A_3 = +23.6$, $A_4 = -8.2$, $A_5 = +22.7$, and $A_6 = +10.9$. At $i = 9\%$ find the unrecovered balance at the end of each year for all 6 years of the life of the investment. Is $i = 9\%$ the unique value of IRR for this investment cash flow profile? If so, why? If not, why not?

P4.23 A home owner borrows $80,000 from a bank and agrees to repay the principal and interest in six uniform annual payments of $18,910 each for the next 6 years. The annual interest rate for borrowing is 11%.

(a) Determine the principal and interest payments in each of the six years.

(b) At the end of year 2, the borrower decides to pay an additional amount of $26,284. What will be the principal and interest payments in future years if the borrower continues the payment schedule until the principal is repaid?

(c) At the end of year 2, the borrower decides to pay an additional amount of $30,000. What will be the principal and interest payments in future years?

P4.24 A small business owner borrows a sum of $10,000 from a bank with an agreement to repay the principal and interest on the loan through a uniform series of annual payments of $2,638 each for five years. The annual interest rate for borrowing is 10%.

(a) Determine the principal and interest payments in each of the five years.

(b) At the end of year 2, the borrower decides to pay an additional amount of $1,982. What will be the principal and interest payments in future years if the borrrower continues the payment schedule until the principal is repaid?

(c) At the end of year 2, the borrower decides to pay an additional $2,000. What will be the principal and interest payments in future years?

P4.25 For Problem P4.2, find the annual overall rate of return if the external rate of return for reinvestment is 12%.

P4.26 For Problem P4.5, find the annual overall rate of return if the external rate of return for reinvestment is 3% per period.

CHAPTER 5

The Economic Feasibility of Independent Projects

5.1 BASIC PRINCIPLES OF ECONOMIC EVALUATION

A prelude to an economic evaluation of investment proposals is the generation of promising proposals for consideration. For the purpose of analysis, such proposals may be classified either as independent or mutually exclusive. Consider, for example, an oil company that plans to build five new refineries in different parts of the United States. The production and marketing operations of each of these new plants are expected to be independent of those of other plants. Then, these proposed new refineries may be regarded as independent projects. On the other hand, if five alternative designs of a refinery reflecting different scales of operation are considered for construction at a given site, then these proposed alternatives are mutually exclusive.

In this chapter, we consider the evaluation of independent proposals to determine whether they are economically feasible. Essentially, we judge whether *each* independent proposal is superior or equal to the *null alternative,* which is defined as an alternative having no benefit and no cost over the entire planning horizon. If each of the independent proposals is feasible, then all of them are acceptable.

In the evaluation of mutually exclusive proposals, we must rank the merits of all proposals and select only the best one, provided that it is economically feasible. It does not make sense economically to select the best among all mutually exclusive alternatives when even the best is not good enough. Thus, this chapter also serves as a prerequisite for the analysis of mutually exclusive proposals. The methods of ranking mutually exclusive proposals for the purpose of selecting the best alternative is treated separately in Chapter 6.

A systematic approach for economic evaluation of independent projects involves the following steps:

1. Generate a set of investment projects for consideration.
2. Establish the planning horizon for economic study.
3. Develop the cash flow profile for each project.
4. Specify the minimum attractive rate of return.
5. Examine the objective and merit measures.
6. Establish the criterion for accepting or rejecting a proposal.
7. Perform sensitivity analysis.
8. Accept or reject a proposal on the basis of the established criterion.

This same approach is applicable to the economical evaluation of mutually exclusive proposals except that the details of application will be different from those for evaluating independent proposals and that only the best among all mutually exclusive proposals will be selected.

It is important to emphasize that many assumptions and policies, some implicit and some explicit, are introduced in economic evaluations by the decision maker. For example, the generation of a set of investment projects for consideration out of an infinite number of possibilities is a matter of judgment. The planning horizon reflects the time span over which the investor wishes to look ahead. The cash flow profile of an investment project represents the forecast of the benefits and costs over the planning horizon.

Consequently, the viewpoint of a decision maker is pertinent in economic evaluation. In fact, many assumptions and policies, some implicit and some explicit, are introduced by the decision maker. For example, the general manager of a subsidiary of a large international corporation will consider only ramifications that are relevant to the profitability of a project under her jurisdiction. Similarly, the executive of a local mass transit authority would consider federal subsidy as "free" in evaluating a new capital project. Thus, we must first establish for whom a project is intended in order to understand whether the assumptions are justified.

5.2 THE PLANNING HORIZON

The period of time to which an investor wishes to look ahead is called the *planning horizon*. Since the future is uncertain, the period of time selected is limited by our ability to forecast. Hence, the planning horizon may be regarded as the foreseeable future during which some reasonable estimates can be made. The factors that influence the selection of the planning horizon include, but are not limited to, the objectives of the organization, the tax status of the organization, the nature of the investment proposals, and technological obsolescence. Several common approaches have been used and are summarized as follows:

1. The planning horizon reflects the period of intended use of the capital.
2. The planning horizon represents the period of intended use of a facility in which the investment is made.
3. The planning horizon is set equal to a period of time that allows coincidence of the expiration of several facilities which have different useful lives.
4. The planning horizon is assumed to be infinite if the facility is to be used indefinitely.

We shall examine these approaches in the context of economic analysis under different conditions.

Consider, for example, a private firm that is in the process of making a long-range plan for capital investment. The management wishes to look as far ahead as possible, but at the same time recognizes that the costs and benefits of various proposals cannot be estimated with any degree of accuracy beyond a certain time period. Consequently, it may arbitrarily choose a planning horizon, say 20 years. If the useful life of a facility is shorter than 20 years, a replacement facility may or may not be necessary for the remaining years, depending on the nature of the facility. On the other hand, if the useful life of a facility is longer than 20 years, then the salvage value of the facility at the end of 20 years must be included in the analysis.

The term *useful life* of a facility refers to the period of time during which the physical condition of the facility, as well as its economic value, continues to be acceptable. For example, an office building may still be structurally sound after 40 years of service; however, if it costs more to remodel the building than to demolish it and construct a new one to provide the level of service and convenience required, then the building is no longer acceptable for economic reasons. In that case, the useful life is regarded to be 40 years because the effects of benefits and costs after 40 years will be negligible. In the economic evaluation of a major investment project having a long useful life, it is often convenient to take the useful life of the facility as the planning horizon, which is also referred to as the *life cycle* of the investment project.

Another example is the consideration of a group of independent projects by an organization which attempts to use a consistent standard in selecting those that are economically feasible. Suppose that each of these projects is independent of the others and they do not have the same useful life. A consistent standard may be established by imposing the same planning horizon for all projects while properly taking into consideration all costs and benefits within the planning horizon. One possibility is to set the planning horizon at the expiration of all projects, i.e., equal to the least common multiple of useful lives of all projects. However, this approach is often unrealistic, although it may be applicable to some special situations. In any case, it is important to choose a planning horizon within which the investment proposals can be realistically represented.

There are situations in which a facility is intended to be used indefinitely. For example, in planning a new sewage system or other types of infrastructure for a city, the facility is expected to be in use as long as it can perform the intended function.

Although no physical asset will last forever, it is possible to provide a sufficient fund at the beginning of the planning horizon so that a physical asset with finite useful life can be constructed and, at the end of its useful life, be replaced by an identical asset, and this process of replacement can be repeated forever. The fund required at the beginning of the planning horizon for achieving such purposes is referred to as the *capitalized cost* of a project. The annual interest U, which is generated by the capitalized cost P_c in perpetuity, is referred to as the *equivalent uniform annual cost*. If i denotes the annual interest rate, then $U = P_c i$, or

$$P_c = \frac{U}{i} \tag{5.1}$$

Example 5.1

An owner of two existing commercial buildings is considering the possibility of retrofitting them for energy conservation. Both buildings are otherwise in good condition and are expected to last for another 30 years. Preliminary investigations indicate that technologically a solar energy system with 25 years of useful life is most suitable for building 1, and a heat recovery system with 18 years of useful life is most suitable for building 2. What planning horizon should be used for the economic analysis of these two projects?

Let the period of intended use of the buildings after retrofitting for energy conservation be $N_0 = 30$ years. Let the useful life of the solar energy system proposed for building 1 be denoted by $N_1 = 25$ years, and the useful life of the heat recovery system proposed for building 2 be denoted by $N_2 = 18$ years. The owner apparently has not found a suitable alternative energy conservation system with a useful life of 30 years to cover the entire period of intended use of the buildings, and she does not expect to do anything else to retrofit the buildings again when the usefulness of the proposed systems expires.

Since the proposed systems for the two separate buildings are independent projects, we can examine the economic feasibility of these projects separately and determine if each of them is acceptable. The owner of the buildings is interested in the profit potential of both projects in the foreseeable future. It is therefore possible to adopt a common planning horizon for these two projects in spite of the fact that they have different useful lives. For example, we may choose a planning horizon of $n = N_0 = 30$ years or $n = N_1 = 25$ years, as shown in parts (a) and (b) of Fig. 5.1, respectively. In both cases, it is implied that the net return from each project in any year will be reinvested at a specified rate of return until the end of the planning horizon. Theoretically, we may also choose a planning horizon of $n = N_2 = 18$ years, provided that the salvage value of the solar energy system at the end of 18 years can be realistically determined. The practical difficulty in determining the salvage value is that the solar system is a valuable asset as long as it remains in service in the building for 7 more years but may be worthless in the market once it is taken out of the building.

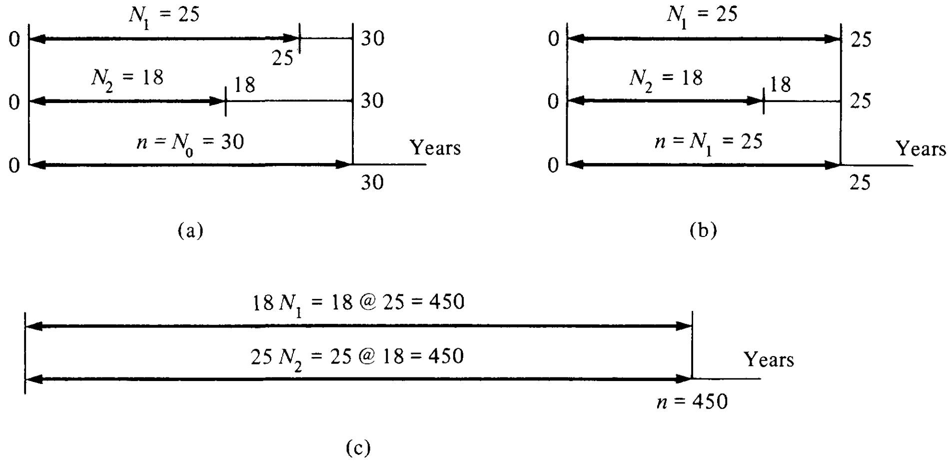

Figure 5.1 Planning horizon for two retrofitting projects.

The use of the least common multiple of $N_1 = 25$ years and $N_2 = 18$ years will lead to $n = (25)(18) = 450$ years as indicated in part (c) of Fig. 5.1. The use of the least common multiple factor as the planning horizon tacitly assumes that each of the two energy conservation systems will be replaced with a system identical to the original one when its useful life expires. It ignores the fact that the buildings themselves will become obsolete after 30 years and neither system will actually be replaced. Consequently, the use of the least common multiple factor of the useful lives of these projects is incorrect for this case.

Example 5.2

A city is spending \$10 million on a new sewage system with the expectation that it will last 50 years with no salvage value at the end of that period. If the MARR specified by the city is 8%, what is the capitalized cost of the system?

Since the cost of the system is \$10 million and it is expected to last 50 years with no salvage value at the end, the equivalent uniform annual cost is

$$U = P(U \mid P, 8\%, 50) = (10)(0.08174286) = \$0.8174286 \text{ million}$$

Note that the compound interest factor has been computed for more significant figures than that given in the table in Appendix A in order to provide the accuracy desired. The capitalized cost for the system can be obtained by Eq. (5.1):

$$P_c = \frac{U}{i} = \frac{0.8174286}{0.08} = \$10.217857 \text{ million}$$

Thus, the additional amount to be raised now is

$$P_0 = 10.217857 - 10 = \$0.217857 \text{ million}$$

Since we are dealing with the difference between two numbers, the value of U must be carried to a sufficient number of digits in order to avoid serious truncation errors. Then, we can expect a present sum of $P_0 = 0.217857$ to produce a sum of $P_c = 10.217857$ at $n = 50$ so that the cycle can be repeated forever. That is,

$$P_c = P_0(F \mid P, 8\%, 50) = (0.217857)(46.9016) = \$10.217842 \text{ million}$$

The discrepancy between \$10.217842 million and \$10.217857 million is caused by the truncation of the digits in the computation.

5.3 CASH FLOW PROFILES

The cash flow profile representing an investment opportunity is sometimes referred to as the *production* cash flow profile since the money will be spent in the acquisition or construction of physical plants and buildings which generate incomes from products and services. By contrast, the cash flow profile representing a borrowing opportunity is referred to as the *financing* cash flow profile. In the economic evaluation of investment projects, we consider first the production cash flow profiles, assuming the funds necessary for investment are available. The effects of borrowing and the treatment of financing cash flow profiles are discussed in Chapter 9.

The economic values of investment proposals are represented by their cash flow profiles over the planning horizon. To simplify the economic analysis, it is assumed that each interest period corresponds to 1 year, and the cash flows are accrued in a lump sum at the end of each year. For a project whose expected life is n years, the cash flows can occur in a time stream with n equal interest periods. The points dividing the interest periods are denoted as $t = 0, 1, 2, \ldots, n$, with $t = 0$ representing the present time.

In many engineering projects, large initial outlays for construction will occur in the early periods, followed by periods of smaller expenditures for operation and maintenance as well as proceeds from the investment. However, this is not necessarily the only pattern. For some investments, such as strip mining or mineral extraction, heavy outlays may be required at the end of the projects to restore the sites to environmentally acceptable conditions. For highways and mass transit systems, heavy outlays for reconstruction or rolling stock replacement may be required intermittently. Consequently, the benefits and costs of each project over the planning horizon should be carefully estimated. The separate cash flow profiles of benefits and costs of an investment are referred to as the *time streams of benefits and costs,* respectively.

The costs incurred through the life cycle of a physical asset generally include the initial acquisition costs, the subsequent operation and maintenance costs, and the cost of disposal at the end of the life cycle. In planning capital investment projects, we are only interested in *present* and *future* costs. *Past costs* refer to the expenditures already spent on an existing project. Unrecoverable past costs are called *sunk costs* and must not be included in the analysis of future investments.

The problem of treating sunk costs properly is very important in economic evaluation. Consider, for example, a transit authority which had spent \$1 million in constructing a tunnel for a new transit line when the work was abandoned 4 years ago because of strong public opposition. Finally a compromise is reached and the project will be revived. Should the \$1 million already spent affect the economic feasibility of the revised proposal which will include this tunnel? The answer depends to some degree on the circumstances. If the tunnel cannot be put to any other use except for the transit line, the \$1 million already spent represents a sunk cost and should not be included in the economic evaluation of the revised proposal. However, if the \$1 million had been spent in the acquisition of land which could be sold to recover part of the original cost, then the sunk cost would be the difference between the original cost and the current market value of the land previously acquired for the project.

The benefits derived from the physical asset throughout the planning horizon may be quantified directly or indirectly. Some benefits, such as rents for an office building or fuel cost savings from energy conservation schemes, are explicitly expressed in monetary values; others, such as the reduction of travel time due to a new transit facility or the water recreation opportunities from a flood control project, are far more difficult to measure or express in terms of money.

Let $B_{t,x}$ be the annual benefit at the end of year t for an investment project x, where $x = 1, 2, \ldots$ denotes projects 1, 2, etc., respectively. Then, a series of values $B_{t,x}$ (for $t = 0, 1, 2, \ldots, n$) represents a time stream of benefits over a planning horizon of n years. Let $C_{t,x}$ be the annual cost at the end of year t for the same investment project x. Then, a series of values $C_{t,x}$ (for $t = 0, 1, 2, \ldots, n$) represents a time stream of costs over the planning horizon. Cash receipts accrued and cash disbursements incurred are the most obvious forms of benefits and costs, respectively; however, benefits and costs may also be measured indirectly by *imputed* prices. For example, in purchasing a new machine to reduce the labor in a manufacturing process, benefits may be estimated by the savings from the reduction of labor costs. On the other hand, the salvage value of the machine, which represents its estimated market value at the time of disposal minus the cost of disposal, may be treated as a benefit to be realized at the time of disposal.

The *annual net cash flow* is defined as the annual benefit in excess of the annual cost. Let $A_{t,x}$ be the net annual cash flow at the end of year t for an investment project x. Then, for $t = 0, 1, \ldots, n$

$$A_{t,x} = B_{t,x} - C_{t,x} \tag{5.2}$$

The value of $A_{t,x}$ is positive if $B_{t,x} > C_{t,x}$ and is negative if $B_{t,x} < C_{t,x}$. A series of values $A_{t,x}$ (for $t = 0, 1, 2, \ldots, n$) represents a time stream of net annual benefits or costs over a planning horizon of n years.

In the economic evaluation of an investment proposal, we measure the profit potential of this proposal against a baseline of no gain and no loss. We define this baseline as the *null alternative,* which will be denoted by $x = 0$ such that $B_{t,0} = 0$, $C_{t,0} = 0$, and $A_{t,0} = 0$ for all values of t. For example, in considering a construction project that will provide a new service, we can measure its profit potential against the prospect of doing nothing; then the status quo alternative is identical to the null alternative. However, if we consider a new project to replace an existing facility already providing a service, the status quo alternative may represent a money-losing proposition which is quite different from the null alternative. Even the abandonment of an existing facility does not necessarily lead to a null alternative since it is possible that the defunct facility must be removed as required by law or local ordinance. Then the removal cost must be included in the abandonment alternative.

In evaluating independent projects, we often encounter conventional annual net cash flow profiles in which $A_{t,x}$ is negative at $t = 0$ but positive at $t = 1, 2, \ldots, n$. However, even under some unsuspected circumstances, we may be confronted with an unconventional net cash flow profile with multiple sign changes in $A_{t,x}$. Consequently, we must be prepared to deal with such problems when they occur.

Example 5.3

The stream of benefits $B_{t,x}$ and the stream of cost $C_{t,x}$ for each of the two independent projects are shown in Table 5.1. Find the annual net cash flow profiles for both projects.

We note in Example 5.1 that although the period of intended use of both buildings is 30 years, it is not practical to retrofit any other energy-conserving system in either building when the usefulness of the current proposed projects expires. Consequently, the net present value of each of the independent projects remains the same as long as the planning horizon is set equal to or greater than 25 years since $B_{t,x}$ and $C_{t,x}$ for both $x = 1$ and $x = 2$ are zero for t greater than 25.

The costs for the solar energy system ($x = 1$) and for the heat recovery system ($x = 2$) in Example 5.1 represent the installation costs for these projects at $t = 0$. The annual benefits of each project for $t = 1, 2, \ldots, n$ result from savings in energy costs. Net cash flow profiles $A_{t,x}$ for both projects ($x = 1$ and $x = 2$) are also shown in Table 5.1. Note that since the retrofitting projects are to be installed in existing buildings, only the costs incurred by and the benefits derived from retrofitting need be considered.

TABLE 5.1 CASH FLOW PROFILES FOR THE ENERGY CONSERVATION PROJECTS

	$x = 1$			$x = 2$		
t	$B_{t,1}$	$C_{t,1}$	$A_{t,1}$	$B_{t,2}$	$C_{t,2}$	$A_{t,2}$
0	0	100,000	−100,000	0	80,000	−80,000
1–18 (each)	12,000	0	+ 12,000	11,000	0	+11,000
19–25 (each)	12,000	0	+ 12,000	0	0	0

Example 5.4

A construction contract of a large power plant requires payments by the owner to the contractor at year ends according to some portions of work completed during the six years of construction. The contractor's expenses at year ends are shown in the C_t column and the owner's payments in the B_t column in Table 5.2. Find the net cash flows A_t for the project.

The contractor's expenses C_t reflect the gradual mobilization of labor, material, and equipment for the construction project, reaching the peak at the end of year 4 and then declining toward the end of the project. The owner's payments reflect the portions of work completed up to the time of payment except in year 6 when the entire project is completed. This type of payment schedule is quite common in construction contracts. However, the resulting cash flow profile as represented by the A_t column is identical (except in thousand dollars) to the unconventional cash flow profile in Example 4.13 in Chapter 4. We know from Example 4.13 that a unique IRR of 10% for this profile exists. Hence, the investment of the contractor has an internal rate of return of 10%.

TABLE 5.2 COMPUTATION OF NET CASH FLOWS (IN $1,000)

t	B_t	C_t	A_t
0	0	20,000	−20,000
1	46,000	34,000	+12,000
2	46,000	65,000	−19,000
3	92,000	69,000	+23,000
4	92,000	92,000	0
5	46,000	61,000	−15,900
6	46,000	15,200	+30,800

5.4 THE MINIMUM ATTRACTIVE RATE OF RETURN

In perfectly competitive capital markets, the organization is free to lend or borrow at the minimum attractive rate of return (MARR). Without any explicit policy to the contrary, the external rate of return for reinvestment of returns in the early periods of a project prior to its operation is assumed to be equal to the MARR. Such an implicit reinvestment policy is consistent with the definitions of net future value and net present value in Section 4.2, and will be the basis of discussion in the next few chapters.

Different approaches have been used by private firms and public agencies to establish the MARR, and no one single approach is indisputably superior. In general, the MARR specified by the top management in a private firm reflects the *opportunity cost of capital* of the firm, the market interest rates for lending and borrowing, and the risks associated with investment opportunities. It is sometimes referred to as the *hurdle rate* for capital investments. For public projects, the MARR is specified by a government agency, such as the Office of Management and Budget or the Congress of the United States, which is authorized by law to act in the public interest. The MARR thus specified reflects social and economic welfare considerations and is referred to as the *social rate of discount*.

Regardless of how the MARR is determined by an organization, the MARR specified for the economic evaluation of investment proposals is critically important in determining whether any investment proposal is worthwhile *from the standpoint of the organization*. Since the MARR of an organization often cannot be measured accurately, it is advisable to use several values of the MARR in assessing the profit potential of an investment proposal so that we can appraise how sensitive the profit potential is to the variation of the MARR.

Example 5.5

Find the present value of $10,000 to be received at year t, for $t = 1, 20, 40, 60, 80$, and 100 years, if the discount rate is $i = 4\%, 7\%$, and 10%. Plot the results of present value versus time in years.

The present value can be obtained by multiplying $10,000 by the discount amount factor $(P \mid F, i, t)$. The results (rounded to the nearest even dollar) are tabulated below and plotted in Fig. 5.2.

i	$t = 1$	$t = 20$	$t = 40$	$t = 60$	$t = 80$	$t = 100$
4%	9,615	4,564	2,083	951	434	198
7%	9,346	2,584	668	173	45	12
10%	9,091	1,486	221	33	5	1

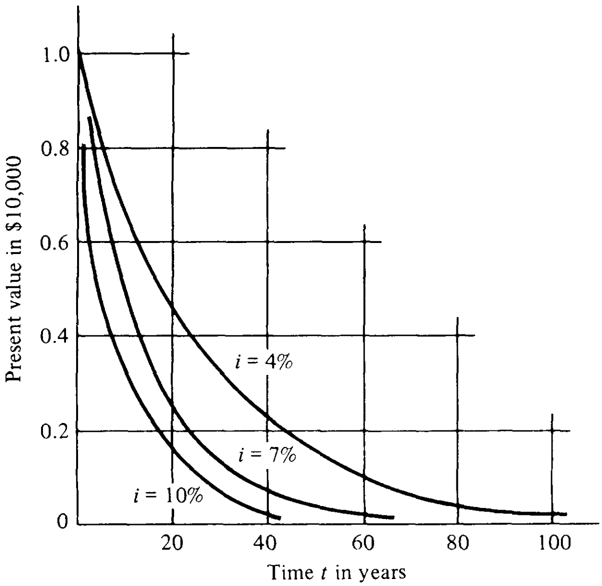

Figure 5.2 Effects of time and discount rate on present value.

The results of this example show how the present value of a sum (say profit) can be affected by the discount rates for a specified planning horizon. For example, for $t = 20$, a present value of 2,584 on the basis of $i = 7\%$ can vary by a wide margin if i is changed by $\pm 3\%$.

5.5 INVESTMENT OBJECTIVES AND MERIT MEASURES

The primary objective of capital investment is to maximize profit or net benefit within a planning horizon. In the context of multiperiod investments, the objective is to maximize the net future value at the end of the planning horizon. It is important to emphasize that real assets are size dependent and each asset is indivisible. Consequently, some projects require more capital outlays than others. As long as an investor can borrow or lend freely at the minimum attractive rate of return (MARR), the size of investment on a proposal project is not constrained. It is therefore worthwhile to invest in an independent project that will generate a gain at the end that is greater than the foregone opportunity regardless of the size of the investment. The effects of capital rationing are discussed in Chapter 9.

Under the assumption of perfect capital markets, the net future value (NFV) is a direct measure of the size of profit or net benefit at the end of the planning horizon that the investor would have gained by having invested in a proposed project relative to investing in the foregone opportunity at the MARR. Hence, the NFV is referred to as a *direct merit measure*. There are other direct merit measures with different time values, e.g., net present value (NPV) which is equal to NFV multiplied by a constant factor $(1 + i)^{-n}$. On the other hand, other merit measures do not directly reflect the size of the profit or net benefit at the end of the planning horizon, e.g., the internal rate of return (IRR). Such a measure is referred to as an *indirect merit measure*. We shall examine in detail both direct and indirect merit measures that are popular with decision makers.

An investment decision criterion consists of two elements: a merit measure and a set of decision rules. Each merit measure requires a set of decision rules to implement the stated objective of profit maximization. For the economic evaluation of independent projects, the decision rules associated with various merit measures have been discussed in detail in Chapter 4. We shall apply them to investment problems in this chapter.

To understand why so many merit measures are widely used, it is important to recognize the decision-making structure in an organization. In a large organization, an analyst may be assigned the task of making a preliminary screening of potential investment opportunities as independent projects without the full knowledge of the investment and financing policies of the organization. The manager at the middle management level is likely to be responsible in making capital investment choices among mutually exclusive proposals on the basis of the full range of organizational policies and recommending them to the top-level management for approval. The top management and the board of directors may approve individual investment proposals one at a time. Knowing that each individual proposal has been scrutinized and chosen for submission by the middle management, the board of directors may judge the merit of a proposal on the basis of the track record of the manager who recommends the proposal instead of the supporting economic analysis.[1] At most, some of them may use a merit measure to make a quick check on the proposals. Since each of them may prefer a different merit measure, the analysts and the managers must be able to anticipate their preferences by providing various merit measures.

5.6 THE NET PRESENT VALUE CRITERION

A capital investment involves the commitment of resources now in anticipation of greater returns in the future. An organization that undertakes a capital project is obviously interested in knowing what return this project will produce at the end of the planning horizon. Hence, the *net future value* of the cash flow profile of the investment compounded at the specified MARR at the end of the planning horizon is a *direct measure* of the profit potential.

For a given series of net cash flows $A_{t,x}$ for a project x over a planning horizon of n years (for $t = 0, 1, 2, \ldots, n$), and a given value of MARR $= i$, the net future value of the series at $t = n$ is given by

$$\text{NFV}_x = \sum_{t=0}^{n} A_{t,x}(1 + i)^{n-t} = \sum_{t=0}^{n} A_{t,x}(F \mid P, i, n - t) \tag{5.3}$$

It is important to note that the NFV_x is dependent on the planning horizon n as well as the MARR $= i$. Since the useful life N_x of a capital project x may be different from the planning horizon n, we must ascertain whether a replacement is necessary when the project expires before the end of the planning horizon. If there is no replacement, the total return at the end of the life of a project will be reinvested at the MARR until the end of the planning horizon; otherwise, the net cash flows of the replacement should be included to cover those years after the expiration of the original project. All these possible situations can be covered by Eq. (5.3) since the net cash flows $A_{t,x}$ may be zero, negative, or positive for some years t as the case may be.

Since the cash flow profile of an investment proposal can be represented by its equivalence at any specified reference point in time, the net present value of a series of cash flows $A_{t,x}$ (for $t = 0, 1, \ldots, n$) for project x can be obtained as follows:

$$\text{NPV}_x = \sum_{t=0}^{n} A_{t,x}(1 + i)^{-t} = \sum_{t=0}^{n} A_{t,x}(P \mid F, i, t) \tag{5.4}$$

It can easily be shown that

$$\text{NFV}_x = \text{NPV}_x(F \mid P, i, n) \tag{5.5a}$$

[1] For a first-hand observation of decision making on capital investments in the boardrooms of large corporations, see Ref. 5.1.

or

$$\text{NPV}_x = \text{NFV}_x(P \mid F, i, n) \tag{5.5b}$$

The net present value is often preferred as a merit measure of a project because its value is not affected by the planning horizon *as long as* the planning horizon is greater than or equal to the useful life of the project *and* there is no replacement when the project expires. However, the assumptions used in the computation of the net *future* value of a project are also implicit in the computation of the net *present* value. If $\text{NFV}_x \geq 0$, it follows that $\text{NPV}_x \geq 0$, and vice versa. When the value is zero, the situation is neutral and may be treated as the limiting condition for which the project is acceptable.

Let BPV_x be the present value of benefits of a project x and CPV_x be the present value of costs of the project x. Also let NPV_x be the net present value of the project x. Then, for MARR $= i$ over a planning horizon of n years,

$$\text{BPV}_x = \sum_{t=0}^{n} B_{t,x}(1+i)^{-t} = \sum_{t=0}^{n} B_{t,x}(P \mid F, i, t) \tag{5.6a}$$

$$\text{CPV}_x = \sum_{t=0}^{n} C_{t,x}(1+i)^{-t} = \sum_{t=0}^{n} C_{t,x}(P \mid F, i, t) \tag{5.6b}$$

Then,

$$\text{NPV}_x = \text{BPV}_x - \text{CPV}_x \tag{5.7}$$

or

$$\text{NPV}_x = \sum_{t=0}^{n} (B_{t,x} - C_{t,x})(P \mid F, i, t) \tag{5.8}$$

In view of Eq. (5.2), the last equation can be replaced by Eq. (5.4).

If there is no budget constraint, the criterion for accepting an independent project x is

$$\text{NPV}_x \geq 0 \tag{5.9}$$

It is implicit that for the null alternative ($x = 0$), $\text{NPV}_0 = 0$. Hence, the project x is superior to or at least as good as the null alternative.

Example 5.6

The cash flow profiles for the solar energy system ($x = 1$) and the heat recovery system ($x = 2$) in Example 5.1 are given in Example 5.3. The MARR is specified to be 10%. Compute the net present value and the net future value of each project over a planning horizon of 30 years.

Let n be the number of years in the planning horizon, and N_x be the useful life of project x. In computing the NPV, the value of n need not be stated explicitly in the case of $N_x \leq n$. Using the data in Table 5.1 of Example 5.3, we compute the NPV at MARR = 10% according to Eq. (5.4) as follows:

$$\text{NPV}_x = \sum_{t=0}^{N_x} A_{t,x}(P \mid F, i, t)$$

Thus, at MARR = 10%

$$[\text{NPV}_1]_{10\%} = -100{,}000 + (12{,}000)(P \mid U, 10\%, 25)$$
$$= -100{,}000 + (12{,}000)(9.0770) = +8{,}924$$
$$[\text{NPV}_2]_{10\%} = -80{,}000 + (11{,}000)(P \mid U, 10\%, 18)$$
$$= -80{,}000 + (11{,}000)(8.2014) = +10{,}215$$

Obviously, the net future value of a project depends on the specific future point in time. Hence, the planning horizon must be specified explicitly if the net future value at the end of n years is desired. Furthermore, in the absence of special circumstances, it is reasonable to use the MARR as the reinvestment rate for benefits received before the end of the planning horizon. Thus, for $n = 30$, we obtain from the cash flow profiles of the two projects the net future values as follows:

$$[\text{NFV}_1]_{10\%} = -(100{,}000)(F \mid P, 10\%, 30) + (12{,}000)(F \mid U, 10\%, 25)(F \mid P, 10\%, 5)$$
$$= -(100{,}000)(17.4494) + (12{,}000)(98.3470)(1.6105)$$
$$= -1{,}744{,}940 + 1{,}900{,}656 = +155{,}716$$
$$[\text{NFV}_2]_{10\%} = -(80{,}000)(F \mid P, 10\%, 30) + (11{,}000)(F \mid U, 10\%, 18)(F \mid P, 10\%, 12)$$
$$= -(80{,}000)(17.4494) + (11{,}000)(45.5992)(3.1384)$$
$$= -1{,}395{,}952 + 1{,}574{,}194 = +178{,}242$$

These results can be obtained readily from the net present values if the latter are computed first. Thus,

$$[\text{NFV}_1]_{10\%} = (\text{NPV}_1)(F \mid P, 10\%, 30)$$
$$= (+8{,}924)(17.4494) = +155.718$$
$$[\text{NFV}_2]_{10\%} = (\text{NPV}_2)(F \mid P, 10\%, 30)$$
$$= (+10{,}215)(17.4494) = +178{,}246$$

Example 5.7

The construction of a commercial building takes two years. During the construction period, the owner has arranged with a bank to pay the construction contractor an amount of \$979,400 at the end of each quarter for eight quarters, and the interest rate is 3% per quarter. At the end of the construction, the owner will pay the bank for the construction expenses and interest costs, and take control of the building. The annual rental income from the building is expected to be one million dollars per year in the next 40 years with no salvage value at the end. If the MARR of the owner is 11%, determine whether this project is worthwhile.

Using the start of the construction of the building as a reference point, the planning horizon would be 42 years. However, the problem can be simplified if the two years before the completion of the building are treated separately. We first find the owner's payment Q to the bank at the end of the construction as follows:

$$Q = (979{,}400)(F \mid U, 3\%, 8) = (979{,}400)(8.8923) = 8{,}709{,}200$$

Hence, upon the completion of construction, the owner invests this amount with a new planning horizon of 40 years. Hence, for the anticipated annual rental income in the next 40 years,

$$\begin{aligned} \text{NPV} &= -8{,}709{,}200 + (1{,}000{,}000)(P \mid U, 11\%, 40) \\ &= -8{,}709{,}200 + (1{,}000{,}000)(8.9511) = 241{,}900 \end{aligned}$$

The investment project is indeed worthwhile.

5.7 THE NPV VERSUS *i* GRAPH

As explained in Section 4.3, the net present value (NPV) versus the discount rate i graph for a cash flow profile is helpful in understanding the nature of the internal rate of return (IRR). Let i^* denote the specific value of the minimum attractive rate of return (MARR) used as the discount rate to compute the NPV. For different values of MARR in the positive range of i, we can compute the corresponding NPV. Then, the sensitivity of the NPV with respect to the MARR can be observed from the NPV versus i graph, which also provides some insight into the relative position of the specified MARR and the IRR.

Example 5.8

A strip-mining operation requires an initial cost of $545,000 which will yield an annual net income of $150,000 from mining for the next 8 years. However, at the end of 8 years, an amount of $150,000 will be required to restore the landscape for environmental protection according to the law, and this amount negates all the income from mining for the year. Is this operation economically sound at MARR = 10%? Also, check the economic feasibility of this project at MARR = 5% and MARR = 15%. Plot the NPV versus i graph for MARR = 0% to 30%.

Let the proposed operation be designated as $x = 1$ (as opposed to $x = 0$ for the null alternative). The net present values of this operation in thousands of dollars is given by

$$\text{NPV}_1 = -545 + (150)(P \mid U, i, 7) + 0$$

For MARR = 0%, 5%, 10%, and 15%, we obtain

$$[\text{NPV}_1]_{0\%} = -545 + (150)(7) = +505$$

$$[\text{NPV}_1]_{5\%} = -545 + (150)(5.7864) = +323.0$$

$$[\text{NPV}_1]_{10\%} = -545 + (150)(4.8684) = +185.3$$

$$[\text{NPV}_1]_{15\%} = -545 + (150)(4.1604) = +79.1$$

It is obvious that this operation is economically sound for the values of MARR computed. Thus, even if the MARR varies from the specified value of 10% by ±5%, it will not affect the merit of accepting this proposal. The NPV versus *i* graph is shown in Fig. 5.3.

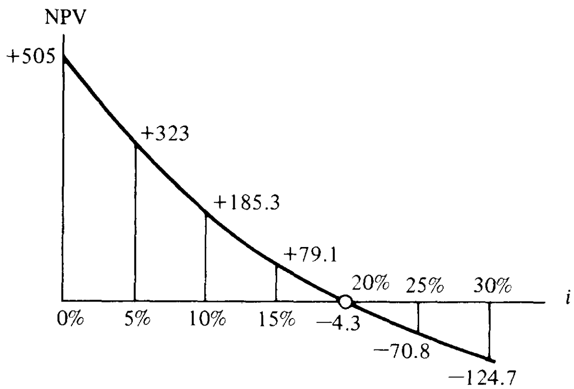

Figure 5.3 NPV versus *i* graph of a conventional investment project.

Example 5.9

Suppose that the strip-mining operation requires an initial cost of $340,000 which will yield an annual net income of $150,000 from mining for the next 8 years. At the end of the eighth year, the cost of restoring the landscape for environmental protection is $980,000, resulting in a net cost of $830,000 after deducting the net income of $150,000 for the year. Plot the NPV versus *i* graph for MARR = 0% to 30%. Is this operation economically feasible at MARR = 10%?

The net present value of this operation (designated as $x = 1$) is given in thousands of dollars as follows:

$$\text{NPV}_1 = -340 + (150)(P \mid U, i, 7) - (830)(P \mid F, i, 8)$$

For i = 0%, 5%, 10%, 15%, 20%, 25%, and 30%, the net present values can be tabulated as in Table 5.3.

The NPV versus *i* graph for this problem is shown in Fig. 5.4. Note that the cash profile for the operation has two sign changes, leading to two values

TABLE 5.3 NET PRESENT VALUE AT THE SPECIFIED MARR

i	0%	5%	10%	15%	20%	25%	30%
−340	− 340	−340	−340	−340	−340	−340	−340
+150($P \mid U, i, 7$)	+1,050	+868	+730	+624	+541	+474	+420
−830($P \mid F, i, 8$)	− 836	−565	−390	−273	− 19	− 1	102
NPV_I	− 126	− 38	0	+ 11	+ 7	− 6	− 22

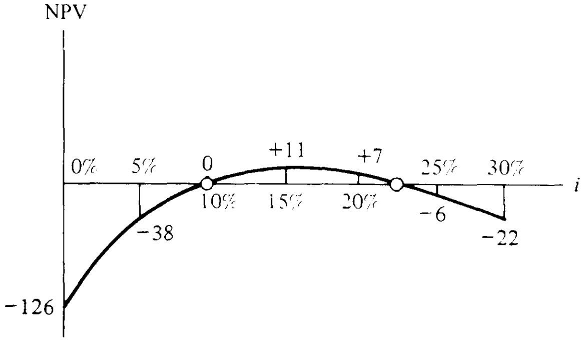

Figure 5.4 NPV versus i graph of an unconventional investment project.

of IRR: 10% and 22.5%. At MARR = 15%, NPV = +11. Even if the MARR varies from the specified value of 15% by ±5%, it will not affect the merit of accepting this proposal.

To confirm that this proposal is acceptable even though the cash flow profile has two values of IRR, we can compute the NFV as represented by R_n in Eq. (4.17). For i = 15%, the values of R_t are computed by Eqs. (4.12) through (4.16), and are tabulated in Table 5.4. Note that for $n = 8$, $R_8 = +32.9$. Consequently, the corresponding net present value is

$$[NPV_I]_{15\%} = (32.9)(P \mid F, 15\%, 8) = 11$$

This result checks with the value obtained by direct computation in Table 5.3.

TABLE 5.4 YEAR-BY-YEAR UNRECOVERED BALANCE FOR MARR = 15%

t	A_t	I_t	Q_t	R_t
0	−340	0	−340	−340
1	+150	− 51	− 99	−241
2	+150	− 36.1	+113.9	−127.1
3	+150	− 19.1	+130.9	+ 3.8
4	+150	+ 0.6	+150.6	+154.4
5	+150	+ 23.1	+173.1	+327.5
6	+150	+ 49.1	+199.1	+526.6
7	+150	+ 79.0	+229.0	+755.6
8	−836	+113.3	−722.7	+ 32.9

5.8 NET EQUIVALENT UNIFORM ANNUAL VALUE CRITERION

The net equivalent uniform annual value refers to a *uniform* series over n years whose net present value is the same as that of a series of net cash flows $A_{t,x}$ for $t = 0, 1, 2, \ldots, n$ representing project x. If the net present value of project x is denoted by NPV_x, then the net equivalent uniform annual value (EUAV) for project x is given by

$$EUAV_x = NPV_x(U \mid P, i, n) \tag{5.10a}$$

or

$$NPV_x = EUAV_x(P \mid U, i, n) \tag{5.10b}$$

Let us elaborate further on the meaning of the term *net equivalent uniform annual value,* which should not be confused with some related terms.[2] When a series of *net annual cash flows* $A_{t,x}$ (for $t = 0, 1, 2, \ldots, n$) is used to represent the cash flow profile of project x, the value of $A_{t,x}$ can be positive, negative, or zero for any given t. The net present value of this series may be designated as NPV_x. The net equivalent uniform annual value (EUAV) of project x simply refers to a uniform series whose annual value is $EUAV_x$ in the years $t = 1, 2, \ldots, n$ such that its net present value equals NPV_x. Note that for a uniform series, the cash flow at $t = 0$ is zero. Thus, the series of cash flows $A_0 = -P$, $A_1 = A_2 = \cdots = A_n = U$ is *not* a uniform series, although the series $A_0 = 0$, $A_1 = A_2 = \cdots = A_n = U$ is. Furthermore, let BPV_x be the present value of a time stream of benefits $B_{t,x}$ (for $t = 0, 1, 2, \ldots, n$) of project x. Then, the *equivalent uniform annual benefit* (EUAB) of project x refers to a uniform series whose annual value is $EUAB_x$ in the years $t = 1, 2, \ldots, n$ such that its present value is the same as BPV_x. Similarly, let CPV_x be the present value of a time stream of costs $C_{t,x}$ (for $t = 0, 1, 2, \ldots, n$) of project x. Then, the *equivalent uniform annual cost* (EUAC) of project x refers to a uniform series whose annual value is $EUAC_x$ in the years $t = 1, 2, \ldots, n$ such that its present value is the same as CPV_x. Thus,

$$EUAB_x = BPV_x(U \mid P, i, n) \tag{5.11a}$$

and

$$EUAC_x = CPV_x(U \mid P, i, n) \tag{5.11b}$$

It should be emphasized that the net equivalent uniform annual value of a series of cash flows is based on a specific planning horizon and a specified MARR. Consequently, the same assumptions used in the computation of the net future value or net present values also prevail.

The decision criterion based on the net equivalent uniform annual value (EUAV) is a direct result of observing the relationship between the EUAV and the

[2] To avoid confusion, the notation U has been used to represent the *uniform* cash flows in the functional notation for the compound interest factors involving the uniform series, while the notation A has been used to denote net *annual* cash flow.

NPV of a cash flow profile. Thus, the criterion for accepting an independent project x can be derived from Eq. (5.9) as follows:

$$\text{EUAV}_x \geq 0 \tag{5.12}$$

Example 5.10

Find the net equivalent annual values of the two projects in Example 5.3 that will not be replaced upon expiration, using (a) a planning horizon of 30 years for both projects, and (b) the useful life of each project as its planning horizon. The MARR is specified to be 10%.

(a) For a planning horizon of 30 years, the net present values of these two projects have been found in Example 5.6 to be $[\text{NPV}_1]_{10\%} = +8{,}924$, and $[\text{NPV}_2]_{10\%} = +10{,}215$. Since the projects will not be replaced upon expiration, the EUAV over 30 years can be obtained as follows:

$$[\text{EUAV}_1]_{10\%} = (\text{NPV}_1)(U \mid P, 10\%, 30) = +947$$

$$[\text{EUAV}_2]_{10\%} = (\text{NPV}_2)(U \mid P, 10\%, 30) = +1{,}084$$

(b) If the useful life of each project is used as its planning horizon, the EUAV for the two projects are, respectively,

$$[\text{EUAV}_1]_{10\%} = (\text{NPV}_1)(U \mid P, 10\%, 25) = +983$$

$$[\text{EUAV}_2]_{10\%} = (\text{NPV}_2)(U \mid P, 10\%, 18) = +1{,}245$$

It can be seen that EUAV is influenced by the planning horizon over which the equivalent uniform annual series is defined in the problem.

Example 5.11

A company plans to be in business for 30 years and has purchased two machines, each operating independently of the other. The first machine costs \$100,000 and is expected to generate a series of uniform net income of \$12,000 per year in the next 25 years with no salvage value at the end. It will be replaced at year 25 by a different machine which will cost \$80,000 and is expected to generate a series of uniform net income of \$11,000 per year in the following 5 years with a salvage value of \$65,000 at the end of 5 years. The second machine costs \$80,000 and is expected to generate a series of uniform net income of \$11,000 per year in the next 18 years with no salvage at the end. It will be replaced at year 18 by an identical machine which has a salvage value of \$15,000 at the end of 12 years. For a MARR of 10%, find the net equivalent uniform annual value (EUAV) for each of the operations over the planning horizon of 30 years.

We can first find the net present value for each of the operations over 30 years for MARR = 10%:

$$
\begin{aligned}
[\text{NPV}_1]_{10\%} &= -100{,}000 + (12{,}000)(P \mid U, 10\%, 25) \\
&\qquad + [-80{,}000 + (11{,}000)(P \mid U, 10\%, 5)](P \mid F, 10\%, 25) \\
&\qquad + (65{,}000)(P \mid F, 10\%, 30) \\
&= 8{,}924 - (38{,}301)(0.0923) + (65{,}000)(0.0573) = +9{,}114 \\
[\text{NPV}_2]_{10\%} &= -80{,}000 + (11{,}000)(P \mid U, 10\%, 18) \\
&\qquad + [-80{,}000 + (11{,}000)(P \mid U, 10\%, 12)](P \mid F, 10\%, 18) \\
&\qquad + (15{,}000)(P \mid F, 10\%, 30) \\
&= +10{,}215 - (5{,}049)(0.1799) + (15{,}000)(0.0573) = +10{,}166
\end{aligned}
$$

The corresponding EUAV over 30 years are obtained as follows:

$$
\begin{aligned}
[\text{EUAV}_1]_{10\%} &= (9{,}114)(U \mid P, 10\%, 30) = +967 \\
[\text{EUAV}_2]_{10\%} &= (10{,}166)(U \mid P, 10\%, 30) = +1{,}079
\end{aligned}
$$

5.9 THE INTERNAL RATE OF RETURN CRITERION

The *internal rate of return* (IRR) refers to the discount rate at which the net present value of the cash flow profile representing a proposed project is zero. The IRR is an indirect but useful merit measure if a unique value exists. For this reason, the discussion here is confined only to the criterion for evaluating a pure investment cash flow profile. Let the IRR of the cash flow profile for project x be i'_x, and let the MARR specified be denoted by i^*. Then, the internal rate of return criterion based on a unique IRR for accepting an independent project x is

$$i^* \le i'_x \tag{5.13}$$

For an unconventional cash flow profile that leads to multiple values of IRR, the decision criteria based on these multiple values are much more complex. The development and applications of such decision criteria are discussed in detail in Chapter 8.

Example 5.12

The cash flow profile of four independent projects are shown in Table 5.5. Determine the acceptability of each of these projects on the basis of the internal rate of return criterion if the MARR is 20%.

There is only one sign change in the cash flow profiles of projects $x = 1$ through $x = 4$, and for each case, IRR is found to be 25%, using the computation procedures discussed in Chapter 4. Since 20% < 25%, each of these four projects is acceptable according to Eq. (5.13).

TABLE 5.5 CASH FLOW PROFILES OF FOUR INDEPENDENT PROJECTS (IN $ MILLION)

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$	$A_{t,4}$
0	− 77.0	−77.0	−75.3	0
1	0	+38.0	+28.0	− 28.0
2	0	+32.0	+28.0	− 28.0
3	0	+26.0	+28.0	− 28.0
4	0	+20.0	+28.0	− 28.0
5	+235.0	+14.0	+28.0	+201.9

5.10 THE OVERALL RATE OF RETURN CRITERION

The overall rate of return (ORR) refers to the overall rate of return to the investor from an investment project. In this section, the application is confined only to conventional investment cash flow profiles.

Let the ORR for project x over a planning horizon of n years be denoted as i_x^0 and let the MARR specified by a firm be denoted by i^*. Then, the criterion for accepting project x based on the ORR is

$$i^* \leq i_x^0 \tag{5.14}$$

Example 5.13

Compute the overall rates of return for the four independent projects in Example 5.12, using a MARR of 20%.

For $x = 1$, there is only a single sum $P_e = -77$ at $t = 0$ and a single sum $F_e = 235$ at $n = 5$. Hence, for NPV = 0,

$$-77 + 235(P \mid F, i_1^0, 5) = 0$$

or

$$(P \mid F, i_1^0, 5) = \frac{77}{235} = 0.3277$$

From the tables in Appendix A, we find $i_1^0 = 25\%$.

For $x = 2$, 3, and 4, the application of Eqs. (4.21) to (4.23) leads to the following equations involving i_2^0, i_3^0, and i_4^0, respectively:

$$-77 + [(38)(F \mid U, 20\%, 5) - (6)(F \mid G, 20\%, 5)](P \mid F, i_2^0, 5) = 0$$

$$-75.3 + (28)(F \mid U, 20\%, 5)(P \mid F, i_3^0, 5) = 0$$

$$-(28)(P \mid U, 20\%, 4) + (201.9)(P \mid F, i_4^0, 5) = 0$$

By interpolation from the tables in Appendix A, we find $i_2^0 = 22.3\%$, $i_3^0 = 22.7\%$, and $i_4^0 = 22.8\%$.

5.11 THE PAYBACK PERIOD CRITERION

The *payback period* refers to length of time within which early receipts from an investment can repay disbursements incurred during the time span. If the MARR is not taken into account in computing the required length of time, the resulting period is referred to as the *simple payback period* (SPP). If the MARR is taken into account in computing the required length of time, the resulting period is referred to as the *discounted payback period* (DPP). In either case, the length of time is a merit measure reflecting the risk involved in an investment. The SPP is applicable only to a conventional investment profile, while the DPP is applicable to a pure investment profile in which a unique value of IRR exists.

Using the annual net cash flow profile $A_{t,x}$ (for $t = 0, 1, 2, \ldots, n$) representing project x, the simple payback period (SPP_x) refers to the smallest positive integer q such that

$$\sum_{t=0}^{q} A_{t,x} \geq 0 \tag{5.15}$$

For the simplest case in which $A_{0,x} = -C_{0,x}$ for $t = 0$ and $A_{t,x} = \bar{B}_x$ for $t = 1$ to $t = n$, i.e., $C_{0,x}$ is the initial cost and $\bar{B}_x$ is the uniform annual benefit from project x, the SPP_x is the smallest positive integer q such that

$$q \geq \frac{C_{0,x}}{\bar{B}_x} \tag{5.16}$$

The determination of the discounted payback period (DPP_x) for an annual net cash flow profile can be facilitated by recognizing that it occurs when the unrecovered balance R_t as defined in Section 4.6 changes from negative to nonnegative. Let $t = q$ be the period in a pure investment profile $A_{t,x}$ (for $t = 0, 1, 2, \ldots, n$) such that

$$R_{q-1} < 0 \qquad \text{and} \qquad R_q \geq 0 \tag{5.17}$$

Then $t = q$ is the discount payback period.

The simple payback periods can be detected by inspection since the numerical computation is extremely simple. The computation of R_t in an investment cash flow profile to determine the discounted payback period is also simple; it can be found by following the procedure outlined in Section 4.6.

Example 5.14

Find the simple payback period for each of the four independent projects in Example 5.12.

The payback period can be obtained according to Eq. (5.15) for each of the projects with only one sign change in the cash flow profile.

1. For $x = 1$, SPP_1 is found to be 5 years because

$$\sum_{t=0}^{4} A_{t,1} = -77 + 0 = -77 \quad (<0)$$

$$\sum_{t=0}^{5} A_{t,1} = -77 + 235 = 158 \quad (>0)$$

2. For $x = 2$, SPP_2 is found to be 3 years because

$$\sum_{t=0}^{2} A_{t,2} = -77 + 38 + 32 = -7 \quad (<0)$$

$$\sum_{t=0}^{3} A_{t,2} = -77 + 38 + 32 + 26 = 19 \quad (>0)$$

3. For $x = 3$, SPP_3 is found to be 3 years because

$$\sum_{t=0}^{2} A_{t,3} = -75.3 + (2)(28) = -19.3 \quad (<0)$$

$$\sum_{t=0}^{3} A_{t,3} = -75.3 + (3)(28) = 8.7 \quad (>0)$$

4. For $x = 4$, SPP_4 is found to be 5 years because

$$\sum_{t=0}^{4} A_{t,4} = -(4)(28) = -112 \quad (<0)$$

$$\sum_{t=0}^{5} A_{t,4} = -(4)(28) + 201.9 = +89.9 \quad (>0)$$

Example 5.15

For the investment cash flow profile of $x = 2$ in Example 5.12, the IRR is known to be 25%. If the MARR is 6%, find the discounted payback period.

The unrecovered balances of the cash flow profile at the ends of periods $t = 1$ through $t = 5$ for the MARR of 6% are obtained by Eqs. (4.12) through (4.16) and recorded in Table 5.6. Since $R_2 = -14.237$ and $R_3 = +10.909$, the discounted payback period for this investment is $t = 3$.

TABLE 5.6 UNRECOVERED BALANCES OF INVESTMENT

t	A_t	I_t	Q_t	R_t
0	−77	0	0	−77
1	+38	−4.620	+33.380	−43.620
2	+32	−2.617	+29.383	−14.237
3	+26	−0.854	+25.146	+10.909
4	+20	+0.655	+20.655	+31.564
5	+14	+1.894	+15.894	+47.458

5.12 COMPARISON OF MERIT MEASURES

Different merit measures are introduced because they are often used to assess enhancements toward different objectives, all of which may be deemed important to a decision maker. With the availability of commercial computer software for computing these measures, there is good reason to determine as many of such measures as one desires. However, the decision maker must understand clearly how these measures are defined and that they should be used only for the restrictive objectives for which they are appropriate.

For the economic evaluation of independent projects without budget constraints, we are interested only in determining whether a proposed project is economically acceptable. The process of maximizing the profit is not involved. Consequently, the decision criteria based on all merit measures introduced in this chapter are straightforward as long as they are correctly applied. However, caution must be voiced that in comparison of mutually exclusive proposals, the maximization process is involved and the decision criteria are generally different from those presented here. This subject is discussed in detail in Chapter 6.

Example 5.16

Compare the merit measures for the four independent projects in Example 5.12, using a value of i = 0%, 8%, and 20% for the MARR.

The results of computation for NPV and ORR are computed on the basis of three values of the MARR, as tabulated in Table 5.7. The IRR results are independent of the MARR and indicate that all four projects are equal in spite of the differences in the cash flow profiles. The payback periods, both simple and discounted, provide no information about the sizes of the profit but can be applied as a measure of the liquidity of the cash flow profiles.

TABLE 5.7 COMPARISON OF MERIT MEASURES OF FOUR INDEPENDENT PROJECTS

Project		$x = 1$	$x = 2$	$x = 3$	$x = 4$
NPV	$i = 0\%$	+158.0	+53.0	+64.7	+89.9
($ million)	$i = 8\%$	+ 82.9	+30.5	+36.5	+44.7
	$i = 20\%$	+ 17.4	+ 7.2	+ 8.4	+ 8.6
ORR	$i = 0\%$	25.0%	11.1%	13.3%	12.5%
	$i = 8\%$	25.0%	15.5%	16.9%	16.9%
	$i = 20\%$	25.0%	22.3%	22.7%	22.8%
IRR		25.0%	25.0%	25.0%	25.0%
SPP (years)		5	3	3	5
DPP (years)	$i = 8\%$	5	3	4	5

5.13 SENSITIVITY ANALYSIS OF NET PRESENT VALUES

Since the net present value of an investment project as represented by Eq. (5.7) or Eq. (5.8) is the primary measure in determining the economic feasibility of the project, it is important to examine how sensitive the NPV may be within the ranges of its parameters which include i, t, $B_{x,t}$, and $C_{x,t}$. For the sake of simplification, we shall consider only the variation of one variable at a time.

We have already emphasized the sensitivity of the net present value with respect to MARR = i in Section 5.7. It can be seen from Eq. (5.8) that the net present value is also sensitive to the timing of future costs and benefits. Because of the nature of the discount amount factor $(P \mid F, i, t)$, costs incurred and benefits received far in the future (those with large t values) are less significant than those near the present. Furthermore, the larger the MARR (the value of i in the factor), the less significant will be the effects of costs and benefits in the future years.

The variation in the estimation of annual benefits $B_{x,t}$ and annual costs $C_{x,t}$ may also affect the net present value to the extent that the acceptability of an investment project may be in jeopardy. The underestimation of the costs or the overestimation of the benefits may result in the acceptance of a project which is not worthwhile, while the underestimation of the benefits or the overestimation of the costs may result in the rejection of an otherwise worthwhile project.

Example 5.17

A motel chain considers building a 300-unit motel in a new location. A planning horizon of 12 years and a MARR of 15% are used for evaluation. The initial cost of the building is \$3.6 million including the land, and the terminal value at the end of 12 years is estimated to be \$1.2 million including the land. The furnishings for the motel cost \$1.2 million and last 4 years with no salvage value, and replacements will be made at 4-year intervals at the same cost. The annual operating and maintenance costs are estimated to be \$600,000. The average rate per unit is anticipated to be \$20 per day, and the motel is expected to be open year-round including weekends and holidays. Evaluate the economic feasibility of this project on the basis of average daily occupancy rates of 60%, 75%, and 90%.

The present value of the cost stream (in millions of dollars) over the planning horizon of 12 years is given by

$$\begin{aligned} \text{CPV} &= 3.6 + 1.2[1 + (P \mid F, 15\%, 4) + (P \mid F, 15\%, 8)] \\ &\qquad - 1.2(P \mid F, 15\%, 12) + 0.6(P \mid U, 15\%, 12) \\ &= 8.907 \end{aligned}$$

The equivalent uniform annual cost is

$$\begin{aligned} \text{EUAC} &= (8.907)(U \mid P, 15\%, 12) \\ &= 1.643 \end{aligned}$$

The annual revenues or benefits based on full occupancy for 365 days are found to be

$$(300)(20)(365) = \$2{,}190{,}000$$

Hence, the net equivalent uniform annual values (in millions of dollars) corresponding to different occupancy rates are as follows:

$$60\% \text{ rate, EUAV} = (0.60)(2.190) - 1.643 = -0.329$$

$$75\% \text{ rate, EUAV} = (0.75)(2.190) - 1.643 = 0$$

$$90\% \text{ rate, EUAV} = (0.90)(2.190) - 1.643 = +0.328$$

Thus, the investment will break even at the 75% occupancy rate.

Example 5.18

A building that cost C_0 at $t = 0$ is expected to have a useful life of n years. The annual maintenance cost is estimated to be C_t and the annual rental income is B_t while the building is in use ($t = 1, 2, \ldots, n$). There is no salvage value at the end of the useful life. For MARR = i, the net present value of the proposal is

$$\text{NPV} = -C_0 + (B_t - C_t)(P \mid U, i, n)$$

For C_0 = \$8 million, find the break-even point for each of the following parameters under the given set conditions:

(a) B_t, given C_t = \$0.5 million, i = 10%, n = 40 years
(b) C_t, given B_t = \$1.5 million, i = 10%, n = 40 years
(c) i, given C_t = \$0.5 million, B_t = \$1.5 million, n = 40 years
(d) n, given C_t = \$0.5 million, B_t = \$1.5 million, i = 10%

The break-even point of each parameter may be found by setting NPV = 0. Thus, for B_t,

$$0 = -8 + (B_t - 0.5)(9.7791), \qquad B_t = \$1{,}318 \text{ million}$$

For C_t,

$$0 = -8 + (1.5 - C_t)(9.7791), \qquad C_t = \$0.682 \text{ million}$$

For i, using extrapolation from i = 11% and 12%,

$$0 = -8 + (1.5 - 0.5)(P \mid U, i, 40), \qquad i = 12.34\%$$

For n, using interpolation between n = 16 and 17,

$$0 = -8 + (1.5 - 0.5)(P \mid U, 10, n), \qquad n = 16.89 \text{ years}$$

Example 5.19

For the problem in Example 5.18 for which C_0 = \$8 million, find the variation of the NPV with respect to the variation in each of the following parameters:

(a) B_t (in \$ million) = 1.20, 1.35, 1.50, 1.65, 1.80
(b) C_t (in \$ million) = 0.40, 0.45, 0.50, 0.55, 0.60

(c) i (%) = 8, 9, 10, 11, 12
(d) n (years) = 30, 35, 40, 45, 50

Plot the results in the percentage variation of the NPV versus the percentage variation of each parameter to show the sensitivity of NPV due to variations of these parameters.

The results of computation are shown in Appendix D as an example of computing NPV using the spreadsheet. These results are plotted in a graph (Fig. 5.5) in which the following ranges of percentage variations are covered for these parameters:

(a) B_t (−20%, −10%, 0%, +10%, +20%)
(b) C_t (−20%, −10%, 0%, +10%, +20%)
(c) i (−20%, −10%, 0%, +10%, +20%)
(d) n (−25%, −12.5%, 0%, +12.5%, +25%)

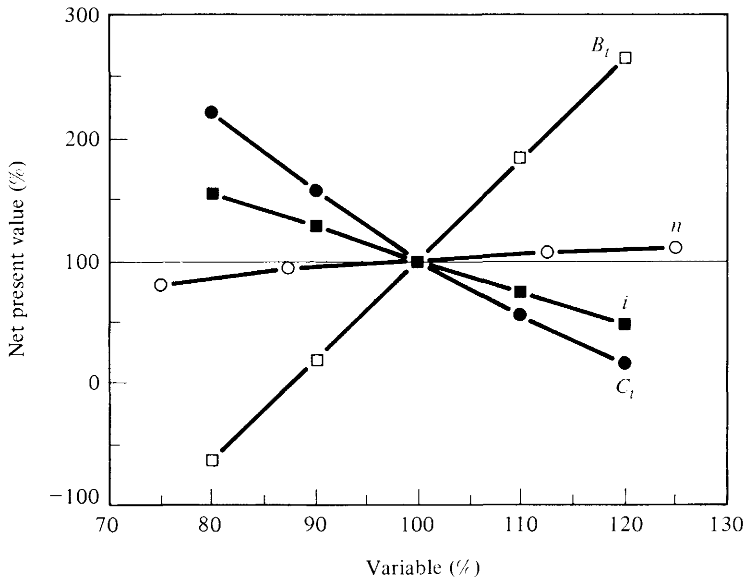

Figure 5.5 Sensitivity analysis of net present value.

5.14 SUMMARY AND STUDY GUIDE

This chapter has presented the basic principles of economic evaluation which are applicable to the evaluation of independent projects or mutually exclusive proposals, although the detail of application to these two categories of projects may be slightly different. In this chapter, we have examined some important concepts in determining whether the independent projects under consideration are economically feasible. It also serves as a prerequisite for the analysis of mutually exclusive proposals in the next chapter.

We have emphasized the importance of the minimum attractive rate of return specified by an organization in determining the profit potential of investment proposals. Since the MARR is so crucial in the computation of the net present value and yet so difficult to pinpoint exactly, it is advisable to compute the net present value of an investment proposal for several values of the MARR.

We have also examined the objectives of investment decisions and their influence on the choice of measures of merit of an investment proposal. Once the decision process in an organization is understood, it is easy to explain the popularity of some merit measures. The net present value criterion is presented as the basis for the most direct method of analysis. Supplementary or secondary measures, such as the overall rate of return, the internal rate of return, and the payback period, which may be desired by the decision makers at different levels of an organization, can be computed in a straightforward manner.

The comparison of the merit measures for four independent projects with substantially different cash flow profiles is shown in Table 5.7. This table gives a summary of the agreements and disagreements of these measures in the evaluation of these projects.

Since the net present value of an investment project is the primary measure in determining the economic feasibility of the project, it is important to examine its sensitivity with respect to the MARR, the timing of future costs and benefits themselves. The NPV versus i graph and other considerations have been suggested for the sensitivity analysis.

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PROBLEMS

P5.1 A company is making a capital investment plan for the next 6 years and a project with a useful life of 6 years is under consideration. The cash flow profile of a proposed project requires an initial cost of $10,000 and a series of uniform annual net benefit of $3,000 over the 6-year period. Determine the net present value, the net future value at the end of 6 years, and the net equivalent uniform annual value of this project if the specified MARR is 12%.

P5.2 A public agency is planning a capital budget for the next 6 years. A proposed project of 4-year duration has the following net annual cash flows: $A_0 = -\$10,000$, $A_1 = A_2 = A_3 = +\$6,000$, and $A_4 = +\$8,000$. Using a MARR of 12%, determine the net equivalent uniform annual value over the 6-year period for each of the following conditions:

(a) There is no replacement when the project expires.

(b) There is a second project of 2-year duration with the following net cash flows: $A_4 = -\$4,000$, $A_5 = A_6 = +\$3,000$.

P5.3 The net cash flow profile of an investment project of 5-year duration is as follows: $A_0 = -\$1,000$, $A_1 = +\$400$, $A_2 = \$600$, $A_4 = -\$200$, and $A_5 = +\$800$. Using a MARR of 8%, determine the net equivalent uniform annual value under each of the following conditions:

(a) Over the 5-year duration of this project

(b) Over a planning horizon of eight years with no replacement when the project expires

P5.4 An investment of $90,000 in automatic equipment at present is expected to save $15,000 per year in the next 5 years and $10,000 per year from the sixth through the tenth year. Using a MARR of 8%, determine whether this investment is worthwhile on the basis of the net equivalent uniform annual value over the 10-year period.

P5.5 The initial cost of a machine is $25,000 and the machine is expected to last for 12 years. The annual maintenance cost is $1,200 per year in the first five years, and goes up according to a linear gradient by an amount of $100 from the sixth year to the twelfth year, i.e., $1,300 for year 6, $1,400 for year 7, etc. The salvage value of the machine after 12 years is estimated to be $3,000. This machine will generate a uniform series of benefits of $5,000 per year for 12 years, which would be received at the end of each year. If the MARR is 8%, determine whether this investment is worthwhile on the basis of the net equivalent uniform annual value over the 12-year period.

P5.6 A machine that costs $15,000 now is expected to generate a uniform annual profit of $2,000 in the first five years, and a linear gradient of $100 less for each of the subsequent five years, i.e., $1,900 for year 6, $1,800 for year 7, etc. until $1,500 for year 10. There is no salvage value when the machine is disposed of at the end of 10 years. If the MARR is 8%, determine whether the project is acceptable on the basis of the net equivalent uniform annual value over the 10-year period.

P5.7 A precision inspection instrument is purchased for $80,000. The operating and maintenance costs are $5,000 for the first year, and increase by $500 from the previous year each year thereafter. The revenues produced by the machine are estimated to be $15,000 in the first year and increase by $1,500 from the previous year each year thereafter. The machine is expected to have a useful life of 10 years with a salvage value of $10,000 at the end of 10 years. Using a MARR of 12%, determine whether the purchase is worthwhile.

P5.8 A firm is considering two independent investment proposals, each requiring an initial expenditure of $10,000 but offering different benefits in the next 5 years. Proposal 1 yields $2,000 per year in the first 2 years and $3,000 per year in the last 3 years. Proposal 2 yields $2,700 in each of the 5 years. Using a MARR of 10%, should any of these proposals be accepted?

P5.9 For the four independent projects in Table 5.7 of this chapter, find the net present values of their cash flow profiles, using a MARR of 8%.

P5.10 An investment project of 6-year duration consists of the following net cash flows (in $ million): $A_0 = -7.2, A_1 = -5.5, A_2 = 0, A_3 = -3.5, A_4 = +6.5, A_5 = +14.7$, and $A_6 = +3.4$. The MARR is 9%. Determine whether the project is worthwhile on the basis of the overall rate of return.

P5.11 For the strip-mining operation in Example 5.9, find the year-by-year unrecovered balance of the investment if (a) MARR = 5% and (b) MARR = 15%.

P5.12 A strip-mining operation requires an initial cost of $340,000, which will yield an annual net income of $150,000 for the next eight years. At the end of the eighth year, the cost of restoring the landscape for environmental protection is $860,000, resulting in a net loss of $710,000 after deducting the net income from operation. Compute the NPV for MARR = 0%, 5%, 10%, 15%, 20%, 25%, and 30%, and plot the NPV graph. Is this operation acceptable at MARR = 10%?

P5.13 The annual benefits and costs of a proposed investment project of six-year duration (in thousand dollars) are

t	0	1	2	3	4	5	6
B_t	0	60.0	60.0	90.0	60.0	60.0	40.0
C_t	30.0	47.3	78.2	76.4	77.3	25.5	18.2

Verify that $i = 9\%$ is the unique value of IRR for this project. If the MARR is 8%, is the project acceptable on the basis of the IRR criterion?

P5.14 For the annual benefits and costs in Problem P5.13, find the net present value using (a) MARR = 8%, (b) MARR = 9%, and (c) MARR = 10%.

P5.15 For the building project in Example 5.19, the overall rate of return can be expressed as follows:

$$i^0 = [(B_t - C_t)(F \mid U, i, n)/C_0]^{1/n} - 1$$

or using interpolation, i^0 can be obtained from

$$(F \mid P, i^0, n) = (B_t - C_t)(F \mid U, i, n)/C_0$$

Find the variation of the overall rate of return with respect to the variation of each of the parameters listed for the problem. Also plot the results in the percentage variation of i^0 versus the percentage variation of each parameter to show the sensitivity of the overall rate of return.

P5.16 The installation of a new pipeline for transporting crude oil between two locations requires an initial cost of $10 million. The useful life of the pipeline is 40 years with no salvage value at the end of its useful life. The net revenues generated by the pipeline per year depend on volume of crude oil transported. Assume that the expectation is to transport 150 million barrels per year in the first 5 years, 200 million barrels per year in the following 25 years, and 100 million barrels per year in the last 10 years. The net revenue for transporting each million barrels through the pipeline is $6,000. The MARR is specified to be 10%. Determine whether the pipeline project is worthwhile if the transport volume is (a) 100% as expected, (b) 90% of the expectation, and (c) 110% of the expectation.

P5.17 A taxicab company plans to replace a retired cab with a new vehicle which is expected to be driven 30,000 miles per year and to have a useful life of 3 years. The new vehicle costs $9,000 with a salvage value of $1,000 at the end of 3 years. The annual operating and maintenance cost of the vehicle is $20,000, excluding the fuel cost. The fuel consumption of the vehicle averages 15 miles per gallon and the fuel cost averages $1.40 per gallon. Using a MARR of 15%, determine the annual revenue from operating the vehicle in order to break even if average fuel cost is (a) $1.40 per gallon as estimated, (b) 20% higher, and (c) 50% higher.

CHAPTER 6

Comparison of Mutually Exclusive Proposals

6.1 THE NATURE OF MUTUALLY EXCLUSIVE PROPOSALS

In the search for capital investment opportunities, we often encounter potential projects that are not independent of each other. If two or more projects are complementary to each other, then they should be grouped together as a single alternative so that they will be accepted or rejected as a group. On the other hand, if two or more projects have a substituting effect on each other, they should be set apart or in different groups so that the acceptance of one will lead to the rejection of the others. Furthermore, several independent projects may also be combined as a single proposal for comparison with other proposals. Thus, we can generate a number of feasible mutually exclusive proposals through the proper grouping of potential investment projects.

In this chapter, we consider only mutually exclusive proposals that are size-dependent and indivisible. The objective is to select the best among these proposals in order to maximize the profit of the investment under the conditions of a perfect capital market without budget constraints. The systematic approach for evaluating independent projects enunciated in Chapter 5 is generally applicable except for the procedure of selecting the best among the competing proposals that are mutually exclusive.

The mutually exclusive proposals are often conceived as possible variations for one or more of the following factors:

1. The size of a facility on a specific site
2. The production capacity of a facility in meeting market demands

3. The life of a facility with respect to the planning horizon
4. The timing of expanding or replacing a facility

For example, given a specific site for the construction of an office building, we may wish to compare several designs of different heights or different base dimensions. Similarly, in the planning of an oil refinery facility, we may wish to study several process designs that are most suitable for various production capacities. In acquiring a major piece of equipment for manufacturing, we want to know how it will fit in the entire production system and what replacement should be sought if it becomes obsolete sooner than the rest of the system. We often want to know also whether to build a large facility in anticipation of future expansion or to begin with a smaller facility and increase its size at some later time. All these considerations require the comparison of a number of well-developed plans and designs and the selection of only one proposal for eventual execution.

It is understood that the best project among a set of mutually exclusive proposals must be superior or equal to the null alternative to be acceptable. Furthermore, if the alternative of maintaining the status quo or abandoning an existing facility is different from the null alternative, it should be considered along with other mutually exclusive proposals. If this is not done, there is the inherent danger of either accepting a new proposal when the status quo alternative is in fact better, or retaining an existing facility which is no longer economically feasible.

6.2 OBJECTIVES AND MERIT MEASURES

Since the objective of investing in a project is profit maximization, the net present value (NPV) is a direct measure of merit. Similarly, the net future value (NFV) and net equivalent uniform annual value (EUAV) are also direct measures of merit. Decision criteria based on direct merit measures for selecting the best project among a group of mutually exclusive proposals are straightforward. Since the NPV can be most easily and unambiguously computed, the NPV criterion is preferred as the method of analysis.

Indirect measures of merit, such as the internal rate of return (IRR), the overall rate of return (ORR), the simple payback period (SPP), and the discount payment period (DPP), are useful for different purposes. However, they do not necessarily produce proper ranking of the merits of mutually exclusive proposals for the objective of profit maximization without budget constraints.

Example 6.1

The cash flow profiles of three mutually exclusive alternatives (values in dollars) are given below:

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−1,000	−2,000	−3,000
1	+1,300	+2,500	+3,600

Compute the net present values of these alternatives at 5% intervals from $i = 0\%$ to $i = 30\%$. Plot a NPV versus i graph for this range of MARR.

The net present values of these alternatives with 1-year duration can easily be computed.

$$NPV_1 = -1{,}000 + (1{,}300)(P \mid F, i, 1)$$

$$NPV_2 = -2{,}000 + (2{,}500)(P \mid F, i, 1)$$

$$NPV_3 = -3{,}000 + (3{,}600)(P \mid F, i, 1)$$

The values based on $i = 0\%$ to $i = 30\%$ at 5% intervals have been computed and tabulated in Table 6.1. The results have also been plotted in the NPV versus i graph in Fig. 6.1.

TABLE 6.1 NET PRESENT VALUES (IN $1,000) FOR EXAMPLE 6.1

i	NPV_1	NPV_2	NPV_3
0%	300	500	600
5%	238	381	429
10%	182	273	273
15%	131	174	131
20%	83	83	0
25%	40	0	−120
30%	0	−77	−231

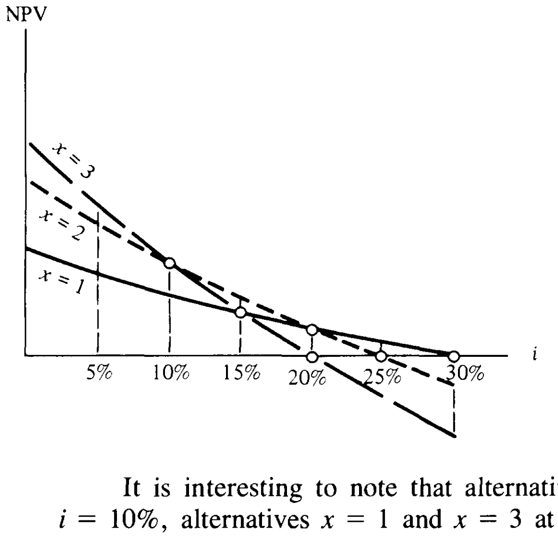

Figure 6.1 NPV versus i graph for Example 6.1.

It is interesting to note that alternatives $x = 2$ and $x = 3$ intersect at $i = 10\%$, alternatives $x = 1$ and $x = 3$ at $i = 15\%$, and alternatives $x = 1$ and $x = 2$ at $i = 20\%$, as indicated by the net present values in Table 6.1. Consequently, the ranks of the alternatives at various intervals of MARR from 0% to 30% are observed from Fig. 6.1 as follows:

Range of i	Best	Second	Last
0%–10%	$x = 3$	$x = 2$	$x = 1$
10%–15%	$x = 2$	$x = 3$	$x = 1$
15%–20%	$x = 2$	$x = 1$	$x = 3$
20%–25%	$x = 1$	$x = 2$	Negative
25%–30%	$x = 1$	Negative	Negative

6.3 DECISION CRITERIA BASED ON DIRECT MERIT MEASURES

The net present value method offers a direct approach for comparing the profit potentials of mutually exclusive proposals. The net present value method for evaluating the economic feasibility of an independent project has been explained in detail in Section 5.6. The same principle may be applied to the evaluation of the economic feasibility of each of a set of mutually exclusive alternatives. In addition, we must rank the merits of all mutually exclusive proposals and select only the one that has the maximum nonnegative net present value.

Let NPV_x be the net present value of the project x for MARR $= i$ over a planning horizon of n years. It is implicit that for the null alternative ($x = 0$), $\text{NPV}_0 = 0$. Then, for a set of m mutually exclusive proposals ($x = 1, 2, \ldots, m$), the criterion for selecting a particular proposal j is that it has the maximum net present value among those projects which are economically feasible. That is,

$$\text{NPV}_j = \max_{x \in m} \text{NPV}_x \tag{6.1}$$

provided that

$$\text{NPV}_j \geq 0$$

Since all mutually exclusive alternatives for comparison must be evaluated at a specified MARR $= i$ over the same planning horizon of n years, we can also compare the net future value or the net equivalent uniform annual value of these alternatives. Thus, the criterion for accepting a particular project j on the basis of the net future values at the end of n years is

$$\text{NFV}_j = \max_{x \in m} \text{NFV}_x \qquad (\text{NFV}_j \geq 0) \tag{6.2}$$

Similarly, the criterion based on the net equivalent uniform annual value is

$$\text{EUAV}_j = \max_{x \in m} \text{EUAV}_x \qquad (\text{EUAV}_j \geq 0) \tag{6.3}$$

The direct approaches using Eq. (6.1), (6.2), or (6.3) are straightforward, and the results thus obtained are unambiguous regardless of the number of possible values for the internal rate of return. To avoid repetitions, further discussion of the direct approaches will be devoted primarily to the net present value method, although the same principles may also be applied to the net future value or the net equivalent uniform annual value.

Since the minimum attractive rate of return is crucially important in the computation of the net present value, it is advisable to compute NPV_x of all competing proposals ($x = 1, 2, \ldots, m$) for several values of MARR. By plotting an NPV versus i graph, the sensitivity of the ranking of the alternatives based on the NPV can be observed. It is important to note that even for proposals with conventional cash flow profiles, the ranking of the alternatives may change with the change of MARR in different ranges of the discount rate i. Such a change is indicated in Fig. 6.1 for the simple cash flow profiles in Example 6.1. Consequently, the intersection between each pair of the NPV curves, which signifies the *incremental* internal rate of return, as well as the intersections of the NPV curves with the horizontal axis, which denote the internal rates of return, are important for the sensitivity analysis of NPV. For unconventional cash flow profiles, the intersection between each pair of the NPV curves becomes even more important because of possible drastic variations in the NPV for different values of the MARR. The solution for such intersections is discussed in detail in Section 6.5.

Example 6.2

Three plans have been proposed for a small commercial building at a leased site. The building is expected to have a useful life of 40 years and no salvage value at the end of the period. The initial construction costs, the annual maintenance costs, and the annual benefits of the three mutually exclusive plans are shown in Table 6.2. If MARR is specified to be 6%, rank these alternatives on the basis of their net present values. Repeat the computation for MARR = 4% and 8%.

TABLE 6.2 COST AND BENEFIT STREAMS (IN $1,000) FOR EXAMPLE 6.2

	$x = 1$		$x = 2$		$x = 3$	
t	$C_{t,1}$	$B_{t,1}$	$C_{t,2}$	$B_{t,2}$	$C_{t,3}$	$B_{t,3}$
0	200	0	250	0	300	0
1–40 (each)	40	61	30	52	55	77

For each alternative x, the uniform annual costs $C_{t,x}$ and the uniform annual benefits $B_{t,x}$ (for $t = 1, 2, \ldots, 40$) may be represented by $\overline{C}_x$ and $\overline{B}_x$, respectively. Then, the net present value of alternative x is given by

$$\text{NPV}_x = -C_{0,x} + (\overline{B}_x - \overline{C}_x)(P \mid U, i, 40)$$

The results of the computation for i = 4%, 6%, and 8% are listed in Table 6.3. A NPV versus i graph for the range of MARR from 0% to 10% is also shown in Fig. 6.2. For all three values of MARR, $x = 1$ is the best, $x = 2$ is the second, and $x = 3$ is the last.

It is interesting to note that for this particular example, the NPV curves do not intersect with each other in the positive range of NPV. Consequently, only the intersections of these curves with the horizontal axis that represent the IRR of these alternatives are important.

TABLE 6.3 NET PRESENT VALUES (IN $1,000) FOR EXAMPLE 6.2

i	NPV_1	NPV_2	NPV_3
4%	215.649	185.442	135.442
6%	115.972	81.019	31.019
8%	50.417	12.341	−37.659

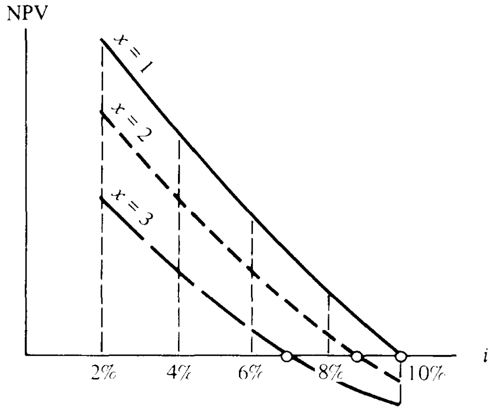

Figure 6.2 NPV versus i graph for Example 6.2.

Example 6.3

The cash flow profiles in Table 6.4 represent the expected annual net costs and benefits of three mutually exclusive alternatives in the selection of a drilling machine. The first alternative ($x = 1$) is the purchase of a new machine that is expected to last 6 years. The second alternative ($x = 2$) is a new machine advertised for lease purchase over a 6-year period with a three-tier payment schedule. The third alternative ($x = 3$) is to continue the use of the existing machine which requires a major overhaul now as well as additional maintenance in the next few years. The annual benefits and costs of these alternatives have been netted to obtain the cash flow profiles as shown. The MARR is specified to be 12%. Determine which of the three alternatives should be se-

TABLE 6.4 CASH FLOW PROFILES FOR THE DRILLING MACHINES

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−10,000	−10,000	−1,500
1–2 (each)	3,000	5,500	1,000
3–5 (each)	3,000	500	1,000
6	3,000	5,500	1,000

lected, using (1) the net present value criterion, (2) the net future value criterion, and (3) the net equivalent uniform annual value criterion.

Using the net present value criterion, we can compare NPV_x for $x = 1, 2, 3$:

$$[NPV_1]_{12\%} = -10{,}000 + (3{,}000)(P \mid U, 12\%, 6)$$
$$= -10{,}000 + (3{,}000)(4.1114) = 2{,}334$$

$$[NPV_2]_{12\%} = -10{,}000 + (5{,}500)(P \mid U, 12\%, 6) - (5{,}000)(P \mid U, 12\%, 3)(P \mid F, 12\%, 2)$$
$$= -10{,}000 + (5{,}500)(4.1114) - (5{,}000)(2.4018)(0.7972) = 3{,}039$$

$$[NPV_3]_{12\%} = -1{,}500 + (1{,}000)(P \mid U, 12\%, 6)$$
$$= -1{,}500 + (1{,}000)(4.1114) = 2{,}611$$

Since all net present values are positive and NPV_2 is the largest, the second alternative should be selected.

Using the net future value criterion, we can compare NFV_x for $x = 1, 2, 3$:

$$[NFV_1]_{12\%} = -(10{,}000)(F \mid P, 12\%, 6) + (3{,}000)(F \mid U, 12\%, 6)$$
$$= -(10{,}000)(1.9738) + (3{,}000)(8.1152) = 4{,}608$$

$$[NFV_2]_{12\%} = -(10{,}000)(F \mid P, 12\%, 6) + (5{,}500)(F \mid U, 12\%, 6) - (5{,}000)(F \mid U, 12\%, 3)(F \mid P, 12\%, 1)$$
$$= -(10{,}000)(1.9738) + (5{,}500)(8.1152) - (5{,}000)(3.3744)(1.12) = 5{,}999$$

$$[NFV_3]_{12\%} = -(1{,}500)(F \mid P, 12\%, 6) + (1{,}000)(F \mid U, 12\%, 6)$$
$$= -(1{,}500)(1.9738) + (1{,}000)(8.1152) = 5{,}155$$

Since NFV_2 is largest, the second alternative should be selected.

Similarly, using the net equivalent uniform annual value, we can compute $EUAV_x$ for $x = 1, 2, 3$ from the corresponding net present values NPV_x as follows:

$$[EUAV_1]_{12\%} = (2{,}334)(U \mid P, 12\%, 6) = (2{,}334)(0.2432) = 568$$
$$[EUAV_2]_{12\%} = (3{,}039)(U \mid P, 12\%, 6) = (3{,}039)(0.2432) = 739$$
$$[EUAV_3]_{12\%} = (2{,}611)(U \mid P, 12\%, 6) = (2{,}611)(0.2432) = 635$$

Obviously, the second alternative should be selected.

6.4 DECISION CRITERIA BASED ON INDIRECT MERIT MEASURES

The net present value not only is a direct merit measure of an individual proposal, but also is a valid basis for ranking the profits from a set of mutually exclusive proposals. Therefore, the net present value criterion is the most straightforward and reliable method in selecting the best among a set of mutually exclusive proposals on the basis of profit maximization without capital rationing. Alternatively, a decision criterion based on the net future values or the net equivalent uniform annual values of mutually exclusive proposals will have the same effects.

Other merit measures that have been used for analyzing the economic feasibility or acceptability of independent projects are not *directly* applicable to the selection of the best among mutually exclusive proposals. It is important to recognize this fact and not to misuse the indirect merit measures when they are not applicable without modifications. Specifically, the internal rate of return is an indirect merit measure of an individual proposal which is independent of the MARR, but it is *not* a valid basis for profit maximization.

Example 6.4

The cash flow profiles for two mutually exclusive proposals $x = 1$ and $x = 2$ are given in Table 6.5. Using MARR = 10%, find the best proposal.

TABLE 6.5 CASH FLOWS AND INCREMENTAL CASH FLOWS (IN DOLLARS)

t	$A_{t,1}$	$A_{t,2}$
0	−1,000	−1,500
1	+ 750	+ 900
2	+ 750	+ 700
3	+ 750	+1,300

The net present values of the two proposals are computed as follows:

$$\text{NPV}_1 = -1{,}000 + (750)(P \mid U, 10\%, 3) = 865.2$$

$$\text{NPV}_2 = -1{,}500 + (900)(P \mid F, 10\%, 1) + (700)(P \mid F, 10\%, 2) + (1{,}300)(P \mid F, 10\%, 3)$$

$$= 873.5$$

On the basis of the NPV criterion, $x = 2$ is the best proposal and should be selected.

It should be noted that IRR = 54.7% for $x = 1$, while IRR = 38.7% for $x = 2$. Clearly, these merit measures cannot be used to rank the proposals. However, if the incremental analysis based on the IRR is used, a correct conclusion can also be reached. We shall consider such a method of analysis in Chapter 8.

Example 6.5

Find the overall rates of return for the three mutually exclusive proposals in Example 6.3. The MARR is specified to be 6%.

For this problem, the transition point for each of the proposals is $t = 0$. The corresponding investment bases and resultant returns are obtained from Eqs. (4.21) and (4.22) as follows:

$$\text{For } x = 1, \quad P_e = -10{,}000, \quad F_e = 24{,}346$$

$$\text{For } x = 2, \quad P_e = -10{,}000, \quad F_e = 25{,}737$$

$$\text{For } x = 3, \quad P_e = -1{,}500, \quad F_e = 8{,}115$$

Hence, the ORR for these proposals are respectively computed by Eq. (4.23) as follows:

$$-10{,}000(1 + i_1^0)^6 + 24{,}346 = 0, \quad i_1^0 = 15.99\%$$

$$-10{,}000(1 + i_2^0)^6 + 25{,}737 = 0, \quad i_2^0 = 17.06\%$$

$$-1{,}500(1 + i_3^0)^6 + 8{,}115 = 0, \quad i_e^0 = 32.50\%$$

Note that $x = 3$ has the highest ORR. However, $x = 2$ is the best proposal because it has the highest NPV, as shown in Example 6.3. The difference in ranking of proposals on the basis of the NPV and of the ORR is attributable to the fact that the former is size dependent while the latter is size independent.

It is often stated that the NPV decision criterion favors large projects because it is based on profit maximization under a perfect capital market. This bias need not occur if the imperfection of the capital market is taken into consideration. For example, if a budget constraint of $2,000 is imposed for the outlay at $t = 0$, then $x = 1$ and $x = 2$ will be automatically eliminated and $x = 3$ becomes the best choice. The detailed discussion of capital rationing is deferred until Chapter 9.

6.5 RELATIONSHIP BETWEEN INCREMENTAL NET PRESENT VALUE AND INCREMENTAL INTERNAL RATE OF RETURN

The underlying principle of economic evaluation is to maximize the profit potential as measured by the discounted net benefit, such as the net present value. When the net present value criterion is used for selecting the best proposal from a set of mutually exclusive alternatives, all that is necessary is to compute the net present values of all alternatives at the specified MARR and select the one that has the maximum nonnegative value.

However, in evaluating the sensitivity of the net present values of mutually exclusive alternatives, we want to know the conditions under which the curves representing different alternatives will or will not intersect with each other; and if they do intersect, we want to find the points of intersection. We can address these issues by examining systematically each pair of alternatives in a set of mutually exclusive alternatives.

Let projects x and y be a pair of mutually exclusive alternatives, and $A_{t,x}$ and $A_{t,y}$ be their respective cash flow profiles for $t = 0, 1, 2, \ldots, n$. Also, let the incremental cash flow profile of x over y be

$$\Delta A_{t,x-y} = A_{t,x} - A_{t,y} \tag{6.4}$$

and the incremental net present value (INPV) of x over y be

$$\Delta \text{NPV}_{x-y} = \text{NPV}_x - \text{NPV}_y \tag{6.5}$$

In general, $A_{t,x}$ and $A_{t,y}$ are either given directly or netted from the annual benefits and costs according to Eq. (5.2). Then, from Eq. (5.3),

$$\text{NPV}_x = \sum_{t=0}^{n} A_{t,x}(1 + i)^{-t}$$

and

$$\text{NPV}_y = \sum_{t=0}^{n} A_{t,y}(1 + i)^{-t}$$

Hence,

$$\Delta \text{NPV}_{x-y} = \sum_{t=0}^{n} \Delta A_{t,x-y}(1 + i)^{-t} \tag{6.6}$$

The internal rate of return i_{x-y} resulting from the incremental cash flow profile x over y is the value of i which sets $\Delta\text{NPV}_{x-y} = 0$. This value $i = i_{x-y}$, which is referred to as the *incremental internal rate of return* (IIRR) of x over y, is obtained by solving for i in the algebraic equation

$$\sum_{t=0}^{n} \Delta A_{t,x-y}(1 + i)^{-t} = 0 \tag{6.7}$$

Since the left side of Eq. (6.7) is a polynomial of i to the nth power, its solution may be obtained in accordance with the discussion in Section 4.3 of Chapter 4. The value(s) of i_{x-y} thus obtained will be the intersection(s) of the curves representing NPV_x and NPV_y as functions of i.

Let us consider first the simple cases in which there is only one sign change in the incremental cash flow profile $A_{t,x-y}$ for $t = 0, 1, 2, \ldots, n$. Then, there exists only one positive real root for Eq. (6.7). It is convenient to order x and y in a NPV-MARR graph with projects x and y such that

$$[\text{NPV}_x]_{0\%} > [\text{NPV}_y]_{0\%}$$

Consequently, in a graph showing the relationship between INPV and IIRR, we have

$$[\Delta\text{NPV}_{x-y}]_{0\%} > 0$$

Two typical examples are shown in Figs. 6.3 and 6.4 in which the value of i_{x-y} corresponding to the intersection of the curves representing NPV_x and NPV_y in part (a) of each figure is identical to the value of i_{x-y} in part (b) showing the relationship between ΔNPV_{x-y} and MARR. Since MARR $= i_{x-y}$ for $\Delta NPV_{x-y} = 0$, the value of i_{x-y} can be obtained from Eq. (6.7). If $[NPV_x]_{0\%} = [NPV_y]_{0\%}$, indicating $[\Delta NPV_{x-y}]_{0\%} = 0$, then $i_{x-y} = 0\%$ is the root of i in Eq. (6.7).

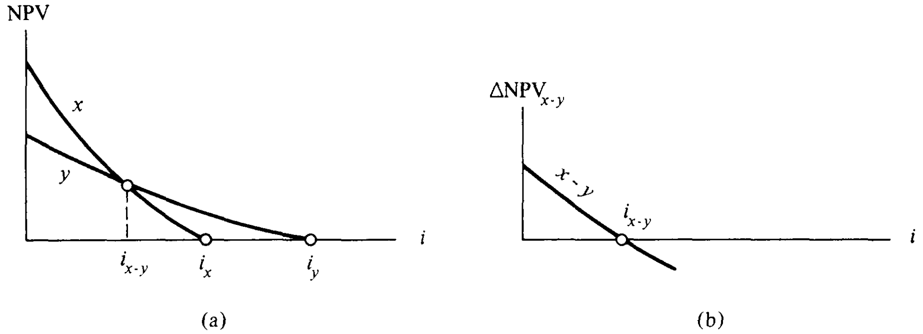

Figure 6.3 Incremental IRR in positive region of NPV.

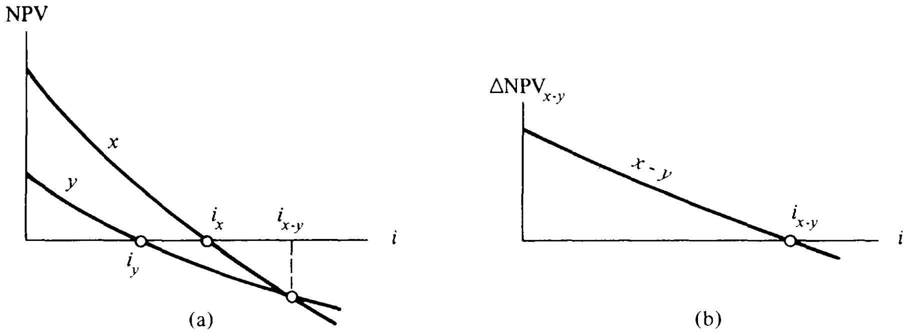

Figure 6.4 Incremental IRR in negative region of NPV.

Since the incremental cash flow profile $\Delta A_{t,x-y}$ is obtained from the difference of cash flow profiles $A_{t,x}$ and $A_{t,y}$, it is quite possible that there is no sign change in $\Delta A_{t,x-y}$ even if both $A_{t,x}$ and $A_{t,y}$ have only one sign change. Two typical examples are shown in Figs. 6.5 and 6.6, in which the curves representing NPV_x and NPV_y do not intersect in the half-plane of positive i in the NPV-MARR graph and therefore no positive i_{x-y} exists.

In general, either the cash flow profile of x or that of y or both may have two or more sign changes. For such cases, the incremental cash flow profile $\Delta A_{t,x-y}$ may lead to multiple roots of i, indicating multiple intersections between the curves rep-

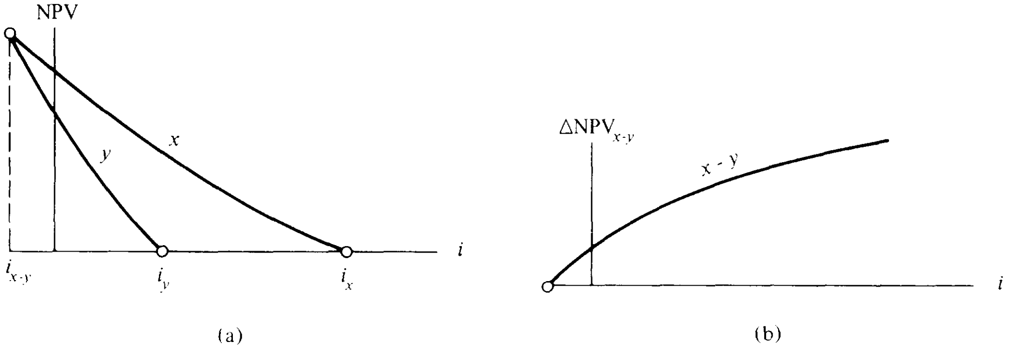

Figure 6.5 No positive incremental IRR.

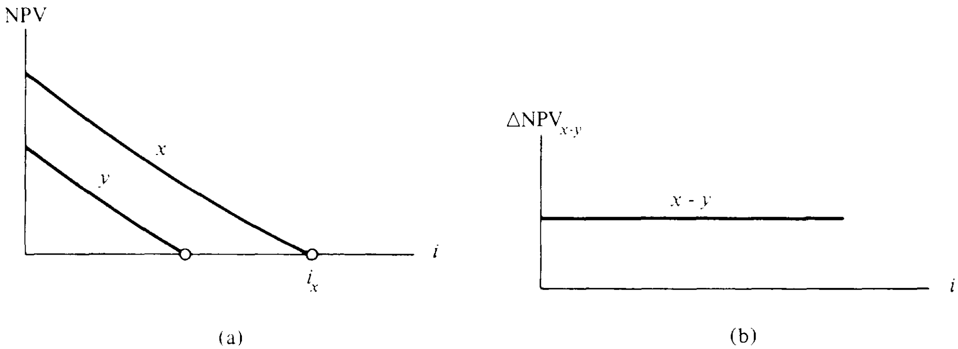

Figure 6.6 Non-existence of incremental IRR.

resenting NPV_x and NPV_y. When that happens, the relationship between INPV and IIRR should be examined in locating the multiple values of i_{x-y}. More detailed discussion of the solution for all possible cases is given in Chapter 8.

Example 6.6

For the three mutually exclusive alternatives in Example 6.2, verify that the curves representing the net present values of any two of these alternatives will not intersect in the half-plane of positive i in the NPV versus i graph.

The cash flow profiles of these mutually exclusive alternatives are first netted out and the results of $A_{t,x}$ for $x = 1$, 2, and 3 are shown in Table 6.6. For each of the alternatives, we can compute

$$[NPV_1]_{0\%} = -200 + (21)(40) = 640$$
$$[NPV_2]_{0\%} = -250 + (22)(40) = 630$$
$$[NPV_3]_{0\%} = -300 + (22)(40) = 580$$

In finding the incremental net present value for each pair of alternatives, we order the pair so that $[\Delta NPV_{x-y}]_{0\%} > 0$. Hence, the incremental cash flow profiles of $x = 2$ over $x = 3$, that of $x = 1$ over $x = 2$, and that of $x = 1$ over $x = 3$ are computed by Eq. (6.4), and the corresponding incremental net present values are computed by Eq. (6.5). These results for $i = 0\%$ are also shown in Table 6.6.

TABLE 6.6 CASH FLOWS AND INCREMENTAL CASH FLOWS FOR EXAMPLE 6.2

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$	$\Delta A_{t,2-3}$	$\Delta A_{t,1-2}$	$\Delta A_{t,1-3}$
0	−200	−250	−300	+50	+50	+100
1–40 (each)	+ 21	+22	+22	0	−1	−1
$[\text{NPV}]_{0\%}$	+640	+630	+580	+50	+10	+60

To find IIRR = i_{x-y}, we set $\Delta\text{NPV}_{x-y} = 0$ for each case. Thus, for $\Delta\text{NPV}_{2-3} = 0$,

$$+50 + (0)(P \mid U, i, 40) = 0 \qquad (P \mid U, i, 40) = -\infty$$

This result indicates that i_{x-y} does not exist. For $\Delta\text{NPV}_{1-2} = 0$,

$$+50 - (1)(P \mid U, i, 40) = 0 \qquad (P \mid U, i, 40) = 50$$

and for $\Delta\text{NPV}_{1-3} = 0$,

$$+100 - (1)(P \mid U, i, 40) = 0 \qquad (P \mid U, i, 40) = 100$$

In both of these cases, the result indicates that i_{x-y} does not exist in the half-plane of positive i of the NPV versus i graph since $(P \mid U, 0\%, 40) = 40$. Therefore, we can conclude that the NPV curves representing the three mutually exclusive alternatives will not intersect in the half-plane of positive i of the NPV versus i graph without actually plotting such a graph.

Example 6.7

The cash flow profiles of two mutually exclusive alternatives are given in Table 6.7. The incremental cash flow profile of $x = 2$ over $x = 1$ is also shown in the table. Find the points of intersection of the curves representing NPV_1 and NPV_2 in the NPV versus i graph.

The cash flow profile of each of these two alternatives has two sign changes and may have up to two roots of i. For $x = 1$, the values of IRR are found to be 11.3% and 40.16% (see Example 4.9 in Chapter 4). Similarly for

TABLE 6.7 CASH FLOWS AND INCREMENTAL CASH FLOWS (IN $ MILLION)

t	$A_{t,1}$	$A_{t,2}$	$\Delta A_{t,2-1}$
0	+18	+57.1	+39.1
1	+10	−18	−28
2	−40	−68	−28
3	−60	−88	−28
4	+30	+2	−28
5	+50	+130	+80
$[\text{NPV}]_{0\%}$	+8	+15.1	+7.1

$x = 2$, the values of IRR are found to be 10.1% and 33.9%. To find the incremental internal rate of return IIRR = i_{2-1}, we set

$$\Delta NPV_{2-1} = 39.1 - (28)(P \mid U, i, 4) + (80)(P \mid F, i, 5) = 0$$

By trial solution, we get

i	$(P \mid U, i, 4)$	$(P \mid F, i, 5)$	ΔNPV_{2-1}
10%	3.1698	0.6209	−0.018
20%	2.5887	0.4019	−1.232
30%	2.1662	0.2693	+0.010
40%	1.8492	0.1859	+2.194

The value of ΔNPV_{2-1} is very close to zero at i = 10% and i = 30%. Hence, the curves representing NPV_1 and NPV_2 intersect at these values as shown in Fig. 6.7.

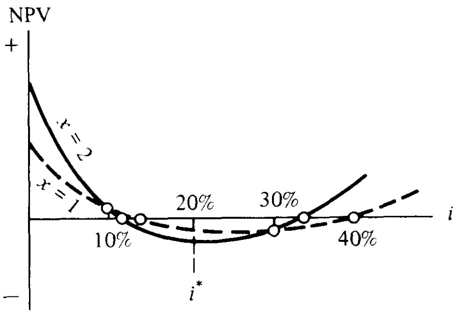

Figure 6.7 Multiple incremental IRR for Example 6.7.

Example 6.8

The cash flow profiles of two mutually exclusive alternatives are given in Table 6.8. The incremental cash flow profile of $x = 2$ over $x = 1$ is also shown in the table. Find the points of intersection of the curves representing NPV_1 and NPV_2 in the NPV versus i graph.

TABLE 6.8 CASH FLOWS AND INCREMENTAL CASH FLOWS (IN $1,000)

t	$A_{t,1}$	$A_{t,2}$	$\Delta A_{t,2-1}$
0	−38.3	−79.9	−41.6
1	+28	+28	0
2	+28	+28	0
3	+28	+28	0
4	+28	+28	0
5	−80	0	+80
$[NPV]_{0\%}$	−6.3	+32.1	+38.4

The cash flow profile of $x = 1$ has two sign changes and the values of IRR for $x = 1$ are found to be 8% and 33.6%. The cash flow profile of $x = 2$ has only one sign change, and the value of IRR for $x = 2$ is found to be 15.0%. To find the incremental internal rate of return IIRR $= i_{2-1}$, we set

$$\Delta NPV_{2-1} = -41.6 + (80)(P \mid F, i, 5) = 0$$

Hence,

$$(P \mid F, i, 5) = \frac{41.6}{80} = 0.52$$

from which we find $i = 14.0\%$ by interpolation between 12% and 15%. Hence, the curves representing NPV_1 and NPV_2 intersect at $i_{2-1} = 14\%$ as shown in Fig. 6.8.

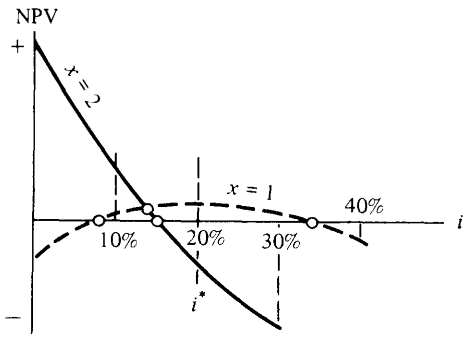

Figure 6.8 Incremental IRR for Example 6.8.

Example 6.9

The cash flow profiles for the three drilling machines in Example 6.3 and their incremental cash flows are shown in Table 6.9. Find the points of intersection of the curves representing NPV_1, NPV_2, and NPV_3 in the NPV versus i graph.

TABLE 6.9 CASH FLOWS AND INCREMENTAL CASH FLOWS (IN $1,000)

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$	$\Delta A_{t,2-1}$	$\Delta A_{t,1-3}$	$\Delta A_{t,2-3}$
0	−10,000	−10,000	−1,500	0	−8,500	−8,500
1–2 (each)	+3,000	+5,550	+1,000	+2,500	+2,000	+4,500
3–5 (each)	+3,000	+ 500	+1,000	−2,500	+2,000	− 500
6	+3,000	+5,500	+1,000	+2,500	+2,000	+4,500
$[NPV]_{0\%}$	8,000	8,000	4,500	0	3,500	3,500

Since $[NPV_1]_{0\%} = [NPV_2]_{0\%} = 8{,}000$, leading to $[\Delta NPV_{2-1}]_{0\%} = 0$, the curves representing NPV_1 and NPV_2 intersect at $i_{2-1} = 0\%$. For $x = 1$ and $x = 3$,

$$\Delta NPV_{1-3} = -8{,}500 + (2{,}000)(P \mid U, i, 6) = 0$$

from which we obtain $i_{1-3} = 10.84\%$. Similarly, for $x = 2$ and $x = 3$,

$$\Delta NPV_{2-3} = -8{,}500 + (4{,}500)(P \mid U, i, 6) - (5{,}000)(P \mid U, i, 3)(P \mid F, i, 2) = 0$$

from which we get $i_{2-3} = 14.39\%$. These points of intersection are shown in Fig. 6.9. Although both $\Delta A_{t,\,2-1}$ and $\Delta A_{t,\,2-3}$ have multiple sign changes, the values of IIRR other than those found here are not pertinent to our discussion.

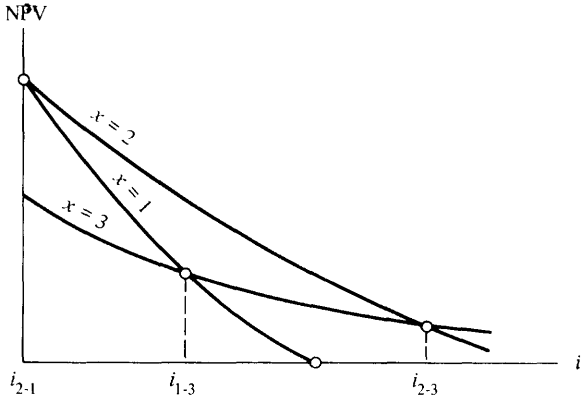

Figure 6.9 Incremental IRR for Example 6.9.

Example 6.10

The cash flow profiles of three mutually exclusive proposals and their incremental cash flow profiles are shown in Table 6.10.

(a) Find the net present values at a MARR of 3%.
(b) Find the points of intersection of the curves representing the net present values.

TABLE 6.10 CASH FLOWS AND INCREMENTAL CASH FLOWS (IN $1,000)

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$	$\Delta A_{t,2-1}$	$\Delta A_{t,3-1}$	$\Delta A_{t,3-2}$
0	−205	−350	−295	−145	−90	+55
1	+50	+100	+70	+50	+20	−30
2	+50	+90	+70	+40	+20	−20
3	+50	+80	+70	+30	+20	−10
4	+50	+70	+70	+20	+20	0
5	+50	+60	+70	+10	+20	+10
$[NPV]_{0\%}$	+45	+50	+55	+5	+10	+5

(a) The net present values of the three alternatives at a MARR of 3% are as follows:

$$[NPV_1]_{3\%} = -205 + (50)(P \mid U, 3\%, 5) = 24.0$$

$$[NPV_2]_{3\%} = -350 + (100)(P \mid U, 3\%, 5) - (10)(P \mid G, 3\%, 5) = 19.1$$

$$[NPV_3]_{3\%} = -295 + (70)(P \mid U, 3\%, 5) = 25.6$$

Since $[NPV_3]_{3\%}$ has the maximum nonnegative value among the three mutually exclusive alternatives, the alternative $x = 3$ should be selected.

(b) The intersections of the curves representing NPV_1, NPV_2, and NPV_3 can be obtained by considering the alternatives pairwise. Thus, for $x = 2$ and $x = 1$,

$$\Delta NPV_{2-1} = -145 + (50)(P \mid U, i, 5) - (10)(P \mid G, i, 5) = 0$$

from which we obtain $i_{2-1} = 1.47\%$. For $x = 3$ and $x = 1$,

$$\Delta NPV_{3-1} = -90 + (20)(P \mid U, i, 5) = 0$$

from which we get $i_{3-1} = 3.62\%$. For $x = 3$ and $x = 2$,

$$\Delta NPV_{3-2} = +55 - (30)(P \mid U, i, 5) + (10)(P \mid G, i, 5) = 0$$

It is found from the last equation that i_{3-2} does not exist in the half-plane of positive i in the NPV versus i graph. The intersections of the curves representing NPV_1, NPV_2, and NPV_3 are shown in Fig. 6.10.

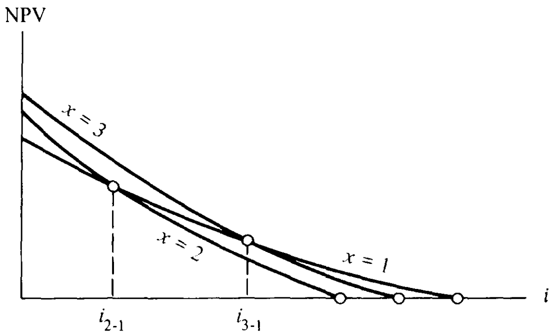

Figure 6.10 Incremental IRR for Example 6.10.

COMPARISON OF MERIT MEASURES

It is important to distinguish between the merit measures and the methods of analysis when we deal with mutually exclusive proposals. The alternatives can be ranked according to the net present value criterion, and, after the best alternative is selected on this basis, other merit measures for the selected proposal can be computed as desired by the decision maker.

It is also important to emphasize that although profit maximization is the primary objective in evaluating mutually exclusive proposals, some secondary objectives such as the payback period may be introduced by decision makers. In fact, an organization may be satisfied with selecting a reasonably profitable proposal without necessarily seeking the most profitable alternatives because of other considerations.

A public agency may even wish to select a proposal with a negative net present value in order to achieve certain socially desirable goals. However, the use of profit maximization as a decision criterion will allow us to know the price of selecting an alternative other than the one that has the maximum nonnegative net present value.

Example 6.11

Compare the measures of merit of the six mutually exclusive projects whose cash flow profits are given in Table 6.11. Use three values of i = 0%, 3%, and 6% for the MARR in computing the net present value, and the overall rate of return. If i = 3% is the MARR specified for the selection of the project, and i = 0% and i = 6% are used for sensitivity analyses only, determine the correct ranking of this set of mutually exclusive projects.

TABLE 6.11 CASH FLOWS PROFILES (IN $1,000) OF SIX MUTUALLY EXCLUSIVE PROPOSALS

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$	$A_{t,4}$	$A_{t,5}$	$A_{t,6}$
0	−205	−350	−295	−223.6	−185.5	0
1	+50	+100	+70	+56	+56	−50
2	+50	+90	+70	+56	+56	−50
3	+50	+80	+70	+56	+56	−50
4	+50	+70	+70	+56	+56	−50
5	+50	+60	+70	+56	0	+243.3

Since the procedures for computing various measures of merit have been described previously in Chapters 4 and 5, only the results of computation are tabulated in Table 6.12.

In comparing the net present values of projects x = 1, 2, . . . , 6, it is noted that ranking the projects at i = 3%, project x = 4 will produce the max-

TABLE 6.12 COMPARISON OF MEASURES OF MERIT FOR SIX MUTUALLY EXCLUSIVE PROPOSALS

Measure of Merit		Project					
		$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$	$x = 6$
NPV	$i = 0\%$	+45.0	+50.0	+55.0	+56.4	+38.5	+43.3
($1,000)	$i = 3\%$	+24.0	+19.1	+25.6	+32.9	+22.7	+24.0
	$i = 6\%$	+5.6	−8.1	−0.1	+12.3	+8.5	+8.6
ORR	$i = 0\%$	4.0%	2.7%	3.5%	4.6%	3.9%	4.0%
	$i = 3\%$	5.3%	4.1%	4.7%	5.9%	5.4%	5.5%
	$i = 6\%$	6.6%	5.5%	6.0%	7.1%	7.0%	7.0%
IRR		7.0%	5.1%	6.0%	8.0%	8.0%	8.0%
Correct ranking based on $[NPV]_{3\%}$		Third	Sixth	Second	First	Fifth	Third

imum profit or net benefit. However, ranking the projects is relatively sensitive within the limits of ±3% since it may be changed within the range of $i = 0\%$ to $i = 6\%$.

On the other hand, the internal rate of return for each project is independent of the knowledge of the MARR. Although the cash flow profiles for projects $x = 4$, $x = 5$, and $x = 6$ are quite different, each of them has an internal rate of return of 8%. The IRR values cannot be used to determine the ranking of the mutually exclusive projects.

6.7 DECISIONS ON THE TIMING OF EXPANSION AND REPLACEMENT

Frequently, the choice of investment alternatives may involve decisions on the timing of investments. Examples of such considerations include multiple-stage acquisition, replacement, and retention of a physical asset.

In planning large-scale facilities, we are often confronted with the decision whether the project should be constructed for the "ultimate capacity" or constructed in multiple stages with limited capacities in the intermediate stages. The choice depends on the projected demand as well as the difference in costs between single-stage and multiple-stage construction. For some facilities, such as urban highways, the demand will grow rapidly as more facilities are added to reduce the congestion. Consequently, there is no such thing as the "ultimate capacity." On the other hand, in constructing a sewage treatment plant for a new community, it is reasonable to expect that the demand will grow gradually over the years until the community reaches its maturity in population growth. Consequently, each situation must be considered individually on the basis of the available information.

In some situations, the decision maker may consider the possibility of delaying the purchasing of a facility by renting it for temporary use. Then, a choice must be made by comparing the alternative of purchase to the alternative of renting. In other cases, such a comparison is appealing because the facility may not be needed continuously and rental arrangement can be made in a short notice.

Example 6.12

In planning the water supply for a new community over the next 40 years, the City Water Authority is considering the following two plans. Using a MARR of 10%, determine which plan is preferable.

1. Build the complete system now at a cost of $220 million. The net revenues from supplying the local residents are expected to grow from $10 million in the first year to $48 million in the twentieth year, increasing at a rate of $2 million per year, and to reach a uniform amount of $50 million per year for the subsequent 20 years.
2. Build the system in two stages at a cost of $140 million now and another $140 million 20 years later. The net revenues from the plan are expected

to grow from \$10 million in the first year to \$48.8 million in the twenty-fifth year, increasing at a rate of \$1.6 million per year and to reach a uniform amount of \$50 million per year for the subsequent 15 years.

The net present values of plans 1 and 2 (in millions of dollars) can be computed from the cash flow profiles shown in Fig. 6.11 as follows:

$$
\begin{aligned}
[\mathrm{NPV}_1]_{10\%} &= -220 + (50)(P \mid U, 10\%, 40) \\
&\qquad - (50 - 10)(P \mid U, 10\%, 20) + (2)(P \mid G, 10\%, 20) \\
&= -220 + (50)(9.7791) - (40)(8.5136) + (2)(55.4069) \\
&= \$39.225 \text{ million} \\
[\mathrm{NPV}_2]_{10\%} &= -140 - (140)(P \mid F, 10\%, 20) + (50)(P \mid U, 10\%, 40) \\
&\qquad - (50 - 10)(P \mid U, 10\%, 25) + (1.6)(P \mid G, 10\%, 25) \\
&= -(140)(1 + 0.1486) + (50)(9.7791) - (40)(9.0770) \\
&\qquad + (1.6)(67.6964) \\
&= \$73.385 \text{ million}
\end{aligned}
$$

Since plan 2 has a higher nonnegative net present value, it is preferable.

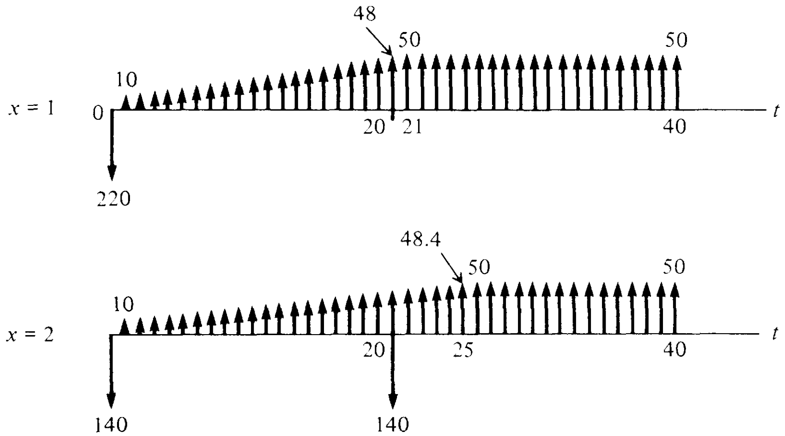

Figure 6.11 Alternatives with different timing for expansion.

6.8 MAXIMUM PERFORMANCE AT A FIXED COST

In some situations, the cost for acquiring a facility is fixed due to budgetary requirements. If the facility is size dependent and indivisible, then the choice is made among the mutually exclusive proposals requiring the same cost in the project's life cycle. It will not be advisable to compare the initial costs only if an alternative with a lower initial cost may require higher operation and maintenance costs in future years. However, if the future costs for all proposals are equal, the initial costs may be used as the basis for comparison.

Consequently, if CPV_x is known to be constant for a set of m mutually exclusive proposals, then the criterion for selecting a particular project j is that it has the maximum discounted total benefit at present. That is, for $x = 1, 2, \ldots, m$,

$$BPV_j = \max_{x \in m} BPV_x \tag{6.8}$$

provided that

$$NPV_j = BPV_j - CPV_j \geq 0$$

Equation (6.8) is referred to as the *maximum benefit criterion.*

Example 6.13

A small community is expected to receive a revenue-sharing grant from the federal government and intends to spend the full amount on one of two capital project improvements. The first project will generate a stream of benefits over a period of 5 years with \$10,000 for the first year, \$9,000 for the second year, \$8,000 for the third year, \$7,000 for the fourth year, and \$6,000 for the fifth year. The second project will generate a stream of uniform benefits of \$10,000 per year over a period of 4 years, and no replacement will be made in the fifth year. If the MARR is 10%, which project will provide greater benefits?

Using a planning horizon of 5 years, the present values of the benefits of these two mutually exclusive projects can be computed as follows:

$$[BPV_1]_{10\%} = (10{,}000)(P \mid U, 10\%, 5) - (1{,}000)(P \mid G, 10\%, 5)$$
$$= (10{,}000)(3.7908) - (1{,}000)(6.8618) = 31{,}046$$
$$[BPV_2]_{10\%} = (10{,}000)(P \mid U, 10\%, 4) = (10{,}000)(3.1699) = 31{,}699$$

The second project produces the maximum benefit and should be selected.

6.9 MINIMUM LIFE CYCLE COST

In the acquisition of a facility, if the level of performance or effectiveness is fixed, the decision becomes one of finding an alternative among the mutually exclusive proposals that requires the minimum cost. Since the cost refers to the present value of acquisition, operation, and maintenance of a facility over the planning horizon, it is often referred to as the *life cycle cost*.

Let BPV_x and CPV_x be the present values of benefits and costs, respectively, of project x. If BPV_x is known to be constant for a set of m mutually exclusive proposals, then the criterion for selecting a particular proposal j is that it has the minimum discounted cost at present. That is, for $x = 1, 2, \ldots, m$,

$$CPV_j = \min_{x \in m} CPV_x \tag{6.9}$$

provided that

$$NPV_j = BPV_j - CPV_j \geq 0$$

Equation (6.9) is referred to as the *minimum cost criterion.*

Very often, the amortized cost of a facility over its life cycle is used. Hence, the minimum cost criterion can also be expressed in the equivalent uniform annual cost (EUAC). Noting that

$$EUAC_x = CPV_x\,(U \mid P, i, n)$$

then, the criterion for selecting a particular proposal j from a set of proposals $x = 1, 2, \ldots, m$ is

$$EUAC_j = \min_{x \in m} EUAC_x \tag{6.10}$$

In determining the EUAC of mutually exclusive proposals, it is important to check carefully the useful lives of the alternatives versus the planning horizon as well as any difference in the annual benefits of these alternatives. If the assumptions on which Eq. (6.10) is based cannot be fulfilled, then the net present value decision criterion can be more conveniently used to account for the realistic replacement conditions.

Example 6.14

A plant engineer is considering the purchase of one of the three lathes that can perform the intended functions equally well. The estimated service lives of the three alternatives ($x = 1, 2, 3$) as well as the cost streams over the service lives are given in Table 6.13. Assume that within a planning horizon of 20 years, each alternative will be perpetually replaced at the end of its service life and the cost stream of each replacement is identical to that of the original lathe. If MARR is 6%, select the alternative that requires the minimum cost.

TABLE 6.13 COST STREAMS OF ALTERNATIVES IN EXAMPLE 6.14

Item	$x = 1$	$x = 2$	$x = 3$
Service life (years)	20	10	5
Purchase cost ($1,000s)	25	20	10
Annual maintenance ($1,000s)	1	1.2	2
Salvage value at end ($1,000s)	5	3	1

The cost streams of the three alternatives are depicted in Fig. 6.12, in which the planning horizon coincides with the expiration of all three alternatives. The present values of the costs of these alternatives are computed as follows:

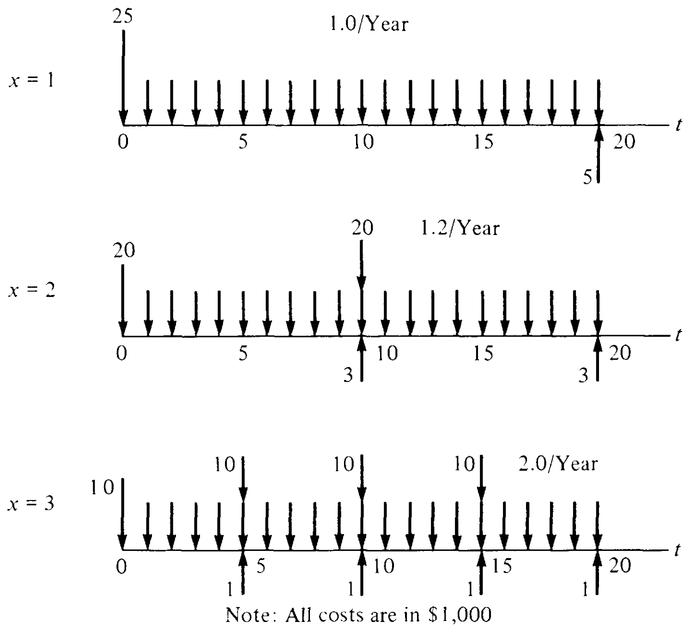

Figure 6.12 Alternatives with different useful lives.

$$
\begin{aligned}
[\mathrm{CPV}_1]_{6\%} &= 25 + (1)(P \mid U, 6\%, 20) - (5)(P \mid F, 6\%, 20) \\
&= 25 + (1)(11.4699) - (5)(0.3118) = 34.911 \\
[\mathrm{CPV}_2]_{6\%} &= [20 + (1.2)(P \mid U, 6\%, 10) \\
&\qquad - (3)(P \mid F, 6\%, 10)][1 + (P \mid F, 6\% \; 10)] \\
&= [20 + (1.2)(7.3601) - (3)(0.5584)](+0.5584) = 42.321 \\
[\mathrm{CPV}_3]_{6\%} &= [10 + (2)(P \mid U, 6\%, 5) \\
&\quad - (1)(P \mid F, 6\%, 5)][1 + (P \mid F, 6\%, 5) + (P \mid F, 6\%, 10) \\
&\qquad + (P \mid F, 6\%, 15)] \\
&= [10 + (2)(4.2124) \\
&\qquad - (1)(0.7473)](1 + 0.7473 + 0.5584 + 0.4173) \\
&= 48.136
\end{aligned}
$$

Since alternative $x = 1$ has the minimum cost, it should be selected.

For this particular problem in which the planning horizon coincides with the expiration of all three alternatives, the solution can be simplified by considering the equivalent uniform annual value of costs of the alternatives. Let $EUAC_x$ be the equivalent uniform annual value of costs of alternative x. Then,

$$
\begin{aligned}
[EUAC_1]_{6\%} &= (25)(U \mid P, 6\%, 20) + 1 - (5)(U \mid F, 6\%, 20) \\
&= (25)(0.0872) + 1 - (5)(0.0272) = 3.044 \\
[EUAC_2]_{6\%} &= (20)(U \mid P, 6\%, 10) + 1.2 - (3)(U \mid F, 6\%, 10) \\
&= (20)(0.1359) + 1.2 - (3)(0.0759) = 3.690 \\
[EUAC_3]_{6\%} &= (10)(U \mid P, 6\%, 5) + 2 - (1)(U \mid F, 6\%, 5) \\
&= (10)(0.2374) + 2 - (1)(0.1774) = 4.197
\end{aligned}
$$

Again, alternative $x = 1$ should be selected.

Note that the simplification by using the equivalent uniform annual values of costs is possible because of the assumption in the problem statement concerning the replacements in the planning horizon. Such an assumption should be used only if it is justified.

Example 6.15

A manufacturing firm plans a line of new products and expects to be in operation for 10 years only. Two machines for consideration in this operation have different characteristics and are expected to produce different results. The cost and benefit cash flow profiles for the two machines are shown in Table 6.14. Assuming that the replacements are identical to the original machines, and that benefits will cease at the end of 10 years, determine which machine should be selected if the MARR is 10%.

TABLE 6.14 OPERATING CHARACTERISTICS OF THE TWO MACHINES

Item	Machine 1	Machine 2
Estimated useful life	5 years	7 years
Initial purchase cost	\$75,000	\$90,000
Salvage value at the end of life	\$5,000	\$10,000
Salvage value at end of 3 years	\$20,000	\$35,000
Uniform annual operating cost	\$9,000	\$6,000
Uniform annual revenue	\$30,000	\$29,000

If Machine 1 is used, it will be replaced by an identical machine at the end of 5 years. Thus, for the planning horizon of 10 years,

$$
\begin{aligned}
BPV_1 &= (30{,}000)(P \mid U, 10\%, 10) \\
&\quad + (5{,}000)[(P \mid F, 10\%, 5) + (P \mid F, 10\%, 10)] \\
&= 189{,}370
\end{aligned}
$$

$$\text{CPV}_1 = 75{,}000 + (9{,}000)(P \mid U, 10\%, 10) + (75{,}000)(P \mid F, 10\%, 5)$$
$$= 176{,}868$$
$$\text{NPV}_1 = 189{,}370 - 176{,}868 = 12{,}502$$

If Machine 2 is used, it will be kept for 7 years and replaced by an identical machine at the end of 7 years. The replacement will serve for 3 years only, at the end of which it will be disposed. By keeping the original machine for 7 years, we get

$$\text{BPV}_2 = (29{,}000)(P \mid U, 10\%, 10) + (10{,}000)(P \mid F, 10\%, 7) + (35{,}000)(P \mid F, 10\%, 10)$$
$$= 196{,}818$$
$$\text{CPV}_2 = 90{,}000 + (6{,}000)(P \mid U, 10\%, 10) + (90{,}000)(P \mid F, 10\%, 7)$$
$$= 173{,}056$$
$$\text{NPV}_2 = 196{,}818 - 173{,}056 = 23{,}762$$

As a practical matter, we try not to spend the money on a new machine until it is necessary. Furthermore, the resulting net present value for the alternative of keeping the original machine for 3 years and replacing it with an identical machine after 3 years is found to be lower. There is no merit in adopting such an alternative. Based on the results of the analysis, Machine 2 should be selected.

6.10 REPLACEMENT ANALYSIS

In the replacement of existing facilities, the question often raised is whether the facility should be replaced now or some time later if the facility can be maintained or overhauled to last until a later date. In such a situation, at least several important issues should be considered. First, assuming that the demand and/or the benefit of having the facility remains unchanged, what is the optimal period that the existing facility should be kept in order to minimize the equivalent uniform annual cost during this period? Second, under the same conditions, what is the likelihood of current advances in technology which may reduce drastically the costs of a new facility as replacement? Third, what is the projected demand and potential increase in benefits if a new facility is introduced as replacement?

In previous sections, we have pointed out the limitation of the equivalent uniform annual cost decision criterion in dealing with mutually exclusive proposals which have unequal lives and/or difference to the annual benefits for the alternatives. In fact, Examples 6.14 and 6.15 touch on the replacement situations inherent in problems involving alternatives of unequal lives. Since the EUAC decision criterion is traditionally used in equipment replacement analysis, the same caution should be exercised to avoid common mistakes of applying the formulas routinely without examining the underlying assumptions.

The replacement analysis may be treated as a problem of comparing mutually exclusive proposals, i.e., the existing alternative (called defender) against potential replacement alternatives (called challengers). The computational procedure for replacement analysis is not different from that for the comparison of mutually exclusive proposals. However, the motivation for making periodic reviews to replace existing equipment is often lacking because insufficient attentions are given to excessive maintenance, declining efficiency, technological obsolescence, and other exogenous factors.

When a piece of equipment is first purchased, we can determine its original economic life by considering the initial acquisition cost, the projected operating and maintenance cost over the years, and the salvage value at the time of its disposal. While such an estimate of the optimal economic life is useful in the decision for the original purchase, subsequent exogenous factors may favor new replacement prior to the end of the estimated optimal economic life. However, a firm is often reluctant to replace a piece of equipment early as long as the existing equipment produces a profit. What the firm may miss is the opportunity to increase its profit by investing in a new and improved model of the equipment.

The reluctance of a firm to replace operational and profitable equipment is understandable. The new technology used in the latest model may incur uncertain operation and maintenance cost, or more technological improvements are expected to be on the market soon to make the latest model obsolete. The forecasting of future demand for the services of the equipment is just as risky as that of technological advances. However, if a firm is not satisfied with the status quo, it must periodically review these issues and act to stay in a competitive market.

Consequently, in the next two sections we discuss separately the determination of the optimal economic life of physical assets and the treatment of contending issues arising from exogenous factors in equipment replacement.

6.11 OPTIMAL ECONOMIC LIFE OF PHYSICAL ASSETS

The concept of minimization of present cost can be applied to assess the optimal economic life of a physical asset that is likely to be replaced by a similar asset when, because of physical deterioration, it can no longer perform the intended function. The category of physical assets includes equipment and facilities that are not subject to rapid technological obsolescence. In such a case, the benefit is considered to be identical whether the existing asset or its new replacement is in use if the existing asset is fully operational.

Let us consider the typical cash flow profile for a piece of equipment as shown in Fig. 6.13. The capital cost includes the initial acquisition cost P at year $t = 0$, less the salvage value S at year $t = N$. The operating and maintenance (O&M) cost consists of a uniform series of magnitude U_1 from year $t = 1$ to year $t = N$, and a linear gradient with a value of 0 at year $t = 1$ to a value of $(N - 1)G$ at year $t = N$.

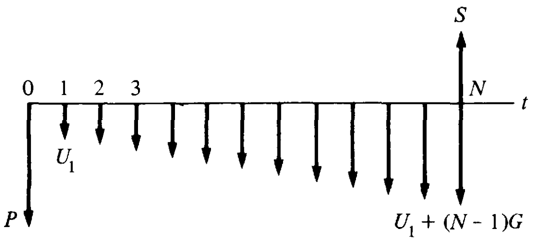

Figure 6.13 Cost profile for equipment acquisition and maintenance.

The O&M cost is expected to increase linearly with time because it will cost more to maintain the equipment in fully operable condition as it gets older. We can of course find the present value of the costs for operating and maintaining the equipment during N years and for disposing of it at the end. The cost expressed in the present value is given by

$$\text{CPV} = P - S(P \mid F, i, N) + U_1(P \mid U, i, N) + G(P \mid G, i, N) \quad (6.11)$$

However, we are more interested in the *equivalent uniform annual cost* because it will give us some idea of the annualized cost over a period of n years. Let EUAC denote the equivalent uniform annual value of cost. Then,

$$\text{EUAC} = P(U \mid P, i, N) - S(U \mid F, i, N) + U_1 + G(U \mid G, i, N)$$

Noting from Eqs. (3.15) and (3.17) in Chapter 3 that

$$(U \mid F, i, N) = (U \mid P, i, N) - i$$

we find

$$\text{EUAC} = (P - S)(U \mid P, i, N) + Si + U_1 + G(U \mid G, i, N) \quad (6.12)$$

Note also

$$(U \mid G, i, N) = (U \mid P, i, N)(P \mid G, i, N)$$

Thus, EUAC can be computed by using the discrete compound interest tables in Appendix A if the values of P, S, U_1, G, i, and N are known.

We note that the EUAC consists of two cost components: the EUAC of capital recovery (CR) and the EUAC of operating and maintenance (O&M). They are given by

$$\text{EUAC of CR} = (P - S)(U \mid P, i, N) + Si \quad (6.13a)$$

$$\text{EUAC of O\&M} = U_1 + G(U \mid G, i, N) \quad (6.13b)$$

We can compute these two components as well as the total value of EUAC and plot the results of EUAC versus N, as shown schematically in Fig. 6.14. In general, for the range of N values under consideration, a particular value N^* exists at which the total value of EUAC is a minimum. Note that the EUAC of operating and maintenance increases with N, while the EUAC of capital recovery decreases as N increases. If the planning horizon n coincides with N^*, then N^* is the optimal economic life of the equipment because the equipment, if acquired, should not be kept beyond N^* years.

Let us examine further the underlying assumptions in the determination of the optimal economic life N^* of a piece of equipment. First, we have assumed that N^*

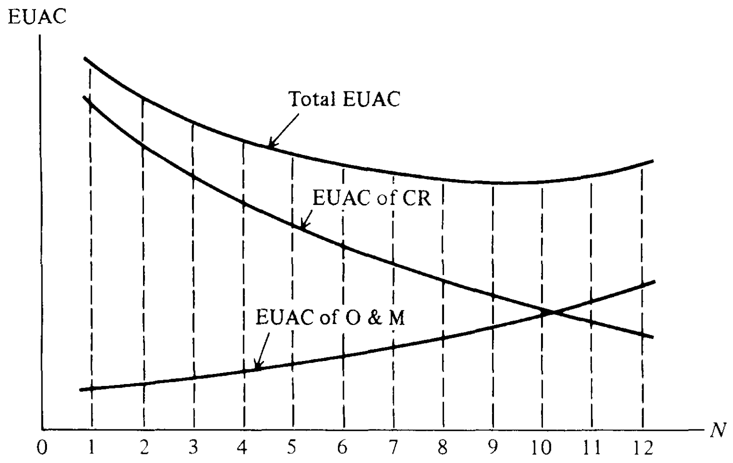

Figure 6.14 Total EUAC for different useful lives.

coincides with the planning horizon n and that the technology remains essentially the same during these N^* years. Neither one of these assumptions is entirely realistic, but they do not negate some useful observations concerning the optimal economic life. Thus, the total EUAC curve for most equipment is relatively flat after N takes on values greater than in the first few years when the capital recovery costs are relatively high. Consequently, if N^* is close to the planning horizon n, we can use n as the practical limit of the economic life of the equipment. Second, even if N^* coincides with n, new equipment with improved technology is expected to appear during the period of N^* years, and the current equipment may become technologically obsolete and/or economically unprofitable before the end of N^* years. Consequently, N^* may be regarded as the maximum number of years that the equipment should be kept, and the economic efficiency of this equipment should be reviewed periodically to determine if it should be replaced before the end of N^* years.

However, we must not extend this idea of finding the optimal economic life of equipment beyond the limits of its original assumptions. This approach should not be used when the assumptions do not reflect the actual conditions of technological obsolescence and therefore are no longer valid. When we use the equivalent uniform annual costs for comparing and/or selecting investment proposals, we can consider the realistic cash flow profiles of these proposals instead of assuming that each proposed physical asset will be replaced by a similar asset until a period equal to the least common multiple of the useful lives of all physical assets has expired. The idea of replacing a physical asset by another similar new asset with an identical cash flow profile is appealing because of its simplicity, but it is very unrealistic in an age when technological advances often produce a significant increase in efficiency and decrease in price. The computer and biomedical technologies are only two dramatic examples of productivity improvements that have characterized many industries. Consequently, it is important to emphasize the meaning of the cash flow profiles in the replacement analysis, particularly when the equivalent uniform annual cost method is used.

Another possible pitfall in the use of the equivalent uniform annual cost method for comparing and/or selecting investment proposals is that we may be comparing alternatives whose benefits are quite different. For example, when we consider a physical asset which deteriorates with usage, we have consciously built in higher future O&M costs in the form of a linear gradient that increases with time. If the asset is eventually replaced by another similar new asset, we expect that similar benefit will continue, although both the capital recovery cost and the operating and maintenance cost may be different. However, because of technological improvements, the efficiency of the new asset may increase as the cost decreases. Consequently, by comparing only the cost, we may in fact "save" money by not replacing the asset and reducing the loss of an unprofitable operation. If we generate enough bad alternatives, even a poor proposal may look very attractive by comparison when we consider only the costs and not the net benefits of the proposals.

Example 6.16

A construction firm is considering the purchase of a heavy-duty tractor which costs $130,000. The salvage value S of the tractor, if it is sold after 1, 2, . . . , 12 years, and the annual O&M cost if it is kept over these years are given in Table 6.15. The tractor cannot be kept in serviceable condition after 12 years, regardless of the O&M cost. Determine the equivalent uniform annual cost if the tractor is kept for $N = 1, 2, \ldots,$ or 12 years, if MARR = 8%.

For this problem, the initial cost $P = 130{,}000$ is the same as the salvage value S at $N = 0$, i.e., the tractor has not been used yet. The O&M cost can be broken up into a uniform series and a linear gradient with $U_1 = 4{,}000$ and $G = 3{,}000$, as indicated by the data in Table 6.16. Hence, the two components of EUAC can be computed by Eqs. (6.13) and the total EUAC computed from Eq. (6.12). The results of this problem follow the typical trends of the EUAC curves shown in Fig. 6.14.

TABLE 6.15 COMPUTATION OF EUAC FOR A RANGE OF *N*

N	*S*	O&M	EUAC of CR	EUAC of O&M	Total EUAC
0	130,000				
1	90,000	4,000	43,200 + 7,200 = 50,400	4,000 + 0 = 4,000	54,400
2	60,000	7,000	39,260 + 4,800 = 44,060	4,000 + 1,440 = 5,440	49,500
3	40,000	10,000	34,920 + 3,200 = 38,120	4,000 + 2,850 = 6,850	44,970
4	30,000	13,000	30,190 + 2,400 = 32,590	4,000 + 4,210 = 8,210	40,800
5	20,000	16,000	27,560 + 1,600 = 29,160	4,000 + 5,540 = 9,540	38,700
6	10,000	19,000	25,960 + 800 = 26,760	4,000 + 6,830 = 10,830	37,590
7	10,000	22,000	23,050 + 800 = 23,850	4,000 + 8,080 = 12,080	35,930
8	10,000	25,000	20,880 + 800 = 21,680	4,000 + 9,290 = 13,290	34,970
9	10,000	28,000	19,210 + 800 = 20,010	4,000 + 10,470 = 14,470	34,480
10	10,000	31,000	17,880 + 800 = 18,680	4,000 + 11,610 = 15,610	34,290
11	10,000	34,000	16,810 + 800 = 17,610	4,000 + 12,720 = 16,720	34,330
12	10,000	37,000	15,920 + 800 = 16,720	4,000 + 13,790 = 17,790	34,510

TABLE 6.16 COSTS OF THE TWO NEW TRACTORS AND THE EXISTING TRACTOR

	Tractor 0		Tractor 1		Tractor 2	
N	*S*	O&M	*S*	O&M	*S*	O&M
0	20,000		120,000		110,000	
1	10,000	19,000	85,000	3,000	80,000	4,000
2	10,000	22,000	55,000	6,000	55,000	7,000
3	10,000	25,000	35,000	9,000	30,000	10,000
4	10,000	28,000	25,000	12,000	20,000	13,000
5	10,000	31,000	15,000	15,000	10,000	16,000
6			10,000	18,000	5,000	19,000
7			10,000	21,000	5,000	22,000
8			10,000	24,000	5,000	25,000
9			10,000	27,000	5,000	28,000
10			10,000	30,000	5,000	31,000

Example 6.17

Five years ago, a construction firm purchased a heavy-duty tractor which had an estimated cash flow profile like the one in Example 6.16 and decided at that time to keep the tractor for 10 years. In the past 5 years, the actual operating and maintenance costs were very close to the estimated values and the firm was quite satisfied with the choice of this tractor. The firm is expected to continue in the construction business for 5 more years and is now confronted with the decision whether to keep this tractor or to replace it with a new model which may be more economical. Two new models available are expected to pro-vide the same level of service as that of the existing tractor. The CR costs and O&M costs of the two new models (1 and 2) in contention for replacement of the old tractor (0) are shown in Table 6.16. Should the existing tractor be replaced now if MARR = 8%?

In reviewing whether the existing tractor should be replaced by the new models available this year, we should use a planning horizon of 5 years since the firm originally intended to keep the existing tractor for 5 more years, even though the tractor may last longer than that. Note that the costs incurred in the past 5 years are sunk costs and therefore should not be considered; what matters is the costs for the next 5 years as duplicated in Table 6.16 from Table 6.15. Consequently, we simply compare the three alternatives over the next 5 years by computing either their net present costs or the equivalent uniform annual costs over the 5-year period. Thus, from Eq. (6.11)

$$[\text{CPV}_0]_{8\%} = 20{,}000 - (10{,}000)(P \mid F, 8\%, 5) + (19{,}000)(P \mid U, 8\%, 5) + (3{,}000)(P \mid G, 8\%, 5)$$

$$= 20{,}000 - (10{,}000)(0.6806) + (19{,}000)(3.9927) + (3{,}000)(7.3724)$$

$$= 20{,}000 - 6{,}810 + 75{,}860 + 22{,}120 = 111{,}170$$

$$[\text{CPV}_1]_{8\%} = 120{,}000 - (15{,}000)(P \mid F, 8\%, 5) + (3{,}000)(P \mid U, 8\%, 5) + (3{,}000)(P \mid G, 8\%, 5)$$

$$= 120{,}000 - (15{,}000)(0.6806) + (3{,}000)(3.9927) + (3{,}000)(7.3724)$$

$$= 120{,}000 - 10{,}210 + 11{,}980 + 22{,}120 = 143{,}890$$

$$[\text{CPV}_2]_{8\%} = 110{,}000 - (10{,}000)(0.6806) + (4{,}000)(3.9927) + (3{,}000)(7.3724)$$

$$= 110{,}000 - 6{,}810 + 15{,}970 + 22{,}120 + 141{,}280$$

From this analysis, it is obvious that the existing tractor has the minimum cost and therefore should be kept.

If we used the EUAC criterion for the replacement decision, we would still adopt a planning horizon of 5 years for all alternatives since all cash flows after year 5 are irrelevant. Therefore, the result would be the same as that reached according to the minimum cost decision criterion. It is incorrect to use 10 years as the useful lives for the new tractors in computing their EUAC.

6.12 EXOGENOUS FACTORS IN EQUIPMENT REPLACEMENT

Since excessive maintenance and declining efficiency of a piece of equipment are quite noticeable, they should be the major sources of concern for the replacement of the equipment. However, exogenous factors such as technological obsolescence, price variations in new equipment, and changing demand of its usage are also influential on replacement consideration, and deserve more attention.

In some mature industries, the technological improvements in a challenger may be marginal and/or the cost of the challenger may not be competitive enough to dislodger the defender. In many other industries, technological advances often produce a significant increase in efficiency and decrease in price. The computer and biomedical technologies are only two dramatic examples of such productivity improvements. In some cases, however, if more dramatic technological breakthroughs in a field are anticipated in the near future, a firm may defer the replacement of a piece of equipment even though the analysis favors such a replacement now. By delaying the replacement, the firm essentially looks at a longer planning horizon and expects to replace the equipment when the breakthroughs materialize.

In making replacement analysis, it is important to scrutinize the cash flow profiles to reflect correctly the true costs for comparison. For example, the sunk costs for the defender should be excluded, or the difference between the salvage value of the defender and its trade-in value for a challenger should be handled properly. Furthermore, the increase in benefits as a result of introducing the challenger should not be lost just because cost reduction is often the major concern of replacement analysis.

Example 6.18

Three years ago, Cadbury Medical Laboratory purchased a testing machine that was expected to have a useful life of five years. It now appears that there will be a dramatic breakthrough in the technology, with a new piece of testing equipment available in the market in another five years. Since the existing machine can last for only two more years, alternatives are analyzed for possible replacement in the interim.

1. Keep the defender for two more years, and rent a comparable machine for services at $8,500 in the next three years.
2. Buy challenger 1 which has a list price of $23,000 but the seller offers a trade-in value of $14,000 for the existing machine which is $1,500 higher than its market salvage value of $12,500.
3. Buy challenger 2 which has a price tag of $25,000 and allows no trade-in, but has a higher productivity which will produce an additional annual benefit of $500 in comparison with the defender or with challenger 1.

The cash flow profiles of these alternatives excluding the trade-in value are shown in Table 6.17.

TABLE 6.17 COSTS OF DEFENDER AND CHALLENGERS

	Defender		Challenger 1		Challenger 2		
t	S	O&M	S	O&M	S	O&M	AAB
0	12,500	0	23,000	0	25,000	0	0
1	10,000	4,000	19,000	1,000	21,000	800	500
2	8,000	5,000	15,500	2,000	17,500	1,600	500
3	0	8,500	12,500	3,000	14,500	2,400	500
4	0	8,500	10,000	4,000	12,000	3,200	500
5	0	8,500	8,000	5,000	10,000	4,000	500

Note that the current market salvage value of the defender is $12,500, which is equivalent to the current purchase price of a used machine if the defender is kept. All sunk costs before $t = 0$ are irrelevant. However, since challenger 1 is accompanied by a higher trade-in value, the difference of $1,500 should be treated as a discount from the market value and subtracted from its list price. Furthermore, the additional annual benefit (AAB) column is listed for challenger 2 while the salvage value S and O&M costs are given for all alternatives. This item (AAB) is to be subtracted from the corresponding annual cost.

If the MARR is 8%, and the defender and the two challengers are designed as $x = 0$, 1, and 2, respectively, then

$$[\text{CPV}_0]_{8\%} = 12{,}500 + (4{,}000)(P \mid F, 8\%, 1) + (5{,}000 - 8{,}000)(P \mid F, 8\%, 2) + (8{,}500)(P \mid U, 8\%, 3)(P \mid F, 8\%, 2) = 32{,}411$$

$$[\text{CPV}_1]_{8\%} = (23{,}000 - 1{,}500) - (8{,}000)(P \mid F, 8\%, 5) + (1{,}000)(P \mid U, 8\%, 5) + (1{,}000)(P \mid G, 8\%, 5) = 28{,}420$$

$$[\text{CPV}_2]_{8\%} = 25{,}000 - (10{,}000)(P \mid F, 8\%, 5) + (800 - 500)(P \mid U, 8\%, 5) + (800)(P \mid G, 8\%, 5) = 26{,}630$$

The results indicate that challenger 1 is better than the defender and challenger 2 is the best of all alternatives.

6.13 SUMMARY AND STUDY GUIDE

This chapter has presented the basic principles in the economic evaluation of mutually exclusive proposals. It has emphasized the direct approaches by which the best among a set of mutually exclusive alternatives can be selected in a straightforward and unambiguous manner. Examples of solutions based on the net present value criterion are given throughout the entire chapter.

Since the minimum attractive rate of return is crucially important in the computation of the net present value but difficult to pinpoint exactly, the NPV versus i graph for mutually exclusive alternatives can be used to examine the sensitivity of ranking the alternatives based on their net present values. Specifically, the relationship between the incremental net present value (INPV) and the incremental internal rate of return (IIRR) is considered in order to find the intersection of the curves representing the net present values of any two of the alternatives. Thus, the values of the MARR at which the ranking of various alternatives may change can be determined.

If the level of performance or effectiveness for all mutually exclusive proposals is fixed, the objective can be regarded as selecting the alternative which requires the minimum total cost. On the other hand, if the cost of all proposals is fixed or constant, the objective then is to select the alternative that produces the maximum benefits. The minimum cost and maximum benefit criteria represent two special cases used in comparing the relative merits of mutually exclusive alternatives without determining the net present values of the alternatives.

Frequently, the choice of investment alternatives may involve decisions on the timing of the investment or decisions between purchase and renting. In the replacement of existing facilities, the question often raised is whether the facility should be replaced now or some time later. The problems associated with replacement analysis have been addressed in detail in the context of comparison of alternatives.

REFERENCES

6.1 Hirschleifer, J., J. C. DeHaven, and J. W. Milliman, *Water Supply Economics, Technology and Policy*. Chicago, IL: University of Chicago Press, 1960.

6.2 Wohl, M., and C. Hendrickson. *Transportation Investment and Pricing Principles*. New York: Wiley, 1984.

PROBLEMS

P6.1 Three mutually exclusive proposals, each with a life of 5 years, have cash flow profiles as follows. Using a MARR of 8%, select the best proposal.

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−$1,597	−$1,650	−$1,500
1	+ 400	+ 500	+ 300
2	+ 400	+ 450	+ 350
3	+ 400	+ 400	+ 400
4	+ 400	+ 350	+ 450
5	+ 400	+ 300	+ 500

P6.2 Three mutually exclusive proposals with cash flow profiles given below are being considered for a planning horizon of 5 years. Using a MARR of 6%, select the best proposal.

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−$223.6	−$185.5	$ 0
1	+ 56	+ 56	− 50
2	+ 56	+ 56	− 50
3	+ 56	+ 56	− 50
4	+ 56	+ 56	− 50
5	+ 56	0	+ 243.3

P6.3 Find the overall rates of return for all three proposals in Problem P6.1.

P6.4 Plot the NPV-MARR graph of the three proposals in Problem P6.1 for i = 0%, 5%, and 10%.

P6.5 Two different oil pumps may be used for the extraction of oil from a well for 2 more years. The more expensive pump will permit the oil to be extracted more quickly and slightly increase the total amount of oil extracted. After 2 years, the well is expected to be dry, and there is no salvage value for either pump. The benefit and cost streams of the two pumps are given below. Using a MARR of 15%, determine which pump should be selected. Also, compare the overall rates of return for both pumps.

t	$A_{t,1}$	$A_{t,2}$
0	−\$100,000	−\$110,000
1	+ 70,000	+ 115,000
2	+ 70,000	+ 30,000

P6.6 Two different designs of a heat exchanger for a chemical processing plant are being considered. Both heat exchangers are expected to last 10 years with no salvage value at the end of 10 years. The benefits and costs associated with these two designs are given below. Using a MARR of 12%, determine which design should be selected. Also compare the overall rates of return for both designs.

t	$A_{t,1}$	$A_{t,2}$
0	−\$250,000	−\$200,000
1–5 (each)	+ 50,000	+ 30,000
6–10 (each)	+ 40,000	+ 40,000

P6.7 The cash flow profiles of three mutually exclusive alternatives of 1-year duration are given as follows:

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−\$100	−\$200	−\$300
1	+ 150	+ 280	+ 395

Find their net present values for the range of MARR with i = 0%, 10%, 20%, 30%, 40%, and 50%, and plot the NPV versus i graph. Determine the IRR for these projects, i.e., i_1, i_2, and i_3. Verify also that the values of IIRR are i_{3-2} = 15%, i_{3-1} = 22.5%, and i_{2-1} = 30%.

P6.8 The cash flow profiles of three mutually exclusive alternatives of 2-year duration are given as follows:

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−\$500	−\$1,000	−\$2,000
1	0	+ 591.7	+ 1,200
2	+ 720	+ 591.7	+ 1,100

Find their net present values for the range of MARR with i = 0%, 5%, 10%, 15%, and 20%, and plot the NPV versus i graph.

P6.9 Find the internal rates of return and the incremental internal rate of return where applicable for the three mutually exclusive projects in Problem P6.8.

P6.10 Plot the NPV versus i graph of the three mutually exclusive projects in Problem P6.2 for i = 0%, 4%, 8%, and 12%.

P6.11 The cash flow profiles of three mutually exclusive alternatives of 1-year duration are given as follows:

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−10,000	−14,800	−20,000
1	+12,100	+17,600	+23,210

(a) Find the net present values for the range of i = 0%, 5%, 10%, 15%, and 20%. Plot the NPV versus i graph.

(b) Determine the incremental internal rates of return (IIRR) for each pair of projects.

P6.12 Three mutually exclusive plans have been proposed for a new building which has a useful life of 30 years.

	$x = 1$		$x = 2$		$x = 3$	
t	$C_{t,1}$	$B_{t,1}$	$C_{t,2}$	$B_{t,2}$	$C_{t,3}$	$B_{t,3}$
0	1,800	0	1,900	0	2,000	0
1–30 (each)	200	500	250	530	300	570

(a) If MARR = 11% for the owner, rank these plans on the basis of their net present values.

(b) Verify that each pair of the NPV versus i curves representing these plans will not intersect in the half-plane of positive i.

P6.13 Two different types of conveyers can be used to transport rocks to a rock crusher at a large construction site. The construction project is expected to last 5 years and the MARR of 6% is specified for the project. Determine which conveyer should be selected on the basis of the following information:

	Type I	Type II
Initial cost	\$3,000	\$4,000
Annual operating and maintenance cost	600	400
Salvage value after 5 years	500	800
Annual benefits for rocks transported	\$1,500	\$1,600

P6.14 An industrial firm is considering the purchase of a machine for the performance of a given task in the next 10 years. Two machines with different physical characteristics but performing equally well are available. Using a MARR of 8%, and assuming that the costs of replacements are identical to the originals, determine which alternative should be selected.

	Machine 1	Machine 2
Estimated useful life	5 years	7 years
Installed cost	\$10,000	\$13,000
Salvage value at the end of life	\$1,000	\$1,000
Salvage values after 3 years	\$3,000	\$6,000
Uniform annual operating cost	\$1,200	\$1,000

P6.15 A culvert of 4-foot diameter under an existing roadway for the runoff from a creek is causing flooding during heavy rains. To remedy the situation, two solutions have been proposed for consideration:

1. Remove the existing culvert and replace it with a 6-foot-diameter culvert at an installed cost of $8,000. This new culvert is expected to last 30 years and has no salvage value when abandoned.
2. Add a new 4-foot-diameter culvert alongside the existing culvert at an installed cost of $5,000. This new culvert is expected to last 30 years and has no salvage value when abandoned. However, the existing culvert will last only 15 more years, at which time it must be replaced by another 4-foot-diameter culvert at an installed cost of $5,000.

Assuming that the roadway will be abandoned 30 years from now and using a MARR of 12%, determine which alternative should be selected.

P6.16 A new suburban area has arranged with the central city to provide water service. The costs of pipeline needed to meet the growing demand for water are as follows:

Pipe diameter	Initial cost for installation	Annual cost for pumping
12 inches	$500,000	$60,000
16 inches	$780,000	$70,000
20 inches	$1,100,000	$80,000

Assume that the water supply system is to last 40 years, and the MARR of the city is 8%. There is no salvage value for the pipe. Three different projections of growth in demand in the first 20 years suggest the three possible alternatives. Determine the present values of costs for each of these systems.

Years from now	Plan 1	Plan 2	Plan 3
0	Install one 12-in.	Install one 12-in.	Install one 16-in.
10	Add one 20-in.	Add one 16-in.	Add one 12-in.
20	Add one 16-in.	Add one 20-in.	Add one 20-in.

P6.17 An irrigation project is proposed for the development of a rural area. Two alternatives in timing are under consideration:

1. If the project is constructed to full capacity now, the initial construction cost will be $1,000,000, and the benefit will be $100,000 per year in the next 40 years. There will be no salvage value at the end of 40 years.
2. If the project is constructed in two stages, the initial construction cost for the first stage will be $400,000 and the benefit will be $40,000 per year until the second stage will be added at the end of 20 years. The construction cost for the second stage will be $800,000 but the benefit will be $100,000 per year from the twenty-first through the fortieth year. There will be no salvage value.

If a MARR of 8% is specified, determine which alternative should be selected.

P6.18 A cable television company plans to construct a system of cable networks in a new community in two stages. The initial installation cost for the first stage will be \$2 million and the benefit will be \$300,000 per year until the second stage is completed. The installation cost for the second stage will also be \$2 million but the benefit from the year after the installation of the second stage to the end of the planning horizon will be \$400,000 per year. The system is planned for 30 years with no salvage value at the end. Assuming a MARR of 6%, determine the net present value for the range of timing of the second stage installation at the end of 10, 15, and 20 years.

P6.19 A line of minicomputers that have a useful life of 4 years is expected to have a new model every 2 years. Although the price of a new model remains the same, the productivity is improved as reflected by the additional annual benefit of the new model. A small consulting firm purchased a minicomputer in this line (model 0) two years ago, and is thinking to get a new model (1) as a replacement. Alternatively, it can keep the existing computer with the intention of buying yet another model (2) two years later. Using a planning horizon of 4 years and the cash flow profiles of these models shown below, determine whether the existing computer should be replaced now. The MARR of the firm is 9%.

	Model 0		Model 1			Model 2		
t	S	O&M	S	O&M	AAB	S	O&M	AAB
0	2,220	0	6,000	0	0	0	0	0
1	1,140	2,000	3,840	1,000	500	0	0	0
2	600	2,500	2,200	1,500	500	6,000	0	0
3			1,140	2,000	500	3,840	1,000	1,000
4			600	2,500	500	2,220	1,500	1,000

P6.20 Four years ago, a building contractor bought a crane that was expected to last 8 years. Recently, he was tempted by an offer of a high trade-in value of \$130,000 for the existing crane if he buys a new one that can perform equally well. While the list price of the new crane is \$200,000, the offer effectively gives a discount of \$10,000 since the market salvage value of the existing crane is \$120,000. Using a planning horizon of 4 years and the cash flow profiles of the defender and challenger shown below (which do not include the trade-in offer), determine whether the existing crane should be replaced. The MARR of the contractor is 9%.

	Defender		Challenger	
t	S	O&M	S	O&M
0	120,000		200,000	
1	100,000	18,000	180,000	10,000
2	80,000	20,000	160,000	12,000
3	60,000	24,000	140,000	14,000
4	40,000	26,000	120,000	16,000

CHAPTER 7

The Benefit-Cost Ratio Method

7.1 A HISTORICAL PERSPECTIVE

The idea that benefits of a capital investment project should be measured and that they should exceed costs in order for the project to be justified was institutionalized in the United States by the Flood Control Act of 1936, which established the principle of benefit-cost analysis for public projects.[1] As a result of this act and subsequent legislation, the principle of benefit-cost analysis was applied in the late 1930s to a number of programs under the jurisdiction of the Corps of Engineers, the Department of Agriculture, and the Bureau of Reclamation of the Department of Interior. By 1950, attempts had been made to standardize the practices for economic evaluation of water projects undertaken by these and other government agencies including the Federal Power Commission and the Department of Commerce. The report to the Federal Inter-Agency River Basin Committee,[2] prepared by the Subcommittee on Benefits and Costs, in May 1950, identifies the ratio of benefits to costs as the criterion to indicate the relative merits of different proposals. The report, which is frequently referred to as *the green book*, provides a rational basis for accepting and selecting public projects to be undertaken by these government agencies. Although the benefit-cost ratio is not necessarily the only measure used for the comparison of projects, the higher the ratio above unity, the more favorably a project is generally looked upon by the federal agency and the U.S. Congress. Consequently, the

[1] See *United States Code*, p. 2964. Washington, DC: Government Printing Office, 1940.

[2] See Ref. 7.2, p. 14.

method of economic evaluation which uses the benefit-cost ratio as a decision criterion is referred to as the *benefit-cost ratio method.*

The use of the benefit-cost ratio as a measure of the worth of public projects is by no means confined to federal agencies or to water projects only. For example, the American Association of State Highway Officials describes only the benefit-cost ratio method in its early versions of a manual on user benefit analysis for highway improvements which is often referred to as the *red book*. In the 1977 version, it was acknowledged that "The 1960 AASHTO report presents three inconsistent criteria for interpreting the benefit-cost ratios that result from highway economy studies." The inconsistencies have been partly overcome in the 1977 manual "by (1) recognizing that the proper comparisons to make are between incremental benefits and costs, and (2) suggesting comparable decision rules that utilize either net present value or benefit/cost ratios."[3]

With this historical background in professional practice, it is important to emphasize again that an investment decision consists of two elements: a merit measure and a set of decision rules. Although benefit-cost ratio is an indirect merit measure, it is well entrenched in the minds of decision makers in the public sector. There is no reason why its use should not be continued as long as it is unambiguously defined. On the other hand, it should be recognized that the decision rule for accepting or rejecting an independent project cannot be applied to the selection of the best among a set of mutually exclusive proposals, which must be dealt with by the rules governing profit maximization based on an *incremental analysis*. Therefore, we examine some causes of misunderstanding and misuse of the method. Specifically, we discuss two aspects of its application in economic evaluation:

1. The use of the benefit-cost ratio as a merit measure for evaluating the acceptability of an independent project (in comparison with the null alternative).
2. The use of the benefit-cost ratio method in the *incremental analysis* through successive pairwise comparisons for the selection of the best among a set of mutually exclusive proposals.

7.2 THE BENEFIT-COST RATIO

The benefit-cost ratio (BCR) may be defined as the ratio of the discounted benefits to the discounted costs of an investment with reference to the same point in time. Since the present is a convenient point of reference, the BCR is often computed on the basis of the present value of benefits (BPV) and the present value of costs (CPV).

The benefit-cost ratio is also referred to as the profitability index, indicating the discounted benefits per unit of discounted costs. It is sometimes called the savings-to-investment ratio when the benefits are derived from the reduction of undesirable

[3] See Ref. 7.3, especially pp. 156–157.

effects. For the sake of simplicity, we confine our discussion to situations in which the investment cash flow profile has only one sign change, although the definition of benefit-cost ratio can be broadened to cover unconventional cash flow profiles.

Using the present as a point of reference, we can compute the present values of benefits and costs for a project x according to Eq. (5.6). Then, the benefit-cost ratio of the project x is given by

$$\frac{B_x}{C_x} = \frac{\text{BPV}_x}{\text{CPV}_x} = \frac{\sum_{t=0}^{n} B_{x,t}(P \mid F, i, t)}{\sum_{t=0}^{n} C_{x,t}(P \mid F, i, t)} \tag{7.1}$$

where BPV_x and CPV_x are present values of benefits and costs, respectively, of project x. It should be noted that the benefit-cost ratio in Eq. (7.1) may also be based on $\text{BFV}_x/\text{CFV}_x$ or $\text{EUAB}_x/\text{EUAC}_x$, where BFV_x and CFV_x are future values of benefits and costs, respectively, compounded to the future time n, while EUAB_x and EUAC_x are the equivalent uniform annual benefits and costs, respectively.

The criterion for accepting an independent project on the basis of the benefit-cost ratio can be obtained by substituting Eq. (5.7) into Eq. (5.9) for the net present value criterion:

$$\text{NPV}_x = \text{BPV}_x - \text{CPV}_x \geq 0$$

Consequently,

$$\frac{B_x}{C_x} = \frac{\text{BPV}_x}{\text{CPV}_x} \geq 1 \tag{7.2}$$

Some items in the cash flow profile of a project can often be interpreted either as additional benefits or reduced costs. In that case the value of B_x/C_x depends on how such items are treated in the computation; hence the ratio B_x/C_x is not an unambiguous merit measure of the project. However, the economic feasibility of an independent project will not be altered by different interpretations of additional benefits or reduced costs in computing the benefit-cost ratio.

Consider, for example, that the cash flow profile of an investment project has been established except for one item. Let the present value of this item be d, which may be interpreted either as an additional benefit or a reduced cost, while the present values of benefits and costs of the remaining items in the cash flow profile are denoted by b and c, respectively. Suppose that we consider the following two cases: (1) d is an additional benefit; (2) d is a reduced cost. By using Eq. (5.7) to compute the net present value, we get

$$(1)\ \text{NPV} = (b + d) - c = b - c + d$$

$$(2)\ \text{NPV} = b - (c - d) = b - c + d$$

On the other hand, by using Eq. (7.1) to compute the benefit-cost ratio we have

$$(1)\ \frac{\text{B}}{\text{C}} = \frac{b + d}{c}$$

$$(2)\ \frac{\text{B}}{\text{C}} = \frac{b}{c - d}$$

It can be seen that if d is identified either as an additional benefit or a reduced cost, Eq. (5.7) yields the same NPV for both cases 1 and 2 whereas Eq. (7.1) leads to different B/C ratios. However, if

$$\frac{b + d}{c} \geq 1 \qquad \text{(i.e., } b + d \geq c\text{)}$$

then

$$\frac{b}{c - d} \geq 1 \qquad \text{(i.e., } b \geq c - d\text{)}$$

That is, if a project is acceptable on the basis of the first ratio, it is also acceptable on the basis of the second ratio. Therefore, the economic feasibility of the project will not be altered.

The same reasoning can be applied to similar situations in which the value of the benefit-cost ratio can be changed by the relative values of benefits and costs used in the computation. For example, the cash flow profile of a project is often represented by the *net* annual cash flow A_t (for $t = 0, 1, 2, \ldots$) instead of separate time streams of benefits B_t and costs C_t (both for $t = 0, 1, 2, \ldots$). Furthermore, some items such as the salvage value of a physical asset may be treated either as benefits or negative costs. To the extent possible, it is consistent to treat all receipts in the time stream as benefits regardless of their sources and all disbursements in the time stream as costs regardless of their designated purposes. Then, the criterion based on the benefit-cost ratio can be used to determine the economic feasibility of a project, even though the ratio itself is not a good indicator of the size of the profit.

Example 7.1

A public agency is considering four independent projects which have different useful lives. The cash flow profile for each project x consists of an initial cost $C_{0,x}$, a stream of uniform annual operating and maintenance costs $\overline{C}_x$, and a stream of uniform annual benefits $\overline{B}_x$ for years $t = 1, 2, \ldots, N_x$ where N_x is the useful life of project x. The salvage value at $t = N_x$ for project x is $S_{N,x}$. There will be no replacement when the useful life of a project expires. The numerical data for these projects are given in Table 7.1. The agency intends to adopt a planning horizon of 10 years and specifies a MARR of 10%. Determine whether these projects are economically feasible on the basis of the benefit-cost ratio using

1. The given time streams of costs and benefits
2. The net annual cash flows resulting from the given data

TABLE 7.1 DATA FOR THE INDEPENDENT PROJECTS IN EXAMPLE 7.1

Project x	Useful life N_x	Initial cost $C_{0,x}$	Uniform annual cost $\bar{C}_x$ for $t = 1, 2, \ldots, N_x$	Uniform annual benefit $\bar{B}_x$ for $t = 1, 2, \ldots, N_x$	Salvage value $S_{N,x}$
1	10	$50,000	$6,000	$25,460	$14,000
2	4	45,000	3,000	13,280	5,000
3	10	40,000	4,000	17,040	10,000
4	6	30,000	3,500	15,120	0

Using the given time streams of costs and benefits, the present value of costs of each project x for case 1

$$\text{CPV}_x = C_{0,x} + \sum_{t=1}^{n} C_{t,x}(P \mid F, 10\%, t)$$
$$= C_{0,x} + \bar{C}_x(P \mid U, 10\%, N_x)$$

The present value of benefits of the project is

$$\text{BPV}_x = \sum_{t=1}^{n} B_{t,x}(P \mid F, 10\%, t) + S_{N,x}(P \mid F, 10\%, N_x)$$
$$= \bar{B}_x(P \mid U, 10\%, N_x) + S_{N,x}(P \mid F, 10\%, N_x)$$

On the other hand, if the net annual cash flow profile is first determined from the given data, we have $C_{0,x}$ at $t = 0$ and $U_x = \bar{B}_x - \bar{C}_x$ for $t = 1, 2, \ldots, N_x$. That is,

$$U_1 = 25{,}460 - 6{,}000 = 19{,}460$$
$$U_2 = 13{,}280 - 3{,}000 = 10{,}280$$
$$U_3 = 17{,}040 - 4{,}000 = 13{,}040$$
$$U_4 = 15{,}120 - 3{,}500 = 11{,}620$$

Since the values U_1, U_2, U_3, and U_4 are positive, they represent net annual benefits. Then, for case 2

$$\text{CPV}_x = C_{0,x}$$
$$\text{BPV}_x = U_x(P \mid U, 10\%, N_x) + S_{N,x}(P \mid F, 10\%, N_x)$$

The benefit-cost ratios for both cases 1 and 2 are computed according to Eq. (7.1) and the results are tabulated in Table 7.2. Thus, while projects 1, 3, and 4 should be accepted, project 2 should be rejected since $\text{BPV}_2/\text{CPV}_2 < 1$. Note that the acceptability of these projects is not affected by using the ratios B_x/C_x in case 1 or 2.

TABLE 7.2 BENEFIT-COST RATIOS FOR PROJECTS IN EXAMPLE 7.1

Project	(1) Ratio based on gross annual benefits and costs			(2) Ratio based on net annual benefits and costs		
x	BPV_x	CPV_x	B_x/C_x	BPV_x	CPV_x	B_x/C_x
1	$161,839	$86,868	1.86	$125,000	$50,000	2.50
2	45,511	54,510	0.83	36,000	45,000	0.80
3	108,559	64,578	1.68	84,000	40,000	2.10
4	65,852	45,244	1.46	50,600	30,000	1.69

Example 7.2

Using the net annual cash flows in the previous example, recompute the benefit-cost ratios for all projects if the salvage value is treated as a negative cost instead of a benefit.

The present values of costs and benefits using the net cash flows can be obtained as follows:

$$CPV_x = C_{0,x} - S_{N,x}(P \mid F, 10\%, N_x)$$

$$BPV_x = \overline{U}_x(P \mid U, 10\%, N_x)$$

The results of the computation are shown in Table 7.3. Note again that acceptability of the projects is not altered by a different treatment of the salvage value in the analysis.

TABLE 7.3 DIFFERENT TREATMENTS OF SALVAGE VALUES IN ANALYSIS

Project	Salvage value as benefit		Salvage value as negative cost			
x	B_x/C_x	Acceptability	BPV_x	CPV_x	B_x/C_x	Acceptability
1	2.50	Yes	119,574	44,603	2.68	Yes
2	0.80	No	32,587	41,585	0.78	No
3	2.10	Yes	80,126	36,145	2.22	Yes
4	1.69	Yes	50,609	30,000	1.69	Yes

7.3 INDEPENDENT PROJECTS VERSUS MUTUALLY EXCLUSIVE PROJECTS

To understand clearly the necessity of using the incremental analysis for evaluating mutually exclusive projects when the benefit-cost ratio is used as a decision criterion, let us first examine the relationship between the *present value* of the benefits and the *present value* of the costs of a group of proposed projects on a benefit-cost graph. For the sake of simplicity, let us introduce the following abbreviations:

C = the present value of the costs (CPV) in the cash flow profile of a proposed project

B = the present value of the benefits (BPV) in the cash flow profile of the same project
N = the net present value (NPV)

A typical benefit-cost graph showing four proposed projects is given in Fig. 7.1. Note that the inclined line $B/C = 1$ passes through the origin and has a slope of 1:1, i.e., at a 45° angle with the horizontal axis C. Hence, the value B_x corresponding to the value C_x for project x should be above the inclined line $B/C = 1$ if $B_x > C_x$, and otherwise if $B_x < C_x$. The algebraic difference between B_x and the point on the inclined line $B/C = 1$ corresponding to C_x represents the net present value N_x for project $x (x = 1, 2, 3,$ or 4 for the four proposed projects.)

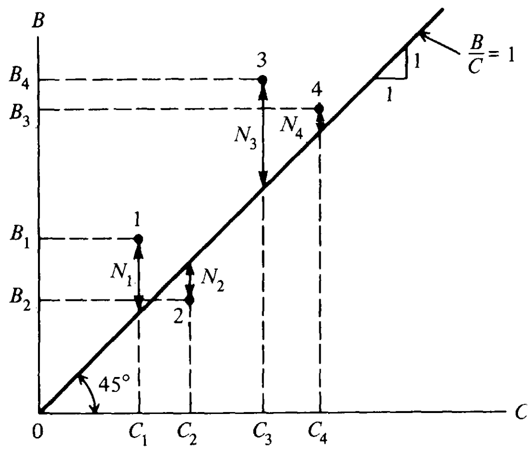

Figure 7.1 Benefit versus cost curve showing NPV of projects.

If the proposed projects are independent of each other and we are only interested in determining whether each of them is economically feasible, we can plot the benefit-cost ratios for all projects on the benefit-cost graph as shown in Fig. 7.2. The inclined line $B/C = 1$ corresponds to NPV = 0 since $B = C$ for all points on the line. Therefore, if the slope B_x/C_x is greater than 1, N_x is positive, as in the case of $x = 1$, 3, and 4; conversely, if the slope B_x/C_x is less than 1, N_x is negative, as in the case of $x = 2$. Hence, this graphical representation is consistent with the benefit-cost ratio criterion for evaluating independent projects discussed in Section 7.2.

However, if the proposed projects are mutually exclusive, we will be interested in determining which project is the best among this group of proposed projects. The ratios B_x/C_x in Fig. 7.2 will not be helpful in finding the best project since the project with the maximum benefit-cost ratio is not necessarily the project with the maximum net present value. Consequently, the ranking of the profit potentials of mutually exclusive projects cannot be based on the ranking of their benefit-cost ratios.

Since the proposal with maximum benefit-cost ratio among a set of mutually exclusive proposals generally does not lead to the maximum net benefit, an incremental analysis must be introduced if the decision criterion is to be based on the

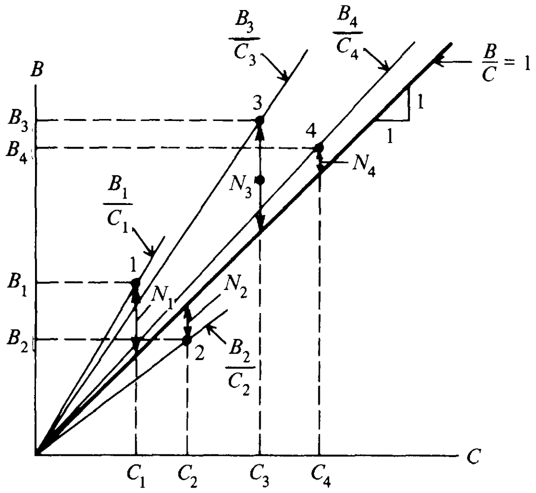

Figure 7.2 Benefit versus cost curve showing BCR of projects.

benefit-cost ratio. You may wonder why it is necessary to learn another method of analysis if you can use the net present value method to find the best among these proposals. The fact is that it is totally unnecessary if you can influence the rest of the world! Unfortunately, many practicing professionals still cling to the notion that to use the benefit-cost ratio as a merit measure, one must also use a method of analysis based on the benefit-cost ratio. You may have to use the method that your employer prefers. Or, if you are a consulting engineer who wants to win a contract from a public agency, you may be informed that the persons who are going to review your work will only accept an analysis based on the benefit-cost ratio criterion. Even though this may not be the method of your choice, you may need it at some point of your career to act defensively.

7.4 OPTIMIZATION OF A CONTINUOUS SET OF ALTERNATIVES

Before discussing the incremental analysis of mutually exclusive proposals, it is informative to consider profit maximization of a continuous set of alternatives. Since the procedure for finding the maximum of a continuous function is well established in calculus, we can easily understand the principle for finding the maximum net present value which is a direct merit measure. It can be shown that the project size (in terms of the present value of cost) that produces the maximum net benefit generally does not lead to maximum benefit-cost ratio, and vice versa.

For example, the alternatives for a proposed project may represent a variation of the project size. Since the cost of an alternative depends on its size, the benefits for a set of mutually exclusive alternatives may be expressed in terms of varying costs. Then the optimal size of the project is the alternative that produces the maximum net benefit. Suppose that the present value of the benefits of a project can be expressed as a continuous function of the present value of the costs. Let us introduce the following notation:

C = the present value of the costs in the cash flow profile of a proposed project
B = the present value of the benefits in the cash flow profile of the same project
N = the net present value

Then, the function representing the benefits of a set of alternatives with continuously increasing costs is given by

$$B = f(C) \tag{7.3}$$

For example, the relationship may be in the form of a polynomial

$$B = 1 + \frac{31}{32}C - \frac{1}{2C} \qquad \text{for } C \geq 1$$

where both B and C are expressed in millions of dollars. If we wish to consider a set of eight alternatives with $C = 1, 2, \ldots, 8$, we can easily find the corresponding values of B.

The net present value of a set of alternatives may also be expressed as a continuous function of C as follows:

$$N = B - C = f(C) - C \tag{7.4}$$

The value of C that will produce an alternative with maximum net benefit may be obtained by taking the first derivative of N with respect to C and setting it equal to zero. That is, if a maximum N exists, x can be found from

$$\frac{dN}{dC} = 0 \tag{7.5}$$

For $N = B - C$, we have

$$\frac{dN}{dC} = \frac{dB}{dC} - 1 = 0$$

or

$$\frac{dB}{dC} = 1 \tag{7.6}$$

The relationships in Eqs. (7.3) through (7.6) are graphically represented in Fig. 7.3. For any given $B = f(C)$, we can plot a graph of B versus C as shown schematically in part (a) of Fig. 7.3. Note that the inclined line $B/C = 1$ passes through the origin and has a slope of 1:1. Hence, the function $B = f(C)$ is above the inclined line when $B > C$. The difference between the curve $B = f(C)$ and the inclined line $B/C = 1$ represents the net present value N. We can also plot a graph

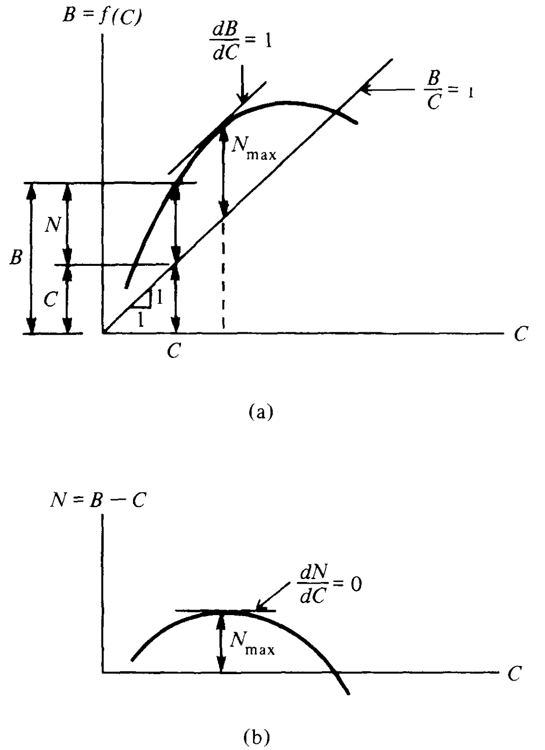

Figure 7.3 Benefit versus cost curve for a continous function.

of N versus C as shown schematically in part (b) of Fig. 7.3 if N is first obtained from Eq. (7.4). Note that the maximum net benefit N_{max} occurs at a value of C corresponding to $dN/dC = 0$ or $dB/dC = 1$, as indicated in Fig. 7.3.

Example 7.3

At a site upstream from a town, a dam is being proposed for flood protection. The benefit from the proposed dam is expected to increase with the increasing height of the dam. The variation of the present value of benefits with respect to the present value of costs may be expressed by

$$B = 1 + \frac{31}{32}C - \frac{1}{2C} \qquad \text{for } C \geq 1$$

where both B and C are expressed in millions of dollars. Determine the value of C that will produce the maximum net benefit N and the value of this net benefit.

Since the maximum net benefit N occurs at the value of C when $dB/dC = 1$ or $dN/dC = 0$, we note that

$$N = B - C = 1 - \frac{1}{32}C - \frac{1}{2C}$$

$$\frac{dN}{dC} = -\frac{1}{32} - \left(\frac{1}{2}\right)(-1)(C^{-2}) = 0$$

Hence,

$$\tfrac{1}{2}C^{-2} = \tfrac{1}{32} \quad \text{or} \quad C = 4$$

The second derivative of N is given by

$$\frac{d^2N}{dC^2} = \left(\frac{1}{2}\right)(-2)(C^{-3}) = -C^{-3}$$

At $C = 4$, $d^2N/dC^2 = -1/64$, indicating a maximum N. Hence, the maximum net benefit at $C = 4$ is

$$N = 1 - \frac{4}{32} - \frac{1}{(2)(4)} = \frac{3}{4} = 0.75$$

If we compute the values of B and N for $C = 1, 2, \ldots, 8$, we obtain the results in \$$10^6$ shown below:

C	B	N
1	1.469	0.469
2	2.688	0.688
3	3.740	0.740
4	4.750	0.750
5	5.744	0.744
6	6.729	0.729
7	7.710	0.710
8	8.688	0.688

7.5 INCREMENTAL ANALYSIS OF MUTUALLY EXCLUSIVE PROPOSALS

For the four projects in Fig. 7.2, for example, it is seen that $N_3 > N_1 > N_4 > N_2$. From the slopes of the lines representing their benefit-cost ratios, however, it is noted that $B_1/C_1 > B_3/C_3 > B_4/C_4 > B_2/C_2$. If we follow the ranking of the benefit-cost ratios and select $x = 1$, which has the highest B/C value, we will deprive ourselves of the opportunity of investing an additional amount of money to realize a bigger net benefit (in terms of the net present value) as represented by N_3 for alternative $x = 3$. Therefore, we should make successive pairwise comparisons to determine whether the *additional* benefits generated by the higher cost alternative outweigh the *additional* costs in each comparison. Since the profit potential of the additional investment is expressed as the ratio of the incremental cost, this approach is referred to as the *incremental analysis* or *incremental benefit-cost ratio* analysis.

The incremental analysis can be carried out systematically by using successive pairwise comparisons. Basically, the set of mutually exclusive proposals under consideration is arranged according to the ascending order of their *present values of costs*. Starting with the lowest cost alternative ($x = 1$), we compare it with the null alternative ($x = 0$) to determine whether this proposal ($x = 1$) is at least as profitable as the best foregone opportunity. If the answer is affirmative, we accept the lowest cost alternative as the tentative choice: otherwise we retain the null alternative. Next, compare this tentative choice with the next lowest cost alternative ($x = 2$) to determine whether the additional investment is profitable. If this answer is affirmative, we accept the latter over the former as the tentative choice; otherwise we retain the former. We can repeat the procedure until the complete set of alternatives is compared in this manner. Then, the final choice is the best among the set of alternatives. The rationale behind this approach is that as long as we can invest an additional amount of money at the MARR and receive a greater net benefit, we should accept the higher cost alternative (based on the present value of costs of a set of mutually exclusive alternatives computed at the specified MARR); otherwise the leftover amount will not earn a return as good as the specified MARR. Hence we must examine in each step whether the profit potential will be increased by changing from the lower cost alternative to the higher cost alternative, until all alternatives have been compared successively.

In order to develop an operating procedure for the incremental benefit-cost ratio analysis, let us introduce the following notation:

C_y = the present value of costs of the alternative y, which refers to the last lower cost alternative accepted prior to the comparison with the higher cost alternative x

ΔC_{x-y} = the incremental cost between the alternatives x and y

B_y = the present value of benefits of the alternative y

ΔB_{x-y} = the incremental benefit between the alternatives x and y

Then, the incremental benefit-cost ratio for a pair of alternatives x and y is defined as follows:

$$\frac{\Delta B_{x-y}}{\Delta C_{x-y}} = \frac{B_x - B_y}{C_x - C_y} \tag{7.7}$$

If the alternatives are arranged in the ascending order of *the present values of their costs*, we can be sure that the denominator in Eq. (7.7) will always be positive. Then, in making a pairwise comparison between a lower cost alternative y and the higher cost alternative x, the higher cost alternative x is preferable if

$$\frac{\Delta B_{x-y}}{\Delta C_{x-y}} \geq 1 \tag{7.8}$$

Otherwise the lower cost alternative y is preferable. Note that the incremental benefit ΔB_{x-y} and consequently the ratio itself may be positive or negative, but Eq. (7.7) holds true even if the ratio is negative (less than zero). Therefore, Eq. (7.8) is the criterion for accepting the higher cost alternative x in the pairwise comparison between the alternatives y and x.

The successive pairwise comparisons of all alternatives will lead to the choice of the best alternative, but the choice is not necessarily economically acceptable since even the best may not be good enough. In order to make sure that the final choice is economically feasible as well as the best among all alternatives, we can always test each alternative for acceptability based on the benefit-cost ratio criterion in Section 7.2 and include only those which are economically feasible in the set of alternatives for selection. However, this extra work is unnecessary if we first compare the lowest cost alternative (say $x = 1$) with the null alternative ($x = 0$). In fact, we simply consider the benefit-cost ratio B_1/C_1 for testing the acceptability of alternative $x = 1$ as the incremental benefit-cost ratio $\Delta B_{1-0}/C_{1-0}$ for the pairwise comparison between $x = 0$ and $x = 1$. If $x = 1$ is preferable, we proceed to compare $x = 1$ with the next higher cost alternative $x = 2$; otherwise, we reject $x = 1$ and compare $x = 0$ with $x = 2$, and so on. We can therefore begin with the lowest cost alternative and conduct pairwise comparisons successively in the ascending order of the present values of costs until all alternatives have been compared and the choice is made from the last pair.

The procedure for pairwise comparisons can best be illustrated by referring to Fig. 7.4 in which the incremental benefit-cost ratios for successive pairwise comparisons are plotted. We begin with the comparison of $x = 1$ with the null alternative ($x = 0$) and accept $x = 1$ as preferable since $\Delta B_{1-0}/\Delta C_{1-0} > 1$, i.e., the slope of

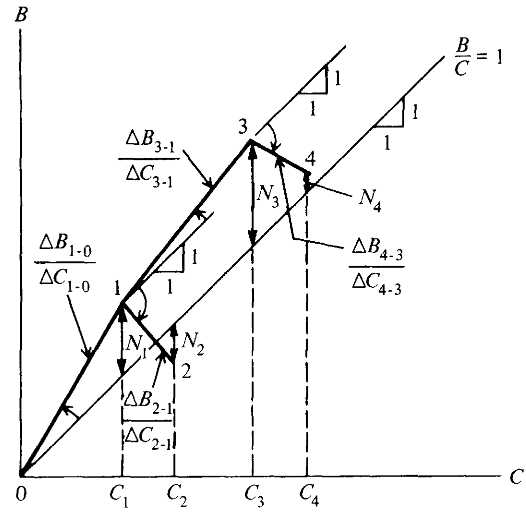

Figure 7.4 Benefit versus cost curve showing incremental BCR.

the line segment 0–1 is greater than that of the line $B/C = 1$. Next, we compare $x = 1$ and $x = 2$ by computing the incremental benefit-cost ratio $\Delta B_{2-1}/\Delta C_{2-1}$, which is represented by the slope of the line segment 1–2 in the figure. Since $\Delta B_{2-1}/\Delta C_{2-1} < 1$, as indicated by the slope of the line segment 1–2, we reject $x = 2$ and next proceed to compare $x = 1$ and $x = 3$. Then we compute the incremental benefit-cost ratio $\Delta B_{3-1}/\Delta C_{3-1}$ because we are interested in knowing how good is $x = 3$ compared with $x = 1$. Thus, the comparison is always between a higher cost alternative x with the *last lower cost alternative accepted,* which is not necessarily the lower cost alternative immediately preceding x, i.e., $y \neq x - 1$ in this case. (In fact, the slope of the line segment 2–3 is irrelevant for the comparison between $x = 1$ and $x = 3$.) Since $\Delta B_{3-1}/\Delta C_{3-1} > 1$, we accept $x = 3$. Finally, we note that $\Delta B_{4-3}/\Delta C_{4-3} < 1$; hence the alternative $x = 3$ is the best among the four mutually exclusive alternatives.

7.6 PROCEDURE FOR INCREMENTAL ANALYSIS

The procedure for the incremental benefit-cost ratio analysis for selecting the best proposal among a set of mutually exclusive alternatives may therefore be summarized as follows:

1. Compute the present values of costs and benefits for the mutually exclusive alternatives, using the specified MARR as the discount rate.
2. Arrange the alternatives in the ascending order of the present values of their costs for the purpose of making pairwise comparisons. If the present values of costs of two or more alternatives are equal, it makes no difference how they are ordered among themselves.
3. Compare the lowest cost alternative ($x = 1$) with the null alternative ($x = 0$). If $\Delta B_{1-0}/\Delta C_{1-0} \geq 1$, accept $x = 1$; otherwise, compare $x = 0$ with the next higher cost alternative.
4. After an alternative is found to be economically feasible, compare it with the next higher cost alternative. If the incremental benefit-cost ratio is greater than or equal to one, accept the higher cost alternative; otherwise retain the lower cost alternative.
5. Repeat the pairwise comparisons by using the criterion in Eq. (7.8) successively until all alternatives have been compared; the alternative accepted in the last pairwise comparison is the best among all alternatives.

Example 7.4

Four mutually exclusive proposals ($x = 1, 2, 3$, and 4) are known to have the following benefits and costs discounted to the present values according to the specified MARR. The benefit-cost ratios have also been computed. Determine the best alternative by using the incremental benefit-cost ratio analysis if (1) the economically infeasible alternative as determined by the benefit-cost ratios is eliminated first, and (2) the economically infeasible alternative is included in the incremental analysis. All costs and benefits are in millions of dollars as follows:

x	CPV_x	BPV_x	BPV_x/CPV_x	Acceptability
1	2	3	1.50	Yes
2	5	4	0.80	No
3	7	11	1.57	Yes
4	9	12	1.33	Yes

In case 1, we eliminate alternative $x = 2$ before performing the incremental analysis. We arrange the remaining alternatives in the ascending order of the alternatives and begin with the comparison of $x = 1$ with the null alternative $x = 0$. The successive pairwise comparisons are as follows:

$$\frac{\Delta B_{1-0}}{\Delta C_{1-0}} = \frac{3 - 0}{2 - 0} = 1.50 \qquad \text{(choose } x = 1\text{)}$$

$$\frac{\Delta B_{3-1}}{\Delta C_{3-1}} = \frac{11 - 3}{7 - 2} = 1.60 \qquad \text{(choose } x = 3\text{)}$$

$$\frac{\Delta B_{4-3}}{\Delta C_{4-3}} = \frac{12 - 11}{9 - 7} = 0.50 \qquad \text{(choose } x = 3\text{)}$$

In case 2, we include alternative $x = 2$ in the incremental analysis. The computation leads to the same final choice. That is,

$$\frac{\Delta B_{1-0}}{\Delta C_{1-0}} = \frac{3 - 0}{2 - 0} = 1.50 \qquad \text{(choose } x = 1\text{)}$$

$$\frac{\Delta B_{2-1}}{\Delta C_{2-1}} = \frac{4 - 3}{5 - 2} = 0.33 \qquad \text{(choose } x = 1\text{)}$$

$$\frac{\Delta B_{3-1}}{\Delta C_{3-1}} = \frac{11 - 3}{7 - 2} = 1.60 \qquad \text{(choose } x = 3\text{)}$$

$$\frac{\Delta B_{4-3}}{\Delta C_{4-3}} = \frac{12 - 11}{9 - 7} = 0.50 \qquad \text{(choose } x = 3\text{)}$$

Example 7.5

Seven mutually exclusive building projects are considered for a site. Each building is expected to have a useful life of 40 years and there is no salvage value at the end of 40 years. The initial construction costs, the uniform annual operating and maintenance (O&M) costs, and the uniform annual benefits (in millions of dollars) for these alternatives for $x = 1, 2, \ldots, 7$ are given in Table 7.4. The MARR is specified to be 7%. Select the best among the seven

TABLE 7.4 CASH FLOW PROFILES OF THE MUTUALLY EXCLUSIVE ALTERNATIVES

Alternative x	Initial cost $C_{0,x}$	Annual O&M costs $C_{t,x} = \bar{C}_x$ for $t = 1, 2, \ldots, 40$	Annual benefits $B_{t,x} = \bar{B}_x$ for $t = 1, 2, \ldots, 40$
1	1.000	0.075	0.202
2	2.000	0.075	0.270
3	1.867	0.160	0.356
4	3.800	0.090	0.420
5	4.600	0.105	0.504
6	5.600	0.105	0.600
7	5.600	0.180	0.666

mutually exclusive alternatives by using the incremental benefit-cost ratio analysis.

We first compute the present values of costs and benefits for each of the mutually exclusive projects by noting that, for any project x,

$$C_x = \text{CPV}_x = C_{x,0} + \bar{C}_x(P \mid U, 7\%, 40)$$
$$B_x = \text{BPV}_x = \bar{B}_x(P \mid U, 7\%, 40)$$

where $C_{x,0}$, $\bar{C}_x$, and $\bar{B}_x$ are given in Table 7.4. For example, we compute for $x = 1$,

$$C_1 = \text{CPV}_1 = 1.0 + (0.075)(13.3317) = 2.0$$
$$B_1 = \text{BPV}_1 = (0.202)(13.3317) = 2.693$$

The results of CPV_x and BPV_x for all alternatives are tabulated in Table 7.5 where the alternatives are arranged in the ascending order of CPV_x. Now we

TABLE 7.5 INCREMENTAL BENEFIT-COST ANALYSIS OF EXAMPLE 7.5

x	CPV_x	BPV_x	$x - y$	ΔC_{x-y}	ΔB_{x-y}	$\frac{\Delta B_{x-y}}{\Delta C_{x-y}}$	Decision
1	2.000	2.693	1 − 0	2.000	2.693	1.347	Choose 1
2	3.000	3.600	2 − 1	1.000	0.907	0.907	Choose 1
3	4.000	4.746	3 − 1	2.000	2.053	1.027	Choose 3
4	5.000	5.599	4 − 3	1.000	0.853	0.853	Choose 3
5	6.000	6.719	5 − 3	2.000	1.973	0.987	Choose 3
6	7.000	7.999	6 − 3	3.000	3.253	1.084	Choose 6
7	8.000	8.879	7 − 6	1.000	0.880	0.880	Choose 6

compare alternative $x = 1$ with the null alternative ($x = 0$) for which both CPV_0 and BPV_0 are zero. Consequently, we obtain

$$\frac{\Delta B_{1-0}}{\Delta C_{1-0}} = \frac{2.693 - 0}{2.000 - 0} = 1.347 \quad (>1)$$

According to the decision criterion in Eq. (7.8), $x = 1$ is preferable to $x = 0$. Note that the column $x - y$ refers to the pairwise comparison between the higher cost alternative x and the alternative y which is the last lower cost alternative accepted. Consequently, we must complete the computation for each line in Table 7.5 before we can decide which pair is to be compared in the next line. That is, because $x = 1$ is chosen at the end of the computation in the first line, we have $y = 1$ in the second line; similarly, because $x = 1$ is also chosen at the end of the computation in the second line, we have $y = 1$ again in the third line. Thus,

$$\frac{\Delta B_{2-1}}{\Delta C_{2-1}} = \frac{3.600 - 2.693}{3.000 - 2.000} = \frac{0.907}{1.000} = 0.907 \quad (<1)$$

Hence, $x = 2$ is unacceptable and $x = 1$ should be retained. Next, we compute

$$\frac{\Delta B_{3-1}}{\Delta C_{3-1}} = \frac{4.746 - 2.693}{4.000 - 2.000} = \frac{2.053}{2.000} = 1.027 \quad (>1)$$

which indicates that $x = 3$ is preferable. The same procedure is repeated until $x = 6$ is chosen as the best when all the alternatives have been compared.

We may note in passing that if the net present value method had been used, we only need to compute for each of the alternatives ($x = 1, 2, \ldots, 7$)

$$NPV_x = BPV_x - CPV_x$$

and the alternative with the highest nonnegative value of NPV_x is the best among all mutually exclusive proposals. The method also yields the alternative $x = 6$ as the best choice.

7.7 TREATMENT OF ADDITIONAL BENEFITS OR REDUCED COSTS

It has been pointed out in Section 7.3 that in the course of assessing benefits and costs, some items may exist that can be interpreted either as additional benefits or reduced costs. The treatment of such an item as a *positive* benefit or a *negative* cost will affect the value of the benefit-cost ratio of a project in the direct analysis but will not affect the acceptability of the project. The same situation may also occur in the incremental benefit-cost ratio analysis. The treatment of such items one way or another will produce different incremental benefit-cost ratios but will not change the selection among the alternatives if the criterion in Eq. (7.8) is followed.

The same reasoning can be applied to similar situations in which the value of the incremental benefit-cost ratio can be changed by the relative values of benefits and costs used in the computation. For example, the cash flow profile of a project x is often represented by the *net* annual cash flows $A_{t,x}$ (for $t = 0, 1, 2, \ldots$) instead of separate time streams of benefits $B_{t,x}$ and costs $C_{t,x}$ (both for $t = 0, 1, 2, \ldots$). The use of $A_{t,x}$ for all mutually exclusive projects will produce the same selection among these alternatives as the use of a combination of $B_{t,x}$ and $C_{t,x}$ for all projects.

We shall introduce a simple example to illustrate that Eq. (7.8) is valid even though some items in each of the mutually exclusive projects may be regarded as either additional benefits or reduced costs. Let d_x be the present value of an item which may be interpreted either as an additional benefit or a reduced cost for the alternative x, while b_x and c_x are, respectively, the present values of benefits and costs of the remaining items in the cash flow profile of the alternative x. Let d_y, b_y, and c_y be similarly defined for the alternative y which refers to the last lower cost alternative accepted. Then, the incremental benefit-cost ratio $\Delta B_x/\Delta C_x$ may be computed for the following two cases: (1) d_x and d_y are treated as additional benefits, and (2) d_x and d_y are treated as reduced costs. Thus, from Eq. (7.7), we get

$$(1)\ \frac{\Delta B_{x-y}}{\Delta C_{x-y}} = \frac{(b_x + d_x) - (b_y + d_y)}{c_x - c_y} = \frac{(b_x - b_y) + (d_x - d_y)}{c_x - c_y}$$

$$(2)\ \frac{\Delta B_{x-y}}{\Delta C_{x-y}} = \frac{b_x - b_y}{(c_x - d_x) - (c_y - d_y)} = \frac{b_x - b_y}{(c_x - c_y) - (d_x - d_y)}$$

However, if

$$\frac{(b_x + d_x) - (b_y + d_y)}{c_x - c_y} \geq 1$$

it follows that

$$(b_x - b_y) + (d_x - d_y) \geq (c_x - c_y)$$

By transposing terms we get

$$(b_x - b_y) \geq (c_x - c_y) - (d_x - d_y)$$

Consequently,

$$\frac{b_x - b_y}{(c_x - c_y) - (d_x - d_y)} \geq 1$$

Thus, in a pairwise comparison, if the alternative x is preferable over the alternative y in the first case, the same condition also holds in the second case. Therefore, the criterion in Eq. (7.8) is valid for either case when the alternatives are arranged in the ascending order of the present values of their costs. However, such items should be treated consistently for all alternatives.

Example 7.6

Suppose that an extra item with a present value of d_x which can be treated either as an additional benefit or a reduced cost is discovered for each alternative in Example 7.4. The values of d_x and the present values of benefits and costs for these alternatives have been computed by treating (1) d_x as an added benefit, and (2) as a reduced cost as shown in Table 7.6. Determine the best alternative by using the incremental benefit-cost method for each case.

TABLE 7.6 TREATMENT OF ADDED BENEFITS OR REDUCED COSTS IN EXAMPLE 7.6

		Case 1		Case 2	
x	d_x	CPV_x	BPV_x	CPV_x	BPV_x
1	0.5	2	3 + 0.5 = 3.5	2 − 0.5 = 1.5	3
2	1.1	5	4 + 1.1 = 5.1	5 − 1.1 = 3.9	4
3	1.2	7	11 + 1.2 = 12.2	7 − 1.2 = 5.8	11
4	0.8	9	12 + 0.8 = 12.8	9 − 0.8 = 8.2	12

The incremental benefit-cost ratios for the two cases may be computed as shown in Table 7.7. Although the incremental benefit-cost ratios are different for the two cases, the decisions are identical.

TABLE 7.7 BENEFIT-COST RATIOS BASED ON BOTH CASES IN EXAMPLE 7.6

$\Delta B_{x-y}/\Delta C_{x-y}$	Case 1	Case 2	Decision
$\dfrac{\Delta B_{1-0}}{\Delta C_{1-0}}$	$\dfrac{3.5-0}{2-0}=1.75$	$\dfrac{3-0}{1.5-0}=2.00$	Choose $x = 1$
$\dfrac{\Delta B_{2-1}}{\Delta C_{2-1}}$	$\dfrac{5.1-3.5}{5-2}=0.53$	$\dfrac{4-3}{3.9-1.5}=0.42$	Choose $x = 1$
$\dfrac{\Delta B_{3-1}}{\Delta C_{3-1}}$	$\dfrac{12.2-3.5}{7-2}=1.74$	$\dfrac{11-3}{5.8-1.5}=1.86$	Choose $x = 3$
$\dfrac{\Delta B_{4-3}}{\Delta C_{4-3}}$	$\dfrac{12.8-12.2}{9-7}=-0.30$	$\dfrac{12-11}{8.2-5.8}=0.42$	Choose $x = 3$

Example 7.7

Suppose that the net annual cash flows of the seven mutually exclusive projects in Example 7.5 are obtained first and used as the basis for computing the incremental benefit-cost ratios. Using a MARR of 7%, select the best alternative from the incremental analysis.

If the net annual cash flows U_x (for $t = 1, 2, \ldots, 40$) are computed first from the given values of $\bar{B}_x$ and $\bar{C}_x$ for each project x in Example 7.5, we have

$$U_x = \bar{B}_x - \bar{C}_x$$

Since U_x (for $t = 1, 2, \ldots, 40$) is positive for each of the seven projects, the only net cost is the initial cost $C_{0,x}$ for a project. Then,

$$\text{CPV}_x = C_{0,x}$$

$$\text{BPV}_x = U_x(P \mid U, 7\%, 40)$$

From the data in Table 7.4 of Example 7.5, we compute CPV_x and BPV_x according to these formulas and the results are arranged in the ascending order of CPV_x as shown in Table 7.8. Note that project $x = 2$ follows project $x = 3$ because $\text{CPV}_3 = 1.867$ and $\text{CPV}_2 = 2.000$. Note also that $\text{CPV}_6 = \text{CPV}_7 = 5.600$, and when a tie exists, it does not matter which project is listed first. However, under such circumstances, either ΔC_{7-6} or ΔC_{6-7} is zero; consequently, the incremental benefit-cost ratio is negative infinity (less than 1). Alternatively, we can examine $\Delta B_{7-6} = -0.120$ or $\Delta B_{6-7} = 0.120$, which means that proposal $x = 7$ is not as good a proposal as $x = 6$. Hence, project $x = 6$ is selected.

TABLE 7.8 INCREMENTAL BENEFIT-COST ANALYSIS OF EXAMPLE 7.7

x	CPV_x	BPV_x	$x - y$	ΔC_{x-y}	ΔB_{x-y}	$\frac{\Delta B_{x-y}}{\Delta C_{x-y}}$	Decision
1	1.000	1.693	1 − 0	1.000	1.693	1.693	Choose 1
3	1.867	2.613	3 − 1	0.867	0.920	1.061	Choose 3
2	2.000	2.600	2 − 3	0.133	−0.013	−0.098	Choose 3
4	3.800	4.400	4 − 3	1.267	1.787	1.410	Choose 4
5	4.600	5.319	5 − 4	0.800	0.919	1.149	Choose 5
6	5.600	6.599	6 − 5	1.000	1.280	1.280	Choose 6
7	5.600	6.479	7 − 6	0	−0.120	$-\infty$	Choose 6

7.8 ORDERING ALTERNATIVES ON THE BASIS OF INITIAL COSTS

A common practice in the application of the incremental benefit-cost analysis for the selection of mutually exclusive proposals is to arrange them in the ascending order of the *initial costs* (at year 0) of the alternatives. In general, however, there is no advantage in ordering the alternatives according to the initial costs, and its use may introduce some unnecessary complications.

Let y be the lower cost alternative and x be the higher cost alternative on the basis of successive pairwise comparisons of *initial costs*. In each step, the alternative tentatively selected in the previous step is compared with the next alternative for consideration. The alternative with the lower initial cost is designated by y while the alternative with the higher initial cost is designated by x. Suppose that the alternatives are first arranged in the ascending order of the *initial costs*. Then the incremental cost $\Delta C_{x-y} = C_x - C_y$ used in Eq. (7.8) may be positive or negative since C_x and C_y are the respective *present values of the costs* of x and y, respectively, and their ranking is not necessarily the same as that based on the initial cost $C_{0,x}$ and $C_{0,y}$ of these alternatives. Therefore, we may have either $C_x > C_y$ or $C_x < C_y$ corresponding to ΔC_{x-y} greater than or less than zero, respectively.

If $C_x > C_y$ while $C_{0,x} > C_{0,y}$, $\Delta C_{x-y} = C_x - C_y$ is positive, and the criterion in Eq. (7.8) can be applied to the new ordering scheme based on initial costs. Note that when $\Delta B_{x-y} = B_x - B_y$ is positive, the incremental ratio $\Delta B_{x-y}/\Delta C_{x-y}$ may be greater than 1 or less than 1 (but greater than 0). When $B_{x-y} = B_x - B_y$ is negative, the incremental ratio is negative (less than 0 and hence less than 1).

If $C_x < C_y$ while $C_{0,x} > C_{0,y}$, $\Delta C_{x-y} = C_x - C_y$ is negative. Hence, the sign of the incremental ratio $\Delta B_{x-y}/\Delta C_{x-y}$ will be negative for a positive ΔB_{x-y}, and positive for a negative ΔB_{x-y}. Then, the criterion in Eq. (7.8) cannot be applied to the new ordering scheme based on initial costs and a different criterion reflecting this situation must be adopted. In that case, the alternative x with a higher initial cost $C_{0,x}$ is preferable if

$$\frac{\Delta B_{x-y}}{\Delta C_{x-y}} = \frac{B_x - B_y}{C_x - C_y} \leq 1 \tag{7.9}$$

Otherwise, the alternative y with a lower initial cost $C_{0,y}$ is preferable. If the initial costs of two or more alternatives are equal, it does not matter how they are ordered among themselves.

Since the ascending order of the initial costs of the alternatives is not always identical to that of their present values of costs, there is no reason to arrange the alternatives in ascending order of the initial cost at all. As a matter of fact, a different set of decision criteria may be developed for selecting the best among a set of alternatives when they are arranged in a random order or according to ascending order of their initial costs. This set of criteria can be stated as follows:

1. If $\Delta C_{x-y} > 0$ while $C_{0,x} - C_{0,y} > 0$, use the criterion in Eq. (7.8). That is, when the denominator of the incremental benefit-cost ratio is positive, the criterion is unambiguous, whether the numerator is positive or negative.
2. If $\Delta C_{x-y} < 0$ while $C_{0,x} - C_{0,y} > 0$, use the criterion in Eq. (7.9). When only the denominator ΔC_{x-y} is negative, the incremental benefit-cost ratio is also negative. Then, the alternative with a higher initial cost is always preferable. When both the numerator and denominator of the ratio are negative, the incremental benefit-cost ratio is positive. Then, the alternative with a higher initial cost is preferable if the ratio is *less than* or equal to 1; otherwise the alternative with a lower cost is preferable.

Example 7.8

The seven mutually exclusive projects in Example 7.5 have been rearranged according to the ascending order of the initial costs as shown in the first and second columns in Table 7.9. Furthermore, since the initial costs of alternatives $x = 7$ and $x = 6$ are equal, they can be ordered either way and $x = 7$ is listed ahead of $x = 6$ in the table. Select the best among the seven mutually

TABLE 7.9 INCREMENTAL BENEFIT-COST ANALYSIS OF EXAMPLE 7.8

x	$C_{0,x}$	CPV_x	BPV_x	$x - y$	ΔC_{x-y}	ΔB_{x-y}	$\frac{\Delta B_{x-y}}{\Delta C_{x-y}}$	Decision
1	1.000	2.000	2.693	1 − 0	2.000	2.693	1.347	Choose 1
3	1.867	4.000	4.746	3 − 1	2.000	2.053	1.027	Choose 3
2	2.000	3.000	3.600	2 − 3	−1.000	−1.146	1.146	Choose 3
4	3.800	5.000	5.599	4 − 3	1.000	0.853	0.853	Choose 3
5	4.600	6.000	6.719	5 − 3	2.000	1.973	0.987	Choose 3
7	5.600	8.000	8.879	7 − 3	4.000	4.133	1.033	Choose 7
6	5.600	7.000	7.999	6 − 7	−1.000	−0.880	0.880	Choose 6

exclusive proposals by using the incremental benefit-cost ratio method and arranging the alternatives according to the ascending order of the initial costs.

The values of CPV_x and BPV_x have already been computed in Example 7.5. The other steps are also similar to those in Example 7.5, except when we encounter a negative sign in ΔC_{x-y}. In such situations, the criterion in Eq. (7.9) instead of the one in Eq. (7.8) should be used. Thus, in the pairwise comparison between $x = 3$ and $x = 2$, we choose $x = 3$ (the alternative with *lower initial cost*) when ΔB_{x-y} is also negative and the ratio is *greater* than 1. Similarly, in the pairwise comparison between $x = 7$ and $x = 6$, we choose $x = 6$ (the alternative with *higher order* even though its initial cost is the same as that of alternative $x = 7$) when ΔB_{x-y} is also negative and the ratio is *less* than 1.

Example 7.9

The seven mutually exclusive projects in Example 7.5 are arranged in a random order as indicated by the first and second columns in Table 7.10. Select the best among the alternatives by using the decision criteria based on successive pairwise comparisons of their initial costs.

TABLE 7.10 INCREMENTAL BENEFIT-COST ANALYSIS OF EXAMPLE 7.9

x	$C_{0,x}$	CPV_x	BPV_x	$x - y$	ΔC_{x-y}	ΔB_{x-y}	$\frac{\Delta B_{x-y}}{\Delta C_{x-y}}$	Decision
3	1.867	4.000	4.746	3 − 0	4.000	4.746	1.187	Choose 3
2	2.000	3.000	3.600	2 − 3	−1.000	−1.146	1.146	Choose 3
4	2.800	5.000	5.599	4 − 3	1.000	0.863	0.853	Choose 3
7	5.600	8.000	8.879	7 − 3	4.000	4.133	1.033	Choose 7
1	1.000	2.000	2.693	7 − 1	6.000	6.186	1.031	Choose 7
6	5.600	7.000	7.999	7 − 6	1.000	0.880	0.880	Choose 6
5	4.600	6.000	6.719	6 − 5	1.000	1.280	1.280	Choose 6

Based on the values of CPV_x and BPV_x computed in Example 7.5, we can apply step by step the set of decision criteria developed in this section. Noting that y is the alternative with the lower initial cost and x is the alternative with the higher initial cost in each pairwise comparison, the values of ΔC_{x-y} and ΔB_{x-y} are computed accordingly. If ΔC_{x-y} is positive, the criterion in Eq. (7.8) is used; if ΔC_{x-y} is negative, as when $x = 2$ and $y = 3$, the criterion in Eq. (7.9) should be applied. When the initial costs of the two alternatives are tied, as in the case of $x = 6$ and $x = 7$, we can adopt either the pair of $x = 7$ and $y = 6$ or that of $x = 6$ and $y = 7$ for comparison. In Table 7.10, the criterion in Eq. (7.8) is used in connection with the pair of $x = 7$ and $y = 6$. If the pair of $x = 6$ and $y = 7$ were used, the criterion in Eq. (7.9) would be invoked to reach the same decision.

7.9 SUMMARY AND STUDY GUIDE

In this chapter, we have examined the problems associated with the use of benefit-cost ratio as a merit measure in accepting or rejecting an independent project. While different interpretations of additional benefits or reduced costs may influence the resulting benefit-cost ratios, they will not alter the decision on acceptability of an independent project.

However, since the computation of benefit-cost ratio is influenced by some items that may be treated as additional benefits or reduced costs, it is important to follow the standards of professional practice in treating these items for the same type of problems consistently such that the resulting benefit-cost ratios for different proposals will not be distorted. For example, in computing the benefit-cost ratio for highway and bus-transit improvements, there are recommended standards for treating reduction of user travel time in the manual published by the American Association of State Highway and Transportation Officials. Similar standards are also established in other fields, such as water resource projects, by pertinent professional organizations.

The consideration of a continuous set of alternatives leads to a better understanding of profit maximization. It can be shown that the project that produces the maximum net benefit generally does not lead to the maximum benefit-cost ratio and vice versa. Consequently, the benefit-cost ratios of mutually exclusive proposals cannot be used to rank their merits, and an incremental analysis must be introduced to find the best among these alternatives if the benefit-cost ratio criterion is used.

If a set of mutually exclusive proposals is first arranged according to the ascending order of their present values of costs, the choice of an alternative in pairwise comparison may be based on Eq. (7.8). On the other hand, if the set is first arranged according to the ascending order of their initial costs, Eq. (7.9) may be needed to supplement Eq. (7.8) in choosing an alternative in pairwise comparison when the incremental cost ΔC_{x-y} is negative. But then, the set of alternatives can be arranged in any random order if the decision criteria include both Eqs. (7.8) and (7.9).

Finally, knowing the complexity of incremental analysis required for selecting the best among a set of alternatives on the basis of the benefit-cost ratio decision criterion, why should we not just use the net present value decision criterion and ignore the roundabout procedure associated with the use of the benefit-cost ratio criterion? If we are interested in the benefit-cost ratio as a profit measure, why not compute it for the best project after that project is selected by using the net present value criterion? These are the questions worth pondering.

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7.3 *Manual on User Benefit Analysis of Highway and Bus-Transit Improvements*. American Association of State Highway and Transportation Officials, 1977.

7.4 Wohl, M., and C. Hendrickson, *Transportation Investment and Pricing Principles*. New York: Wiley, 1984.

PROBLEMS

P7.1 The benefit for a set of alternative projects with continuously increasing costs may be expressed by the function

$$B = 2\sqrt{C}$$

where the cost C is the present value of costs and B is the present value of benefits, both expressed in millions of dollars.

(a) Determine the maximum net benefit N and the corresponding cost C.

(b) Plot a graph showing the relation of net benefit N versus cost C for the range of values of $C = 0, 1, 2, 3, 4$, and 5.

P7.2 The benefit for a set of alternative projects with continuously increasing costs may be expressed by the function

$$B = 3C - \tfrac{1}{4}(C^2 + 7) \qquad C \geq 1$$

where B is the present value of benefits and C is the present value of costs over the planning horizon, both expressed in millions of dollars. Determine the value of C that will maximize the net benefit N and find the corresponding maximum value of N.

P7.3 Repeat Problem 7.2 if the benefit for a set of alternative proposals can be expressed by the continuous function:

$$B = 2C - C^2/6$$

P7.4 Select the best proposal among the alternatives in Problem P6.1 (Chapter 6), using the benefit-cost ratio method.

P7.5 Select the best proposal among the alternatives in Problem P6.2 (Chapter 6), using the benefit-cost ratio method.

P7.6 Select the best proposal among the alternatives in Problem P6.7 (Chapter 6), using the benefit-cost ratio method for the following values of MARR: (a) 10%, (b) 20%, and (c) 30%.

P7.7 Select the best proposal among the alternatives in Problem P6.8 (Chapter 6), using the benefit-cost ratio method for the following values of MARR: (a) 5%, (b) 10%, and (c) 15%.

P7.8 Six mutually exclusive projects are considered as alternatives for a recreation facility. The present values of benefits and costs based on an appropriate interest rate over the entire planning period for all four alternatives are given below. Select the alternative with maximum net benefits, using the benefit-cost ratio method.

Project alternative	CPV in $\$10^6$	BPV in $\$10^6$
1	2.0	2.688
2	3.1	3.300
3	4.2	4.550
4	5.2	5.610
5	6.0	6.724
6	7.1	7.975

P7.9 Three plans have been proposed for a small commercial building at a leased site. The building is expected to have a useful life of 40 years and no salvage value at the end of the period. The initial construction costs, the annual maintenance costs, and the annual benefits are shown below. The MARR is specified to be 6%. Select the best of three plans using the benefit-cost ratio method and computing the present values of benefits and costs in each of the following manners: (a) the benefit stream and the cost stream are treated as given, and (b) the net benefits for years 1 to 40 are first obtained from the difference between the given benefits and costs, and the net benefits are regarded as the benefit stream. All costs and benefits are expressed in thousands of dollars.

	$x = 1$		$x = 2$		$x = 3$	
t	$C_{t,1}$	$B_{t,1}$	$C_{t,2}$	$B_{t,2}$	$C_{t,3}$	$B_{t,3}$
0	200	0	250	0	300	0
1–40 (each)	40	63	30	52	55	77

P7.10 Repeat both (a) and (b) of Problem P7.9 if the alternatives are arranged according to the *initial costs* instead of the present values of costs for the incremental analysis in using the benefit-cost ratio method.

P7.11 A company is considering investing in a water purification system. Three alternatives are under consideration and their cash flow profiles are given below. The MARR specified is 10%. Select the best proposal among these alternatives, using the benefit-cost ratio method.

Year	System 1	System 2	System 3
0	−80,000	−120,000	−100,000
1	+40,000	+60,000	+35,000
2	+35,000	+50,000	+35,000
3	+30,000	+40,000	+35,000
4	+25,000	+30,000	+35,000

P7.12 For the building project in Example 5.19, the benefit-cost ratio can be expressed as follows:

$$\frac{\text{BPV}}{\text{CPV}} = \frac{B_t(P \mid U, i, n)}{C_0 + C_t(P \mid U, i, n)}$$

Find the variation of the benefit-cost ratio with respect to the variation of each of the parameters listed for the problem. Also, plot the results in the percentage variation of BPV/CPV versus the percentage variation of each parameter to show the sensitivity of the benefit-cost ratio.

CHAPTER

8 The Internal Rate of Return Method

8.1 NATURE AND SCOPE OF CURRENT USAGE

The concept of the internal rate of return and its applications to economic evaluation was first introduced in Chapter 2, and the computation of internal rate of return for multiperiod cash flows was discussed in Chapter 4. However, the use of the internal rate of return (IRR) as a method of economic evaluation for independent projects and mutually exclusive proposals was mentioned only briefly in Chapters 5 and 6, respectively. The internal rate of return method refers not only to the use of the IRR as a merit measure, but also the adoption of a set of decision rules based on the concept of IRR including the incremental analysis of mutually exclusive proposals. In this chapter, we consider thoroughly the basic principles and applications of the internal rate of return method for investment and borrowing opportunities involving conventional and unconventional cash flows.

The use of the internal rate of return (IRR) as a profit measure is very popular in the industrial and financial world. To the extent that a unique value of IRR exists for an investment or borrowing cash flow profile, its popularity is understandable. On the other hand, the application of the internal rate of return method to economic evaluation is often misunderstood and misused, particularly when the investment or borrowing cash flow profile is unconventional. The results of many surveys of private corporations indicate the widespread use of the internal rate of return method, and a summary of some surveys showing the percentages of the firms in many countries that used the IRR method was assembled by Rapp[1] in 1980 and is reproduced in

[1] See Ref. 8.4.

TABLE 8.1 SUMMARY OF SURVEYS ON THE USE OF THE IRR METHOD

Year of publication	Author of survey	Percentage of firms using the IRR methods
1967	Renck (Sweden)	50%
1973	Abdelsamad (U.S.)	69%
1973	Fremgen (U.S.)	71%
1973	Rockley (England)	59%
1974	Berry and Tanner (England)	76%
1975	Honko and Virtanen (Finland)	78%
1977	Andersson (Sweden)	54%
1978	Tell (Sweden)	53%

Source: Rapp, B., "The Internal Rate of Return Method—A Critical Study." *Engineering Costs and Production Economics,* Vol. 5, p. 44. Amsterdam: Elsevier Scientific, 1980.

Table 8.1. Other surveys in the United States and the United Kingdom also indicate similar trend of usage. However, it is not clear whether the IRR is merely used as a merit measure or is adopted as a method of analysis as well. For example, in a survey of the Fortune 1000 largest industrial corporations conducted in 1979, which yielded 200 complete returns, 49% of the firms reported using the IRR and 19% using the NPV as evaluation techniques[2]; similarly, in a survey based on the Fortune Service Directory conducted in 1983, which yielded 149 responses, 67% reported using the IRR and 57% using the NPV or both.[3] The survey of 100 large UK firms indicated that 42% used the IRR and 23% used the NPV while adopting the electronic spreadsheet as a computation tool.[4]

As this chapter unfolds, it will become clear why the treatment of the IRR in Chapter 4 is insufficient and why it is necessary to develop a complete set of decision criteria based on the multiple value of IRR for unconventional cash flows if the internal rate of return method is claimed to be as versatile as the net present value method for evaluating the acceptability of independent projects. It can also be seen that the set of decision rules established in Chapter 2 for the incremental analysis based on IRR for selecting mutually exclusive proposals of one-period duration is generally inadequate for selecting alternatives of multiperiod duration. That is why a more complete set of decision criteria must be developed for incremental analysis if the internal rate of return method is to be applied to obtain the same conclusion that can be reached directly and unambiguously by using the net present value method.

We shall define the internal rate of return method as the method of economic evaluation which primarily uses the internal rate of return as the decision criterion. We shall treat it as completely as possible so that it will at least be correctly used if an analyst so chooses. We shall adopt a complete set of decision criteria that will in-

[2] See Ref. 8.2.

[3] See Ref. 8.1.

[4] See Ref. 8.3.

clude all possible conditions whether they occur often or rarely.[5] Specifically, we will discuss two major aspects of application in order to differentiate their characteristics:

1. The use of the internal rate of return method in determining the acceptability of an independent project (in comparison with the null alternative)
2. The use of the internal rate of return method in the incremental analysis through successive pairwise comparisons for the selection of the best proposal among a set of mutually exclusive proposals

As you probably have already surmised, you do not need another method of analysis that requires roundabout procedures but leads to the same conclusion that could be reached by using the net present value decision criterion. However, in view of the popularity of the internal rate of return method, whether or not it is fully understood by its users, you should be prepared to protect yourself when you need it.

A case in point is the examination for registered professional engineers which many aspiring engineers take as a rite of passage. If you look at the examination files in engineering economics, you will find questions that need answers based on the internal rate of return method. The most telling state of mind of some examiners is a "tutorial" written by an influential engineer who served two 3-year terms on the Certification Board of the American Association of Cost Engineers. The author of this tutorial observed,[6] "The fact that only 17 percent of recent certification candidates understood the need for incremental analysis in capital budgeting has led to this short tutorial." Then he went on to give an example of finding the best alternative among four mutually exclusive proposals A, B, C, and D, using the incremental analysis based on IRR. Finally, he created an analysis based on "incremental net present value" and concluded, "The highest net present value is for C, which is the preferred choice as shown by the incremental net present value (which is the only positive one). *This is coincidental.*" Actually, since C has the highest net present value for the example cited, it is the preferred choice; and the incremental net present value has nothing to do with the decision. You may wonder whether 17% of the candidates do not know the incremental analysis based on IRR, or whether they know too well that incremental analysis is unnecessary if the NPV decision criterion is used!

Unfortunately, this is not an isolated incident. Some professionals who use IRR as a merit measure still insist that the choice of the best among a set of mutually exclusive proposals be based on an incremental analysis related to the IRR. You must be able to recognize incorrect and/or misleading interpretations and counter such arguments with understanding.

[5] A correct procedure covering virtually all possible conditions in application has been stated in Ref. 8.6. The rationale of this procedure is developed further in this chapter. The authors are indebted to M. Wohl for sharing the idea about this procedure through private communication prior to its publication. See also Ref. 8.7.

[6] See Ref. 8.5.

Example 8.1

Consider two mutually exclusive investment proposals $x = 1$ and $x = 2$ with the cash flow profiles given in the $A_{1,t}$ and $A_{2,t}$ below:

t	$A_{1,t}$	$A_{2,t}$	$\Delta A_{1-2,t}$	$\Delta A_{2-1,t}$
0	−\$1,500	−\$1,500	0	0
1	+750	+705	+45	−45
2	+750	+800	−50	+50
3	+750	+745	+5	−5

Find the incremental cash flow profile of proposal 1 over proposal 2.

The incremental cash flow profile resulting from the two investment proposals is obtained from the difference of the two profiles (project 1 over project 2) as indicated by the column $\Delta A_{1-2,t}$ in the above tabulation. If the incremental cash flow profile of proposal 2 over proposal 1 is sought, all signs of the coefficients will be reversed as shown in the column $\Delta A_{2-1,t}$. Suppose that we attempt to find the best alternative on the basis of the decision criterion for incremental internal rate of return analysis in Section 2.8 (Chapter 2). We immediately encounter some unexpected difficulties.

First, the example shows that the incremental cash flow profile resulting from two conventional investment proposals can be unconventional with multiple sign changes. Second, it illustrates the dilemma of ordering the two proposals according to their initial costs in order to use the IRR decision criterion in Section 2.8 when the initial costs are equal. That is why we need to develop robust and ironclad procedures in this chapter if the IRR method is to be applied to such problems.

8.2 THE INTERNAL RATE OF RETURN AS A PROFIT MEASURE

It was pointed out in Section 4.3 that the IRR resulting from a pure investment opportunity represents the *internal rate of return to the investor,* while the IRR resulting from a pure borrowing opportunity represents the internal rate of borrowing (or the *internal rate of return to the lender*), even though both are customarily referred to as the internal rate of return. The unique value of the IRR for a pure investment or pure borrowing opportunity is a useful merit measure of the percentage of return for the funds committed in accordance with the cash flow profile. However, the IRR is an absolute measure that is independent of the minimum attractive rate of return (MARR), and must be compared to the MARR to evaluate the acceptability of an independent project.

The set of decision rules developed in Section 4.3 for pure investment and pure borrowing opportunities are as follows:

1. For a pure investment opportunity, accept if IRR $\geq$ MARR; reject otherwise.
2. For a pure borrowing opportunity, accept if IRR $\leq$ MARR; reject otherwise.

Using mathematical notation, let the IRR for an independent investment project x be denoted by i'_x and the specified MARR be denoted by i^*. Ignoring the limiting case of $i^* = i'_x$ the criterion for accepting project x is

$$i^* < i'_x \tag{8.1}$$

Otherwise, it should be rejected. Such a decision criterion leads to the same result obtained on the basis of the net present value criterion, which can be verified by noting that NPV_x is positive at i^* when $i^* < i'_x$ as shown in the NPV versus i graph in Fig. 8.1. In the case of borrowing or financing, the criterion for accepting a loan x by the borrower is

$$i^* > i'_x \tag{8.2}$$

where i'_x is the IRR of the loan x to the lender. The meaning of this criterion is illustrated in the NPV versus i graph in Fig. 8.2.

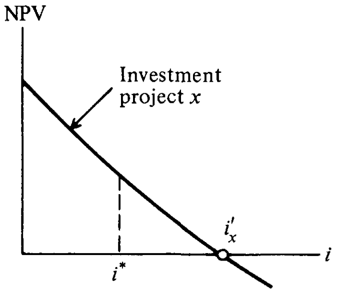

Figure 8.1 Conventional investment project.

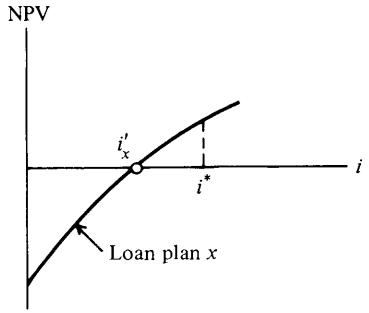

Figure 8.2 Conventional loan plan.

When multiple values of IRR occur for an unconventional investment or borrowing opportunity, none of these values can be used as a merit measure because each IRR represents the internal rate of return for some periods and the internal rate of borrowing for other periods of the cash flow profile. This does not mean that a set of decision criteria cannot be developed on the basis of the multiple values of the IRR; it only means that we need to use a roundabout approach that requires more computation than the net present value method. After all, when you are presented with an unconventional investment opportunity or an unconventional borrowing opportunity, you do not want to turn it down just because you cannot find a unique value of IRR; you should be able to evaluate the opportunity on the basis of the net present value decision criterion, or indirectly, a set of internal rate of return criteria that is compatible with the net present value decision criterion.

8.3 DECISION CRITERIA FOR ACCEPTING INDEPENDENT PROJECTS

Because of the possibility of multiple internal rates of return for investment and financing projects, a more complete set of decision criteria is needed to cover all possible conditions, as shown in Fig. 8.3. Furthermore, the availability of a complete set of criteria for determining the economic feasibility of independent projects

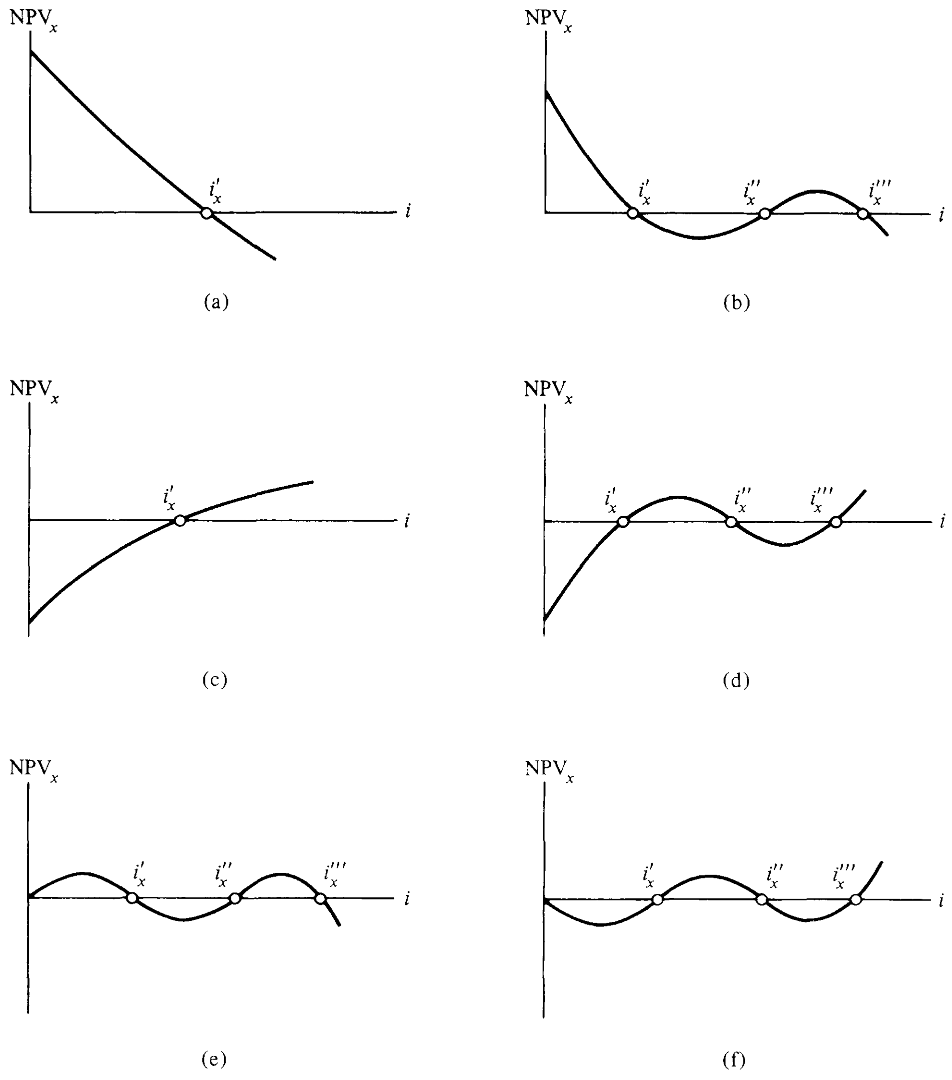

Figure 8.3 NPV versus i graph for various types of cash flows.

will facilitate the development of a consistent procedure for incremental analysis in selecting the best proposal from a set of mutually exclusive alternatives.

In examining the conditions in Fig. 8.3, we note that the net present value NPV at $i = 0\%$ is positive for cases (a) and (b), negative for cases (c) and (d), and zero for cases (e) and (f). Therefore, we use $[\text{NPV}]_{0\%}$ as the basis for classifying various conditions. When $[\text{NPV}]_{0\%}$ is positive or negative, we can examine the case with a single value of IRR and that with multiple values of IRR separately. When $[\text{NPV}]_{0\%} = 0$, we divide the conditions into the case with a positive initial slope and one with a negative initial slope of the curve at $i = 0\%$. Since the slope of the curve can be represented by the derivative, $d(\text{NPV})/di$, or simply dN/di, the sign of the initial slope will be indicated by $[dN/di]_{0\%}$. The calculation of this derivative from a given cash flow profile will be discussed later.

Theoretically, we may also encounter the case of a zero initial slope at $i = 0\%$ when $[\text{NPV}]_{0\%} = 0$, i.e., we may have the situation that both $[\text{NPV}]_{0\%} = 0$ and $[dN/di]_{0\%} = 0$. The set of criteria can be extended to cover this situation by considering the higher derivatives at $i = 0\%$. The calculation of the second derivative from a given cash flow profile will also be discussed later, and by using the same principle, higher derivatives can also be obtained if necessary. That is, we should consider whether the second derivative at $i = 0\%$ is positive, negative, or zero. If it is zero, we should test the third derivative, etc. Since the occurrence of such cases is very unlikely, they are not included here.

Basically, the decision criteria based on the internal rate(s) of return for all these conditions must be consistent with the net present value criterion represented by Eq. (5.6) in Chapter 5. Let IRR_x be the internal rate of return on a project x. If there is only a single value of IRR_x, it will be denoted as i'_x; if there are multiple values of IRR_x, they will be designated by i'_x, i''_x, etc., starting from the lowest to the highest value except when one of the values is zero. In that case, the zero value will not be counted in the designation. Furthermore, if a value of i_x occurs at the tangent point of NPV_x curve to the horizontal axis, it represents a repeated root and should be counted as two equal values of IRR_x. Since the MARR is used as the discount rate i in computing the NPV, the MARR specified for the proposed project x is denoted by i^*. Then, we may observe from Fig. 8.3 the following conditions based on the net present value criterion:

(a) When NPV_x is positive at $i = 0\%$ and there is only a single value of IRR_x, the project x is acceptable if $i^* \le i'_x$.

(b) When NPV_x is positive at $i = 0\%$ and there are multiple values of IRR_x, the project x is acceptable if $i^* \le i'_x$, unacceptable if $i'_x < i^* < i''_x$, and acceptable again if $i''_x \le i^* \le i'''_x$, etc.

(c) When NPV_x is negative at $i = 0\%$ and there is only a single value of IRR_x, the project x is acceptable if $i^* \ge i'_x$.

(d) When NPV_x is negative at $i = 0\%$ and there are multiple values of IRR_x, the project x is unacceptable if $i^* < i'_x$, acceptable if $i'_x \le i^* \le i''_x$, and unacceptable again if $i''_x < i^* < i'''_x$, etc.

(e) When $NPV_x = 0$ at $i = 0\%$ and the slope dN_x/di is positive at $i = 0\%$, the project x is acceptable if $i^* \leq i'_x$, unacceptable if $i'_x < i^* < i''_x$, acceptable again if $i''_x \leq i^* \leq i'''_x$, etc. The value of IRR = 0% will not be counted.

(f) When $NPV_x = 0$ at $i = 0\%$ and the slope dN_x/di is negative at $i = 0\%$, the project x is unacceptable if $i^* < i'_x$, acceptable if $i'_x \leq i^* \leq i''_x$, and unacceptable again if $i''_x < i^* < i'''_x$, etc. The value of IRR = 0% will not be counted.

It should be noted that for $i^* = i'_x, i''_x, \ldots$, we are really indifferent to accepting or rejecting project x. The equality signs associated with the acceptable limits in the above discussion merely reflect the fact that at $i^* = i'_x, i''_x, \ldots$ project x is at least as good as the best opportunity foregone and therefore is acceptable. These equality signs can be omitted if so desired. Furthermore, since we are primarily interested in the range of i^* that is nonnegative, the condition of $0 < i^*$ can be added to the above inequalities wherever appropriate. Thus, the decision criteria for accepting independent projects can be summarized as in Table 8.2.

TABLE 8.2 DECISION CRITERIA FOR ACCEPTING INDEPENDENT PROJECTS

			$[NPV_x]_{0\%} = 0$ ($i_x = 0$ is not counted)	
Range of i^*	$[NPV_x]_{0\%} > 0$	$[NPV_x]_{0\%} < 0$	$[dN_x/di]_{0\%} > 0$	$[dN_x/di]_{0\%} < 0$
$0 \leq i^* < i'_x$	Accept	Reject	Accept	Reject
$i'_x < i^* < i''_x$	Reject	Accept	Reject	Accept
$i''_x < i^* < i'''_x$	Accept	Reject	Accept	Reject
$i'''_x < i^* < i''''_x$	Reject	Accept	Reject	Accept
Additional i_x	Alternate accept/reject	Alternate reject/accept	Alternate accept/reject	Alternate reject/accept
Negative or indeterminate	Accept	Reject		

The decision criteria based on Fig. 8.3 do not cover the special cases in which the cash flow profile has no sign change. For such cases, the internal rate of return is either indeterminate or negative as shown in Figs. 8.4 and 8.5. When the IRR is indeterminate, the project x may lead to a positive NPV_x for all values of i_x as represented by the parallel line above the horizontal axis in Fig. 8.4, or to a negative NPV for all values of i_x, as represented by a parallel line below the horizontal axis. When the value(s) of the IRR are negative, the project x may lead to a $[NPV_x]_{0\%}$ greater than zero, as shown in Fig. 8.5, or to a $[NPV_x]_{0\%}$ less than zero, as represented by the mirror image of Fig. 8.5 with respect to the horizontal axis. Conse-

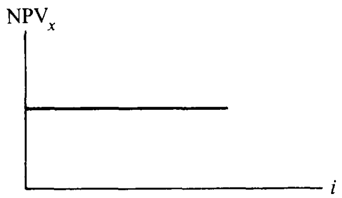

Figure 8.4 NPV versus i graph for no finite IRR.

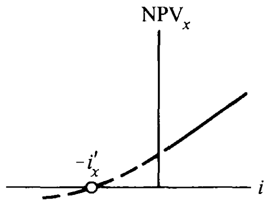

Figure 8.5 NPV versus i graph for negative IRR.

quently, when all IRR are either indeterminate or negative, the decision criterion is to accept project x if $[\text{NPV}_x]_{0\%} > 0$ and to reject project x if $[\text{NPV}_x]_{0\%} < 0$.

The use of the decision criteria developed in this section will lead to the same results of project acceptance as determined by the net present value criterion. However, the amount of work involved is quite substantial since the determination of the internal rate(s) of return requires the numerical solution of a polynomial equation of the nth power where n is the number of years in the planning horizon.

8.4 COMPUTATION OF NET PRESENT VALUE AND ITS DERIVATIVES AT ZERO DISCOUNT RATE

The decision criteria for evaluation of independent projects based on the internal rate(s) of return discussed in the previous section require the knowledge of $[\text{NPV}_x]_{0\%}$ and, in some cases, $[dN_x/di]_{0\%}$ also in the ordering of independent projects. In order to compute these quantities without making reference to the NPV versus i graph, we begin with Eq. (5.8) in Chapter 5, i.e.,

$$\text{NPV}_x = \sum_{t=0}^{n} (B_{t,x} - C_{t,x})(1 + i)^{-t}$$

or

$$\text{NPV}_x = \sum_{t=0}^{n} A_{t,x}(1 + i)^{-t}$$

Then, the net present value of project x at $i = 0\%$ is

$$[\text{NPV}_x]_{0\%} = \sum_{t=0}^{n} (B_{t,x} - C_{t,x}) \tag{8.3}$$

or

$$[\text{NPV}_x]_{0\%} = \sum_{t=0}^{n} A_{t,x} \tag{8.4}$$

To find the slope of the net present value curve of project x at $i = 0\%$, we first find the derivative of NPV_x. Thus,

$$\frac{dN_x}{di} = \frac{d}{di}\left[\sum_{t=0}^{n} (B_{t,x} - C_{t,x})(1 + i)^{-t}\right]$$

$$= -\sum_{t=1}^{n} t(B_{t,x} - C_{t,x})(1 + i)^{-(t+1)}$$

For $i = 0\%$,

$$\left[\frac{dN_x}{di}\right]_{0\%} = -\sum_{t=1}^{n} t(B_{t,x} - C_{t,x}) \tag{8.5}$$

or

$$\left[\frac{dN_x}{di}\right]_{0\%} = -\sum_{t=1}^{n} tA_{t,x} \tag{8.6}$$

If we want to find the second derivative of NPV_x for solving the case when both $[NPV_x]_{0\%} = 0$ and $[dN_x/di]_{0\%} = 0$, we get

$$\frac{d^2N_x}{di^2} = +\sum_{t=1}^{n} t(t + 1)(B_{t,x} - C_{t,x})(1 + i)^{-(t+2)}$$

For $i = 0\%$,

$$\left[\frac{d^2N_x}{di^2}\right]_{0\%} = \sum_{t=1}^{n} t(t + 1)(B_{t,x} - C_{t,x}) \tag{8.7}$$

or

$$\left[\frac{d^2N_x}{di^2}\right]_{0\%} = \sum_{t=1}^{n} t(t + 1)A_{t,x} \tag{8.8}$$

Consequently, the computation of the net present value and its derivatives of a project can be carried out systematically by using these equations.

Example 8.2

The cash flow profiles of three independent projects are shown in Table 8.3. Using a MARR of 10%, apply the decision criteria based on the internal rate of return to determine if each of these projects is acceptable.

TABLE 8.3 CASH FLOWS (IN $1000) FOR THREE INDEPENDENT PROJECTS

Year t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−545	−340	−340
1–7 (each)	+150	+150	+150
8	0	−830	−710

For $x = 1$, there is only a single rate of return because of one sign change in the cash flow profile. It is seen that

$$[NPV_1]_{0\%} = -545 + (7)(150) = 505$$

The internal rate of return i_1 can be obtained by interpolation from $[NPV_1]_{15\%} = +79.1$ and $[NPV]_{20\%} = -4.3$. Thus,

$$i = 15\% + (20\% - 15\%)\frac{79.1}{79.1 + 4.3} = 19.7\%$$

Since the specified MARR is $i^* = 10\%$, we have $19.7\% > 10\%$. Hence, the proposed project is acceptable.

For $x = 2$, $[NPV_2]_{0\%} = -120$, and two values of i_2 have been found (9.6% and 22.9%). For the specified MARR $i^* = 10\%$, we have $9.6\% < 10\% < 22.9\%$. Hence, the proposed project is again acceptable. However, it should be cautioned again that 10% is very close to 9.6%, and any slight change in the MARR to a lower value (say to 9%) would affect the acceptability of the proposed project.

For $x = 3$, $[NPV_3]_{0\%} = 0$, and two values of i_3 have been found (0% and 28.4%). We therefore will compute from Eq. (8.6) the following:

$$\left[\frac{dN_3}{di}\right]_{0\%} = -[(1 + 2 + 3 + 4 + 5 + 6 + 7)(150) + (8)(-710)]$$
$$= -[(28)(150) - (8)(710)] = +1{,}480$$

Ignore the value of 0% and consider 28.4% as the lowest value of i_3. Then, for the specified MARR $i^* = 10\%$, we have $28.4\% > 10\%$. Hence, the proposed project is acceptable.

8.5 THE CONCEPT OF INCREMENTAL INTERNAL RATE OF RETURN ANALYSIS

In the selection of the best proposal from a set of mutually exclusive alternatives, the alternative with the highest internal rate of return is not necessarily the alternative with the maximum net present value. This disagreement can easily be seen in the NPV versus MARR graph in Fig. 8.6(a). Let us assume that each of the three alternatives ($x = 1, 2, 3$) has only a single value of $IRR_x = i_x$. If we consider each project separately and note that $i^* < i_1$, $i^* < i_2$, and $i^* < i_3$, we conclude that all three alternatives are economically acceptable. However, since we are interested in selecting only the best one among these three alternatives, we have to rank them on the basis of their profit potential and select the one that has the greatest merit. At the specified MARR of i^*, the alternative $x = 2$ is clearly the best, $x = 3$ the second best, and $x = 1$ is the last in spite of the fact that $i_1 > i_2 > i_3$.

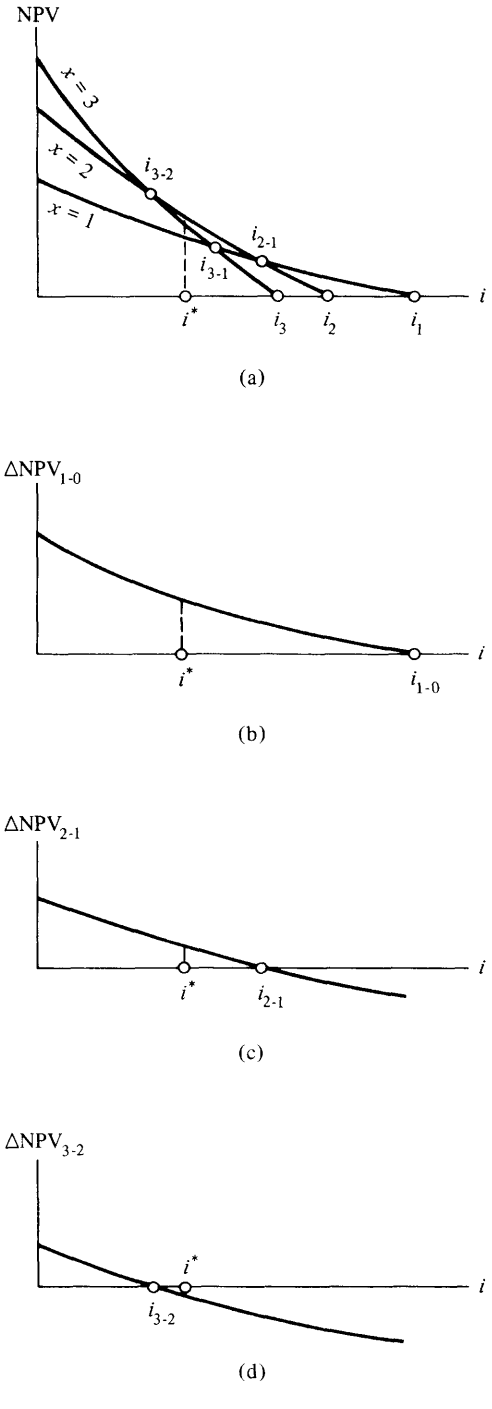

Figure 8.6 Incremental IRR.

The reason why the ranking $i_1 > i_2 > i_3$ cannot be used as the ranking of the merits of alternatives $x = 1$, $x = 2$, and $x = 3$ is that the internal rate of return is not a *direct* measure of the profit potential expressed in the discounted cash flows at a specified time, e.g., the net present value. If we select $x = 1$ which has the highest IRR $= i_1$, we will deprive ourselves of the opportunity of investing an additional amount of money to realize a bigger profit or net benefit (in terms of the net present value) as represented by the alternative $x = 2$. Therefore, we should make successive pairwise comparisons to determine whether the incremental internal rate of return (IIRR) of the additional amount of investment is worthwhile. However, in using the IIRR analysis, we do not compute the present values of costs of the alternatives at the specified MARR $= i^*$ as we do in the incremental benefit-cost analysis. Consequently, we cannot arrange the alternatives in the ascending order of the present values of their costs for the purpose of making successive pairwise comparisons.

One way to overcome this problem is to arrange the alternatives in the ascending order of their net present value at MARR = 0%, since $[\text{NPV}_x]_{0\%}$ for any alternative x can readily be obtained from its cash flow profile without discounting. Referring again to Fig. 8.6(a), the net present values at $i = 0\%$ for $x = 1, 2$, and 3 are in the ascending order. We shall defer the discussion of ordering the alternatives under special conditions, such as when the net present values at $i = 0\%$ for two alternatives are equal. In this section, we shall only examine the basic concept of the IIRR analysis for the example in Fig. 8.6(a).

Let us emphasize again that each of the three alternatives ($x = 1, 2$, and 3) in Fig. 8.6(a) is assumed to have only a single value of IRR_x. Then, we may consider the internal rate of return i_1 as the incremental internal rate of return i_{1-0} for the pairwise comparison between $x = 1$ and the null alternative $x = 0$. Thus, we can plot a ΔNPV_{1-0} versus i graph as shown in Fig. 8.6(b). Note that because $\text{NPV}_0 = 0$ for the null alternative $x = 0$, we have

$$\Delta\text{NPV}_{1-0} = \text{NPV}_1 - \text{NPV}_0 = \text{NPV}_1$$

Since $i_{1-0} > i^*$ as indicated in the figure, we choose $x = 1$ over $x = 0$. Next, we compare $x = 1$ with the next higher order alternative $x = 2$ and obtain a ΔNPV_{2-1} versus i graph as shown in Fig. 8.6(c) in which

$$\Delta\text{NPV}_{2-1} = \text{NPV}_2 - \text{NPV}_1$$

With $i_{2-1} > i^*$ as shown in the figure we choose $x = 2$ over $x = 1$. Finally, we compare $x = 2$ with the next higher order alternative $x = 3$ and obtain a ΔNPV_{3-2} versus i graph as shown in Fig. 8.6(d) in which

$$\Delta\text{NPV}_{3-2} = \text{NPV}_3 - \text{NPV}_2$$

Because $i_{3-2} < i^*$ as shown in the figure, we choose $x = 2$ over $x = 3$; hence $x = 2$ is the best among the three mutually exclusive alternatives.

Note that i_{1-0} in Fig. 8.6(b) is identical to i_1 in Fig. 8.6(a), and that i_{2-1} in Fig. 8.6(c) and i_{3-2} in Fig. 8.6(d) are identical to i_{2-1} and i_{3-2} in Fig. 8.6(a), respectively. Since we have not compared alternative $x = 1$ and $x = 3$ directly as a pair, i_{3-1} in Fig. 8.6(a) is not compared to i^*. Thus, for the example in Fig. 8.6(a), we have made pairwise comparisons in the ascending order of the net present values at $i = 0\%$ for a set of mutually exclusive alternatives, including the null alternative. At the successive stages of the IIRR analysis, we must find the values of i_{1-0}, i_{2-1}, and i_{3-1} because they are used to measure against the MARR = i^*. The computation of these IIRR values can be quite complicated, depending to a large extent on the cash flow profiles of the alternatives, but the methods of computation have been discussed in detail in Chapter 4.

Example 8.3

Select the best among the three mutually exclusive alternatives in Table 8.4, using the incremental internal rate of return analysis, if the MARR specified is 20%.

For the given alternatives, the net present values at $i = 0\%$ as represented by the algebraic sums of the respective cash flow profiles are also computed as shown in Table 8.4.

Since the planning horizon for this simple example consists of only one interest period, the computation of IRR and IIRR is very simple, i.e., to find the percentage gain over a 1-year period since each alternative has only a single value of IRR. The alternatives $x = 1$, 2, and 3 are arranged in the ascending order of $[NPV_x]_{0\%}$. Thus, we begin by comparing the lowest order alternative $x = 1$ with the null alternative $x = 0$. That is,

$$i_{1-0} = i_1 = \frac{150 - 100}{100} = \frac{50}{100} = 50\% \quad (>20\%)$$

Hence, the alternative $x = 1$ is preferable. Next we compare $x = 1$ with the next higher order alternative by computing

$$i_{2-1} = \frac{(280 - 150) - (200 - 100)}{200 - 100} = \frac{30}{100} = 30\% \quad (>20\%)$$

Hence, the alternative $x = 2$ is preferable. Finally, we compare $x = 2$ with the next higher order alternative by computing

TABLE 8.4 CASH FLOW PROFILES OF THREE ALTERNATIVES

t	$A_{1,t}$ $(x = 1)$	$A_{2,t}$ $(x = 2)$	$A_{3,t}$ $(x = 3)$
0	−100	−200	−300
1	+150	+280	+395
$[NPV_x]_{0\%}$	+ 50	+ 80	+ 95

$$i_{3-2} = \frac{(395 - 280) - (300 - 200)}{300 - 200} = \frac{15}{100} = 15\% \quad (<20\%)$$

Again, we choose the alternative $x = 2$, which is the best among the three mutually exclusive alternatives.

Note that by taking the first step to compare $x = 1$ with the null alternative, we are assured that the final choice is economically feasible as well as being the best among the set of mutually exclusive alternatives.

8.6 ORDERING MUTUALLY EXCLUSIVE ALTERNATIVES FOR PAIRWISE COMPARISONS

After introducing the basic concept of incremental internal rate of return analysis, we can now consider the ordering of a set of mutually exclusive alternatives for successive pairwise comparisons in detail. We shall adopt the following notation:

y = the lower order alternative previously accepted prior to the comparison with the higher order alternative x

i_{x-y} = the incremental internal rate of return for the pairwise comparison between the alternatives x and y

ΔNPV_{x-y} = the difference of the net present values of the cash flow profiles of alternatives x and y, i.e., $NPV_x - NPV_y$

$d(\Delta N)/di$ = the first derivative of ΔNPV_{x-y} with respect to i

We recommend arranging the alternatives in the ascending order of their net present values at $i = 0\%$ because this arrangement is the basis of a correct and workable procedure covering all possible conditions for determining the economic acceptability of the alternatives as well as for selecting the best among all alternatives.

In arranging the mutually exclusive alternatives in the ascending order of $[NPV_x]_{0\%}$, we start with the alternative having the lowest *algebraic* value. This arrangement is always possible as long as the values of $[NPV_x]_{0\%}$ for all alternatives are distinct (unequal). In a pairwise comparison between alternatives x and y, the alternative with the *lower algebraic value* can have a $[NPV_x]_{0\%}$ that is positive, negative, or zero. Figure 8.7 shows examples of pairwise combinations of the algebraic values of $[NPV_x]_{0\%}$ which are unequal. Figure 8.8 also shows cases with unequal $[NPV_x]_{0\%}$, but the NPV curves representing the two alternatives do not intersect in the positive range of i.

If the values of $[NPV_x]_{0\%}$ for two or more alternatives are equal, they may be all positive, all negative, or all zero. In either case, we must seek a way to break the tie in ordering them. We can first compute the values of $[dN_x/di]_{0\%}$ for these alternatives, using Eq. (8.5) or (8.6). Then, we arrange them according to the ascending order of $[dN_x/di]_{0\%}$, starting with the alternative having the *lowest* algebraic value. Recalling that dN_x/di represents the slope of the NPV versus i curve, this arrangement will simply follow the ascending order of the *algebraic values of the initial slopes* of these curves. Figure 8.9 shows examples of pairwise combinations of the

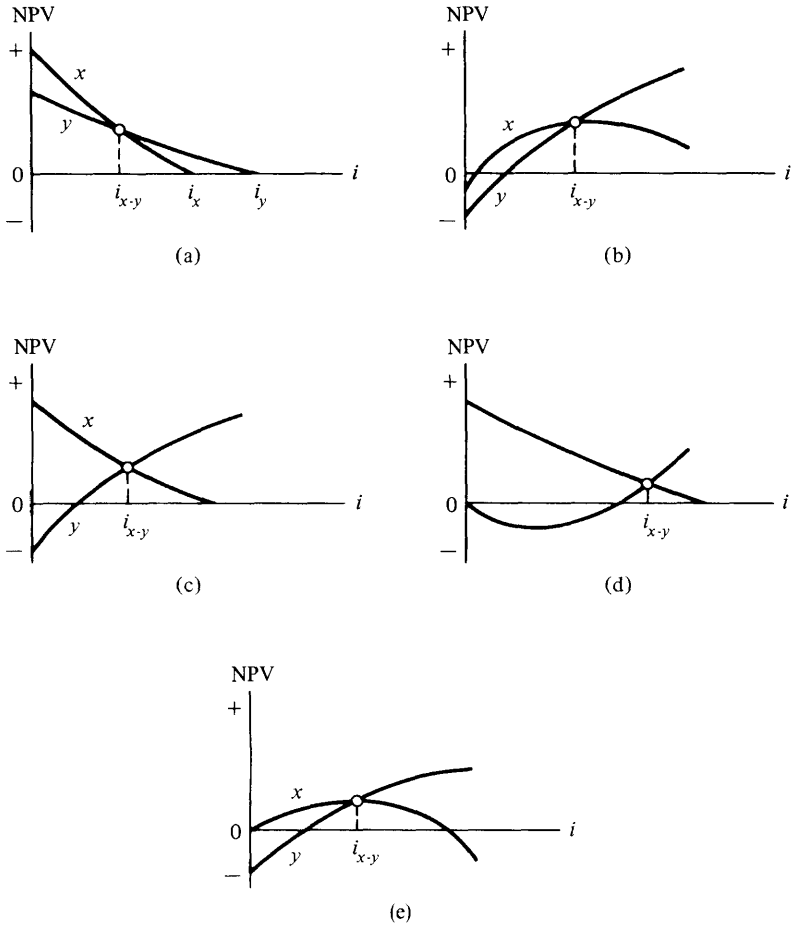

Figure 8.7 Incremental IRR resulting from various types of cash flows.

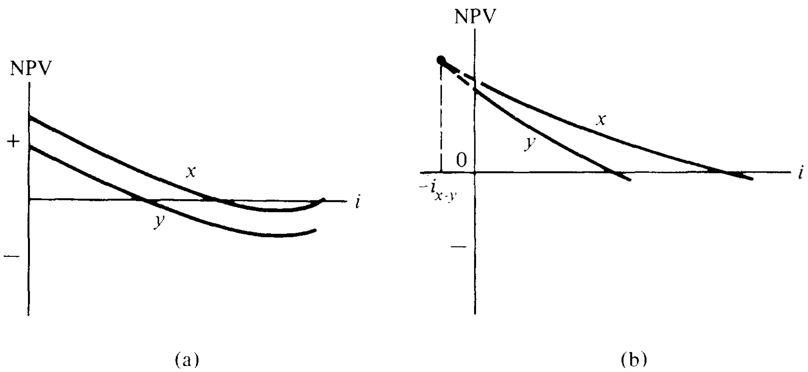

Figure 8.8 No positive incremental IRR for special cases.

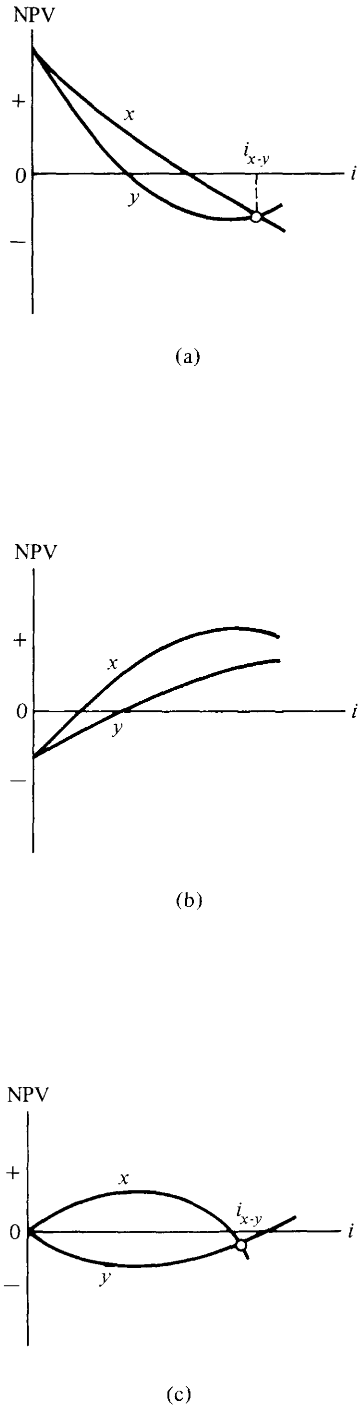

Figure 8.9 Projects with same NPV at $i = 0$.

algebraic values of $[\text{NPV}_x]_{0\%}$ that are equal, and hence the alternatives are arranged according to the algebraic values of the initial slopes $[dN_x/di]_{0\%}$.

To recapitulate, the ordering of a set of mutually exclusive alternatives can be based on the following rules:

1. Arrange the alternatives according to the ascending order of the *algebraic values* of $[\text{NPV}_x]_{0\%}$ if these values are unequal.
2. Among those whose algebraic values of $[\text{NPV}_x]_{0\%}$ are equal, arrange them in the ascending order of the *algebraic values* of $[dN_x/di]_{0\%}$.

These rules for ordering the alternatives will cover all possible conditions shown in Figs. 8.7, 8.8, and 8.9.

To compute i_{x-y} in a pairwise comparison between x and y, we note that the incremental benefit and the incremental cost at any time period t may be obtained from the cash flow profiles of alternatives x and y as follows:

$$\Delta B_{t,x-y} = B_{t,x} - B_{t,y} \tag{8.9}$$

$$\Delta C_{t,x-y} = C_{t,x} - C_{t,y} \tag{8.10}$$

Alternatively, the incremental net value (between benefit and cost) at any time period is given by

$$\Delta A_{t,x-y} = A_{t,x} - A_{t,y} \tag{8.11}$$

Then, for a discount rate i,

$$[\Delta\text{NPV}_{x-y}]_i = [\text{NPV}_x]_i - [\text{NPV}_y]_i \tag{8.12}$$

The incremental internal rate of return i_{x-y} is the discount rate i at which ΔNPV_{x-y} is zero. To differentiate the multiple values of the incremental internal rate of return, they will be designated as i'_{x-y}, i''_{x-y}, etc., starting from the lowest and reaching the highest value except when one of the values is zero. In the case, the zero value will not be counted in the designation.

In making pairwise comparisons of two alternatives x and y, the following situations may be encountered:

1. When $[\text{NPV}_x]_{0\%}$ and $[\text{NPV}_y]_{0\%}$ are unequal, the former is greater than the latter according to the ordering rule. Whether one or more values of IIRR exists, $[\Delta\text{NPV}_{x-y}]_{0\%}$ is always positive, as shown in Fig. 8.10(a) or (b).
2. When $[\text{NPV}_x]_{0\%}$ and $[\text{NPV}_y]_{0\%}$ are unequal but the value(s) of IIRR is either indeterminate or negative, the former is still greater than the latter according to the ordering rule. Hence, $[\text{NPV}_{x-y}]_{0\%}$ is also positive, as shown in Fig. 8.10(c) or (d).

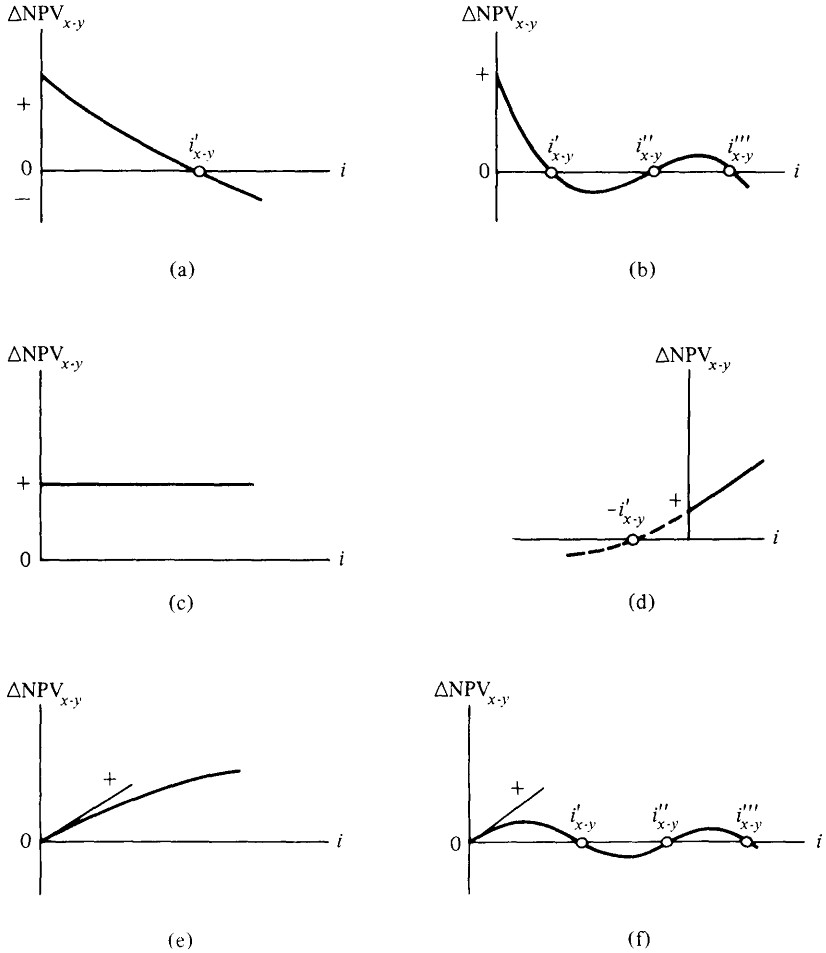

Figure 8.10 Incremental NPV versus i graph for various types of cash flows.

3. When $[\text{NPV}_x]_{0\%} = [\text{NPV}_y]_{0\%}$, then $[dN_x/di]_{0\%}$ is greater than $[dN_y/di]_{0\%}$ according to the ordering rule. Hence, the first derivative of the incremental net present value at $i = 0\%$ must also be positive, as shown in Fig. 8.10(e) or (f), because

$$[d(\Delta N)/di]_{0\%} = [dN_x/di]_{0\%} - [dN_y/di]_{0\%} > 0$$

4. When both $[\Delta\text{NPV}_{x-y}]_{0\%}$ and $[d(\Delta N)/di]_{0\%}$ equal zero, we can check whether the second derivative at $i = 0\%$ is positive, negative, or zero. If it is zero, we should test the third derivative, etc. However, the occurrence of such cases is very unlikely, and therefore their discussion is not included.

Consequently, we need a set of criteria instead of a single criterion for selecting the best among a set of mutually exclusive alternatives. This set of criteria may be stated as follows:

(a) When $[\Delta NPV_{x-y}]_{0\%}$ is positive and there is only a single positive value of i_{x-y}, the higher order alternative x is preferable if $i^* \leq i'_{x-y}$; otherwise the lower order alternative y is preferable.

(b) When $[\Delta NPV_{x-y}]_{0\%}$ is positive but there are multiple values of $i(i'_{x-y}, i''_{x-y}, i'''_{x-y},$ etc., starting from the lowest to the highest value), the following rules hold:
  (1) If the MARR i^* is less than i'_{x-y}, the project x is preferable.
  (2) If the MARR i^* is higher than i'_{x-y} but less than i''_{x-y}, the project y is preferable.
  (3) If the MARR i^* is higher than the i''_{x-y} but less than i'''_{x-y}, the project x is preferable.
  (4) The preference of x or y alternates with each successive value of i_{x-y}.

(c) When $[\Delta NPV_{x-y}]_{0\%}$ is positive but i_{x-y} is indeterminate, the higher order alternative x is preferable.

(d) When $[\Delta NPV_{x-y}]_{0\%}$ is positive but all values of i_{x-y} are negative, the higher order alternative is preferable.

(e) When $[\Delta NPV_{x-y}]_{0\%}$ is zero and there is only a single value of $i_{x-y} = 0$, the higher order alternative x is always preferable.

(f) When $[\Delta NPV_{x-y}]_{0\%}$ is zero but there are multiple values of i_{x-y}, use the same rule as (b) except ignore the value of $i_{x-y} = 0$. That is, the first i_{x-y} value which is greater than zero is defined as the lowest value.

8.7 DECISION CRITERIA FOR SELECTION AMONG MUTUALLY EXCLUSIVE ALTERNATIVES

The decision criteria for selecting an alternative in the successive pairwise comparisons can be summarized as in Table 8.5. The final choice will be the best among the mutually exclusive alternatives. If the null alternative is included in the successive pairwise comparisons, then the best alternative must necessarily be economically feasible without making a separate investigation.

Example 8.4

The cash flow profiles for the three mutually exclusive alternatives are shown in the top part of Table 8.6. The MARR is specified to be 12%. Using the incremental internal rate of return analysis, select the best alternative that is economically feasible.

TABLE 8.5 DECISION CRITERIA FOR ACCEPTING OR REJECTING HIGHER ORDER ALTERNATIVE x IN SUCCESSIVE PAIRWISE COMPARISONS

Range of i^*	Condition: $[\Delta NPV_{x-y}]_{0\%} \geq 0$
$0 \leq i^* < i'_{x-y}$	Accept x
$i'_{x-y} < i^* < i''_{x-y}$	Reject x; accept y
$i''_{x-y} < i^* < i'''_{x-y}$	Accept x
$i'''_{x-y} < i^* < i''''_{x-y}$	Reject x; accept y
Additional i_{x-y}	Alternate accepting and rejecting x
Negative or indeterminate	Accept x

For the purpose of ordering alternatives, we first compute $[NPV_x]_{0\%}$ for $x = 1$, 2, and 3 using Eq. (8.4):

$$[NPV_1]_{0\%} = -10{,}000 + (6)(3{,}000) = 8{,}000$$

$$[NPV_2]_{0\%} = -10{,}000 + (3)(5{,}500 + 500) = 8{,}000$$

$$[NPV_3]_{0\%} = -1{,}500 + (6)(6{,}000) = 4{,}500$$

Thus, $x = 3$ is the lowest order alternative, but $x = 1$ and $x = 2$ are tied. Therefore, we proceed to compute $[dN_x/di]_{0\%}$ for $x = 1$ and $x = 2$, using Eq. (8.6).

TABLE 8.6 CASH FLOW PROFILES FOR THREE MUTUALLY EXCLUSIVE ALTERNATIVES

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−10,000	−10,000	−1,500
1	3,000	5,500	1,000
2	3,000	5,500	1,000
3	3,000	500	1,000
4	3,000	500	1,000
5	3,000	500	1,000
6	3,000	5,500	1,000
$[NPV_x]_{0\%}$	+ 8,000	+ 8,000	+4,500
$[dN_x/di]_{0\%}$	−63,000	−55,500	
Ascending order	Middle	Highest	Lowest

$$[dN_1/di]_{0\%} = -(1 + 2 + 3 + 4 + 5 + 6)(3{,}000) = -63{,}000$$

$$[dN_2/di]_{0\%} = -(1 + 2 + 6)(5{,}500) = (3 + 4 + 5)(500) = -55{,}000$$

Since the *algebraic value* of the slope for $x = 1$ is smaller (most negative), $x = 1$ is regarded as having a lower order. Hence, the ascending order of these alternatives is $x = 3$, $x = 1$, and $x = 2$. The results of the computation are tabulated in the bottom part of Table 8.6.

We therefore first compare $x = 3$ with the null alternative ($x = 0$) by computing $\Delta A_{t,3-0}$ as tabulated in Table 8.7. Since this incremental cash flow profile has only one sign change, we expect a single value for i_{3-0}. Noting that $[\Delta \text{NPV}_{3-0}]_{0\%}$ is positive, we proceed to solve for $i = i_{3-0}$ by letting $[\Delta \text{NPV}_{3-0}]_i = 0$. That is,

$$-1{,}500 + (1{,}000)(P \mid U, i, 6) = 0$$

or

$$(P \mid U, i, 6) = \frac{1{,}500}{1{,}000} = 1.5$$

We know that i is greater than $i^* = 12\%$ since its value is outside the range of the discrete compound interest tables in Appendix A. It has been found by an electronic spreadsheet to be 53.13%. Since this value of $i_{3-0} = 53.13\%$ is greater than $i^* = 12\%$, the alternative $x = 3$ is preferable over the null alternative ($x = 0$).

Next, we compare $x = 3$ with the next higher order alternative $x = 1$ by computing $\Delta A_{t,1-3}$ as tabulated in Table 8.7. Since this incremental cash flow profile has only one sign change, we expect a single value for i_{1-3}. Noting that

TABLE 8.7 INCREMENTAL CASH FLOW PROFILES

t	$\Delta A_{t,3-0}$	$\Delta A_{t,1-3}$	$\Delta A_{t,2-3}$
0	−1,500	−8,500	−8,500
1	1,000	2,000	4,500
2	1,000	2,000	4,500
3	1,000	2,000	− 500
4	1,000	2,000	− 500
5	1,000	2,000	− 500
6	1,000	2,000	4,500
$[\Delta \text{NPV}_{x-y}]_{0\%}$	+4,500	+3,500	+3,500
i_{x-y}	53.13%	10.86%	14.43%

$[\Delta NPV_{1-3}]_{0\%}$ is positive, we solve for $i = i_{1-3}$ by letting $[\Delta NPV_{1-3}]_i = 0$. That is,

$$-8{,}500 + (2{,}000)(P \mid U, i, 6) = 0$$

or

$$(P \mid U, i, 6) = \frac{8{,}500}{2{,}000} = 4.25$$

From the discrete compound interest tables in Appendix A, the value of i is found to lie between 10% and 12%; using interpolation we find

$$i = 10\% + (12\% - 10\%)\frac{4.3553 - 4.25}{4.3553 - 4.1114} = 10.86\%$$

Since $i_{1-3} = 10.86\%$ is less than $i^* = 12\%$, the alternative $x = 3$ is preferable.

Finally, we compare $x = 3$ with $x = 2$ by computing ΔA_{2-3} as tabulated in Table 8.7. Since this incremental cash flow profile has two sign changes, it is possible to have 0, 1, or 2 values for i_{2-3}. Noting that $[\Delta NPV_{2-3}]_{0\%}$ is positive, we solve for $i = i_{2-3}$ by letting $[\Delta NPV_{2-3}]_i = 0$. That is,

$$-8{,}500 + (4{,}500)(P \mid U, i, 6) - (5{,}000)(P \mid U, i, 3)(P \mid F, i, 2) = 0$$

Try $i = 12\%$:

$$-8{,}500 + (4{,}500)(4.1114) - (5{,}000)(2.4018)(0.7972) = +427.7$$

Try $i = 15\%$:

$$-8{,}500 + (4{,}500)(3.7845) - (5{,}000)(2.2832)(0.7561) = -101.3$$

By interpolation,

$$i = 12\% + (15\% - 12\%)\frac{427.7}{427.7 + 101.3} = 14.43\%$$

Since $i_{2-3} = 14.43\%$ is greater than $i^* = 12\%$, the alternative $x = 2$ is preferable. The final choice of $x = 2$ is not only the best among the three mutually exclusive alternatives but it must necessarily be economically feasible because $x = 3$ is proved to be economically feasible (compared to the null alternative $x = 0$) and $x = 2$ is ultimately proved to be better than $x = 3$ through successive pairwise comparisons.

Note that the pair in each column (except the pair of $x = 3$ and $x = 0$) cannot be determined until a choice has been made in the previous column. In other words, we select $x = 3$ over $x = 0$ and then compare $x = 3$ with the next higher order alternative $x = 1$; similarly, we select $x = 3$ over $x = 1$ before comparing it with the next higher alternative $x = 2$.

Example 8.5

Suppose that $x = 1$ and $x = 2$ in Example 8.4 are the only mutually exclusive alternatives. Rework the problem using the incremental internal rate of return analysis.

Since these two alternatives have the same value for $[NPV_x]_{0\%}$, they are ordered according to the algebraic value of $[dN_x/di]_{0\%}$. Thus, $x = 1$ is regarded as the lower order alternative and $x = 2$ the higher order one.

In comparing $x = 1$ with the null alternative $x = 0$, we find the incremental cash flow profile $\Delta A_{t,1-0}$, as shown in Table 8.8. With only one sign change and $[\Delta NPV_{1-0}]_{0\%}$ being positive, we can find $i = i_{1-0}$ by letting $[\Delta NPV_{1-0}]_{0\%} = 0$. That is,

$$10{,}000 + (3{,}000)(P \mid U, i, 6) = 0$$

or,

$$(P \mid U, i, 6) = \frac{10{,}000}{3{,}000} = 3.3333$$

From the discrete compound interest tables in Appendix A, we find that i equals 20% approximately. Since $i_{1-0} = 20\%$ is greater than $i^* = 12\%$, the alternative $x = 1$ is preferable.

TABLE 8.8 INCREMENTAL CASH FLOW PROFILES FOR TWO ALTERNATIVES

t	$\Delta A_{t,1-0}$	$\Delta A_{t,2-1}$
0	−10,000	0
1	3,000	2,500
2	3,000	2,500
3	3,000	−2,500
4	3,000	−2,500
5	3,000	−2,500
6	3,000	2,500
$[\Delta NPV_{x-y}]_{0\%}$	+ 8,000	0
i_{x-y}	20%	0%

Next, we compare $x = 1$ with $x = 2$ by finding the incremental cash flow profile $\Delta A_{t,2-1}$ as shown in Table 8.8. Although there are two sign changes in this cash flow profile, $[\Delta NPV_{2-1}]_{0\%} = 0$ leads to $i_{2-1} = 0\%$. Then, the higher order alternative $x = 2$ is preferable without the necessity of making any further analysis. (The alternatives are arranged in the increasing order of the algebraic values of the initial slopes of the NPV versus MARR curve; hence the initial slope of the ΔNPV_{2-1} versus MARR curve is positive for a practical range of i, indicating that $x = 2$ is preferable.)

It is tempting to dismiss the formal treatment of equal value of $[NPV_x]_{0\%}$ for two alternatives as trivial. After all, the cash flow profiles are estimates, and we can change the value of one of the two alternatives in this problem by one dollar to break the tie. Suppose that you arbitrarily assume $[NPV_2]_{\%} = 8{,}001$ and $[NPV_1]_{0\%} = 8{,}000$. Then $[\Delta NPV_{2-1}]_{0\%} = 1$ and $i'_{2-1} > 0\%$ very slightly. According to the decision rule, the lower order alternative $x = 1$ is preferable since $[\Delta NPV_{2-1}]_{0\%}$ is positive and i'_{2-1} is less than the MARR. However, this conclusion is incorrect. Hence, you cannot arbitrarily order the alternatives for expediency.

Example 8.6

A small oil company is deciding between two different oil pumps for extracting oil from a well. The more expensive pump ($x = 1$) would permit the oil to be extracted more quickly and increase slightly the total amount of oil extracted than the less expensive pump ($x = 2$). The cash flow profiles of these two mutually exclusive alternatives (in thousands of dollars) are shown in Table 8.9. The MARR is specified to be 15%. Using the incremental internal rate of return analysis, determine which pump, if any, should be selected.

TABLE 8.9 CASH FLOW PROFILES FOR TWO OIL PUMPS (IN $1,000)

	$x = 1$			$x = 2$			
t	$B_{t,1}$	$C_{t,1}$	$A_{t,1}$	$B_{t,2}$	$C_{t,2}$	$A_{t,2}$	$\Delta A_{t,2-1} = A_{t,2} - A_{t,1}$
0	0	110	−110	0	100	−100	+10
1	115	0	115	70	0	70	−45
2	30	0	30	70	0	70	+40
$[NPV_x]_{0\%}$			+ 35			+ 40	+ 5

Based on the values of $[NPV_x]_{0\%}$ for $x = 1$ and $x = 2$ in Table 8.9, $x = 1$ is regarded as the lower order alternative and $x = 2$ the higher order one. Therefore, we first compare $x = 1$ with the null alternative ($x = 0$), using the cash flow profile $A_{t,1}$ which is identical to the incremental cash flow profile $A_{t,1-0}$. Since there is only one sign change in the cash flow profile, we can find $i = i_{1-0}$ by letting $[\Delta NPV_{1-0}]_i = 0$. That is,

$$-110{,}000 + 115{,}000(1 + i)^{-1} + 30{,}000(1 + i)^{-2} = 0$$

from which we get $i = 26.16\%$. Since $i_{1-0} = 26.16\%$ is greater than $i^* = 15\%$, the alternative 1 is preferable to the null alternative $x = 0$.

Next, we compare $x = 1$ with $x = 2$ by computing the incremental profile $A_{t,2-1}$ as shown in Table 8.9. Since there are two sign changes, we may have 0, 1, or 2 values of i that will satisfy $[\Delta NPV_{2-1}]_i = 0$. That is,

$$+10{,}000 - 45{,}000(1 + i)^{-1} + 40{,}000(1 + i)^{-2} = 0$$

from which we get i = 21.92% and 228.08%. Thus, for a positive value of $[\Delta NPV_{2-1}]_{0\%}$, alternative 2 is preferable between i = 0% and 21.92%, but not between 21.92% and 228.08%. Hence, alternative 2 is preferable since i^* = 15% is less than i'_{2-1} = 21.92%. The relationships of NPV versus MARR and NPV_{2-1} versus MARR are shown in Figure 8.11. Therefore, x = 2 is economically acceptable and is the better of the two alternatives.

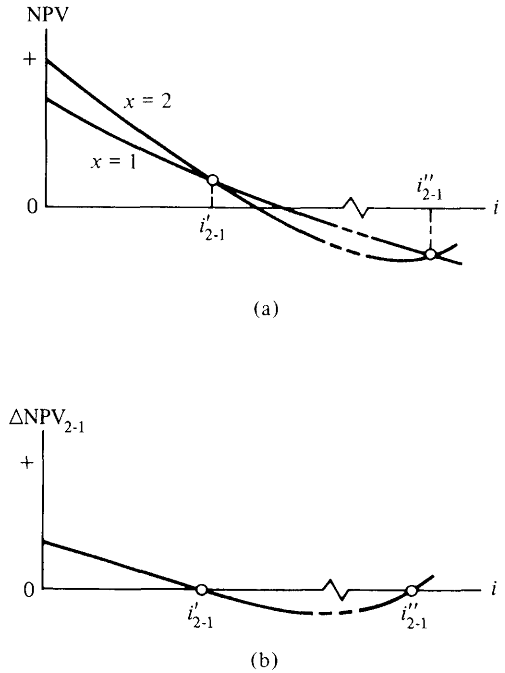

Figure 8.11 NPV versus i graph for Example 8.6.

Example 8.7

Three plans have been proposed for a small commercial building at a leased site, and the time streams of costs and benefits for these plans in the next 40 years are shown in Table 8.10. Using a MARR of 6%, select the best plan on the basis of incremental internal rate of return analysis. (Compare this example with Example 6.2 in Chapter 6, which is based on the net present value method.)

The values of $[NPV_x]_{0\%}$ for the three mutually exclusive alternatives have been computed according to Eq. (8.4), and the ascending order of the alternatives is found to be x = 3, x = 2, and x = 1.

Comparing with the null alternative x = 0, we proceed to solve for $i = i_{3-0}$ by letting $[\Delta NPV_{3-0}]_i = 0$. That is,

$$-300 + (77 - 55)(P \mid U, i, 40) = 0$$

TABLE 8.10 COST AND BENEFIT STREAMS (IN $1,000) OF THREE PLANS

	$x = 1$		$x = 2$		$x = 3$	
t	$C_{t,1}$	$B_{t,1}$	$C_{t,2}$	$B_{t,2}$	$C_{t,3}$	$B_{t,3}$
0	200	0	250	0	300	0
1–40 (each)	40	63	30	52	55	77
$[\text{NPV}_x]_{0\%}$	720		630		580	

from which we find $i_{3-0} = 6.82\%$. Since this is greater than $i^* = 6\%$, the alternative $x = 3$ is preferable to the null alternative.

Next, we try to solve $i = i_{2-3}$ from the equation representing $[\Delta\text{NPV}_{2-3}]_i = 0$. We obtain

$$-(250 - 300) + [(52 - 77) - (30 - 55)](P \mid U, i, 40) = 0$$

or

$$(P \mid U, i, 40) = \frac{-50}{0} = -\infty$$

According to the criteria in Table 8.5, we accept the higher order alternative, which is $x = 2$ in this case.

Finally, we try to solve for $i = i^*_{1-2}$ from the equation representing $[\Delta\text{NPV}_{1-2}] = 0$. We find

$$-(200 - 250) + [(63 - 52) - (40 - 30)](P \mid U, i, 40) = 0$$

or

$$(P \mid U, i, 40) = -50$$

Therefore, we accept the higher order alternative $x = 1$, which is the best among the three plans.

8.8 THE APPLICATION OF THE INTERNAL RATE OF RETURN METHOD

It is often argued by the advocates of the internal rate of return method that the multiple values of the internal rate of return seldom occur in practice. Let us examine whether this statement is correct.

In the evaluation of independent projects, cash flow profiles with multiple sign changes may be found most often in the following situations:

1. Capital projects that require periodic rehabilitation as well as equipment replacement, such as highway resurfacing or bridge repair projects, over the entire planning horizon or life cycle

2. Capital projects that require large expenditures for restoring the natural environment at the end of their useful lives, such as the restoration of a strip-mining site after the removal of coals or the disposal of hazardous materials upon the termination of the operation of a chemical plant
3. Overseas construction projects or military equipment orders by foreign governments for which large partial payments are made in advance as well as at the end of the projects

The criteria presented in Table 8.2 are intended to cover virtually all possible situations that may occur in practice.

However, the multiple sign changes in a cash flow profile occur most frequently in the incremental internal rate of return analysis for selecting the best among the mutually exclusive alternatives. The cash flow profiles used in Examples 8.4 and 8.6 are certainly quite conventional, but in each example, multiple sign changes appear in an incremental cash flow profile obtained from the given data. Furthermore, the existence of equal values of $[NPV_x]_{0\%}$ for some or all of the alternatives is not at all uncommon, as in the case of Example 8.5. Consequently, the criteria presented in Table 8.5 are needed to cover different situations in the incremental analysis.

It is a common current practice to arrange the projects in the ascending order of the *initial costs* in the application of the internal rate of return method, as discussed in Section 2.8 (Chapter 2). However, such an ordering scheme is not always applicable, particularly when some of the cash flow profiles of the mutually exclusive proposals have multiple internal rates of return. Consequently, only the ordering scheme described in this chapter is recommended.

From the examples in this chapter, it becomes clear that the traditional set of decision rules for incremental analysis based on IRR presented in Chapter 2 is generally inadequate for selecting the best among mutually exclusive proposals. Ironclad procedures recommended in this chapter are needed to ensure that problems involving all types of cash flow profiles can be properly analyzed by the internal rate of return method. When you read the engineering economics literature claiming that all methods of analysis, if applied correctly, lead to the same conclusions, you can judge for yourselves whether some purportedly comprehensive IRR decision rules proposed by these authors can be applied to solve conventional problems such as Examples 8.4, 8.5, and 8.6. If they cannot logically lead to correct conclusions even for such simple problems, then any claim that such decision rules can lead to the same conclusions as the net present value method is invalid.

More importantly, even if we have a complete set of IRR decision criteria for evaluating an independent project or for selecting the best among mutually exclusive proposals, the procedures will be indirect and lengthy. If these procedures were fully understood by their users, one wonders if they would still be used in economic evaluation.

8.9 SUMMARY AND STUDY GUIDE

In this chapter, we have considered the use of the internal rate of return (IRR) criteria in the determination of the acceptability of an indepedent project. In particular, we have considered a set of decision criteria needed to cover virtually all conditions that may be encountered in the analysis. We can use $[NPV]_{0\%}$ as the basis for classifying various conditions, and when $[NPV]_{0\%} = 0$, we can further divide the conditions into the case with a positive initial slope and that of a negative initial slope of the NPV curve at $i = 0\%$. The cases in which the IRR is indeterminate or negative were also considered.

The concept of incremental internal rate of return (IIRR) was introduced for successive pairwise comparisons in selecting the best among a set of mutually exclusive alternatives. The alternatives were arranged in the ascending order of $[NPV_x]_{0\%}$ and, when there was a tie between two alternatives, in the ascending order of $[dN_x/di]_{0\%}$. A set of decision criteria has also been introduced to cover such conditions in the incremental analysis. If the null alternative is included in the successive pairwise comparisons, then the best alternative must necessarily be economically feasible without making a separate investigation.

The internal rate of return method is lengthy whether it is applied to the evaluation of the acceptability of independent projects or to the incremental analysis of mutually exclusive proposals. You should be able to understand the significance of the arguments in this chapter without being bogged down by the details.

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8.4 Rapp, B., "The Internal Rate of Return Method—A Critical Study." *Engineering Costs and Production Economics,* Vol. 5, pp. 43–52. Amsterdam: Elsevier Scientific, 1980.

8.5 Rose, A., "Tutorial: Incremental Analysis in Capital Budgeting." *Cost Engineering* 30 (10): 13–15 (1988).

8.6 Wohl, M., "New Ranking Procedure and Set of Decision Rules for Method of Internal Rate of Return." *Transportation Research Record,* No. 828 (1981), 3–5.

8.7 Wohl, M., "A New Ordering Procedure and Set of Decision Rules for the Internal Rate of Return Method." *The Engineering Economist* 30 (4): 363–386 (1985).

PROBLEMS

P8.1 Select the best proposal among the alternatives in Problem P6.1 (Chapter 6), using the internal rate of return method.

P8.2 Select the best proposal among the alternatives in Problem P6.2 (Chapter 6), using the internal rate of return method.

P8.3 Select the best proposal among the alternatives in Problem P6.6 (Chapter 6), using the internal rate of return method.

P8.4 Select the best proposal among the alternatives in Problem P6.8 (Chapter 6), using the internal rate of return method if MARR = 4%.

P8.5 Select the best proposal among the mutually exclusive alternatives given below, using the internal rate of return method. The MARR specified is 6%.

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−200	−350	−300
1	+50	+100	+70
2	+50	+90	+70
3	+50	+80	+70
4	+50	+70	+70
5	+50	+60	+70

P8.6 The benefits and costs of the alternatives for a stagewise improvement of an existing bridge are given below (in thousands of dollars). Select the best proposal using the internal rate of return method. The MARR specified is 10%.

	$x = 1$		$x = 2$	
t	$C_{t,1}$	$B_{t,1}$	$C_{t,2}$	$B_{t,2}$
0	100	0	800	0
1–10 (each)	0	50	0	200
11	900	50	0	200
12–20 (each)	0	200	0	200
21	0	200	500	200
22–30 (each)	0	200	0	150

P8.7 The public works department of a city is considering three mutually exclusive proposals for a road-resurfacing program. The cash flow profiles for the benefits and costs of the program are given below. The MARR specified is 8%. Select the best proposal using the internal rate of return method.

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−40,000	−20,000	−70,000
1	+30,000	+25,000	+40,000
2	+30,000	+25,000	+35,000
3	+30,000	+25,000	+30,000
4	−30,000	−25,000	+25,000
5	+30,000	+25,000	+20,000
6	+30,000	+25,000	+15,000

P8.8 Select the best proposal among the alternatives in Problem P7.8 (Chapter 7), using the internal rate of return method.

P8.9 Select the best among the mutually exclusive borrowing opportunities listed below if the MARR is 6%.

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	+10,000	+10,000	+10,000
1	0	−800	−2,505
2	0	−800	−2,505
3	0	−800	−2,505
4	0	−800	−2,505
5	−14,693	−10,800	−2,505

P8.10 For the mutually exclusive borrowing opportunities in Problem P8.9, select the best alternative if the MARR is 10%.

CHAPTER

9 Capital Budgeting

9.1 BASIC CONCEPTS

In previous chapters, we have assumed that the MARR specified by an organization is the appropriate discount rate for computing the net present values of investment proposals. Thus, the measure of the profit potential of investment proposals and subsequently the decision for accepting or rejecting investment projects are dependent on the specified MARR. The basic idea underlying the decision criteria for accepting and rejecting investment projects is that if an organization can lend or borrow as much money as it wishes at the MARR, the goal of profit maximization of the organization is best served by accepting all independent projects whose net present values based on the MARR are nonnegative, or by selecting the project with the maximum nonnegative net present value among a set of mutually exclusive proposals. When an organization can lend or borrow freely at the MARR, the acceptance and selection of investment projects does not depend on the amount of internal funds available in the organization since it can always finance worthwhile projects through borrowing if necessary.

As stated in Section 5.4, an organization may acquire the necessary funds for an investment project in a variety of ways, usually through a combination of internal funding, borrowing, and lending. The internal funds can be made available by using the working capital generated from its operations, by liquidating part of its business, or by selling its holdings in marketable securities such as government bonds. Borrowing refers to the acquisition of capital from outside sources such as issuing additional securities, e.g., long-term notes and bonds, or securing short-term loans from banks. Lending refers to investing the surplus funds in outside agents, such as acquiring the stock of other companies, purchasing certificates of deposit, or depositing money in banks or money market funds. The borrowing rate refers to the interest rate at which investment capital can be acquired from outside and the lending rate refers to the interest rate at which a firm or an agency is willing to loan or invest its assets outside of the organization. In a perfectly competitive market, the borrowing rate and the lending rate are equal, and this market interest rate may be regarded as the MARR.

In reality, the lending rate and the borrowing rate are different because of market imperfections, and the divergence of these two rates may occur for a variety of reasons, such as financial transaction costs, brokerage fees, and the credit ratings of the firms. Consequently, the determination of the MARR by an organization is to some degree arbitrary. In the private sector, the MARR reflects the *opportunity cost of capital* of a firm, the market interest rates for lending and borrowing, and the risks associated with investment opportunities. In the public sector, the MARR is referred to as the *social discount rate* since it reflects social and economic welfare considerations in the public interest as well as the opportunity cost of capital. Although the discussion of major issues influencing the determination of the MARR for private firms and public agencies will be deferred until Chapters 16 and 17, it is possible to examine the effects of market imperfections on economic evaluation.

In this chapter, we consider various issues arising from the allocation of resources for capital investments in an organization because of the less than perfect capital market. An organization must determine the internal funds available and the amount to be borrowed, if any, in financing capital projects. Such considerations are referred to as *capital budgeting*.

The use of internal funds for a capital project is referred to as *equity financing*, and the use of funds borrowed from outside sources is referred to as *debt financing*. If an organization does not wish to borrow to finance its investment opportunities, it must be selective in funding profitable projects within the budget constraints. The decision to impose budget constraints is referred to as *capital rationing*. On the other hand, the organization may borrow in the capital market to enlarge its capacity to take up more investment opportunities. The choice of various debt instruments (notes, bonds, mortgages, etc.) to finance capital projects has significant impact on interest rates and timing of repayments. Such decisions require extensive knowledge of public finance or corporate finance in the public and private sectors respectively, and are best left to financial experts. However, it is important to understand the interactions of investment and financing that will affect the acceptability of proposed capital projects.

We should distinguish between the finance at the organization level and that at the project level. An organization may periodically raise substantial sums of money through long-term debts. It may also lend its idle funds to outside agents to earn interest. Consequently, the internal funds of an organization often refer to a mix of equity and debt that can be used to finance available investment opportunities. At the project level, an investment project is regarded as all-equity financed unless debts are incurred specially for undertaking the project or are allocated on a prorated basis to the project by the organization. In a broad sense, the term *project financing* denotes the process of obtaining funds to implement a project, whether or not borrowing is involved; in a narrow sense, it refers to the means of borrowing from outside sources. We are primarily interested in financing at the project level with only occasional references to the finance at the organization level.

9.2 CAPITAL RATIONING

In budgeting, capital rationing often arises from the desire of the management of an organization to avoid borrowing for a variety of reasons. Consequently, some profitable projects may be rejected as a result of budget constraints. In general, outlays and budget constraints may be applied to later years as well as the beginning of a project, and the mathematical programming formulation of the problem can be quite complicated.[1] Nevertheless, the basic objective remains the maximization of the NPV subject to the specified constraints.

Frequently, only the initial outlay for acquiring a physical asset is subjected to constraints in a capital budget because the operation and maintenance costs of the asset in future years are budgeted separately to offset incomes from those years. The investment base is not a mandate for providing a reserve at the beginning of a project. While an organization must make a firm commitment to funding a project until its completion, funds needed in later years can be provided in future budgets. In other words, an organization need not place a sum corresponding to its investment base in an escrow earmarked for the project only. As long as the organization can invest the excess funds at a rate equal to or better than the MARR and supply the necessary funds for outlays in later periods, the concept of introducing an investment base for the economic evaluation of investment opportunities with outlays at $t > 0$ is valid. Consequently, a common approach is to specify budget constraints on the basis of the initial outlays or costs attributable to investment opportunities. For the sake of simplicity, our consideration is limited to the treatment of conventional net cash flow profiles with outlay at $t = 0$ only.

The maximum amount that can be made available in the capital budget is the budget constraint on the initial cost of a project or a group of projects in the selection process. Under the assumption that physical assets are size dependent, the particular project or group of projects selected should produce the maximum net benefit subject to the specified budget constraint. Then, for a set of m mutually exclusive proposals ($x = 1, 2, \ldots, m$), only one project will be selected. Then the criterion for selecting a particular proposal j is as follows:

$$\begin{aligned} &\text{NPV}_j = \max \text{NPV}_x \qquad (\text{NPV}_j \geq 0) \\ &\text{subject to } C_{0,j} \leq C_q \end{aligned} \tag{9.1}$$

[1] See Ref. 9.1

where C_q is the maximum initial cost permitted during the budgeting period $t = 0$, and $C_{0,j}$ is the initial cost of proposal j. For projects with conventional cash flow profiles in which the outlay occurs only at time $t = 0$ and net incomes are expected in subsequent periods, the budget constraint can also be expressed in terms of the present value of cost as follows:

$$\text{NPV}_j = \max \text{NPV}_x \qquad (\text{NPV}_j \geq 0)$$
$$\text{subject to } \text{CPV}_j \leq \text{CPV}_q \tag{9.2}$$

where CPV_q is the maximum present value of the costs that may be invested in any given project and CPV_j is the present value of costs for proposal j. When the outlay occurs only at $t = 0$, it is obvious that $\text{CPV}_j = C_{0,j}$ and $\text{CPV}_q = C_q$.

For independent projects, Eq. (9.1) or Eq. (9.2) can easily be applied only if the sizes of these projects are large relative to the budget constraint so that only one project can be selected. Since the budget constraints often allow the funding of a number of independent proposals, we need to list all possible combinations of profitable independent proposals and select the combination that produces the maximum net benefit. Such an approach is not practical when the number of profitable projects is large, and a trial-and-error approach for selecting independent projects is presented in the next section.

Example 9.1

Three mutually exclusive building projects are proposed for a given site. All have a useful life of 40 years and no salvage value at the end. The initial construction cost at $t = 0$ for project x is $C_{0,x}$ and the net uniform annual net benefit is U_x in years $t = 1, 2, \ldots, 40$. These values for alternatives $x = 1$, 2, and 3 (in \$ million) are as follows: $C_{0,1} = 4.6$, $C_{0,2} = 6.0$ and $C_{0,3} = 6.4$ while $U_1 = 0.550$, $U_2 = 0.666$ and $U_3 = 0.700$. The MARR is specified to be 7%. Select the best alternative if there is a constraint of \$6.1 million for the initial cost.

The present value of costs CPV_x and benefits BPV_x for any alternative x are given by

$$\text{CPV}_x = C_{x,0}$$
$$\text{BPV}_x = U_x(P \mid U, 7\%, 40)$$

Then,

$$\text{NPV}_x = \text{BPV}_x - \text{CPV}_x = -C_{x,0} + U_x(P \mid U, 7\%, 40)$$

The results for $x = 1$, 2, and 3 (in \$ million) are

$$\text{NPV}_1 = -4.6 + (0.550)(13.3317) = 2.732$$
$$\text{NPV}_2 = -6.0 + (0.666)(13.3317) = 2.879$$
$$\text{NPV}_3 = -6.4 + (0.700)(13.3317) = 2.932$$

Since $C_{0,3}$ = \$6.4 million exceeds the budget constraint of \$6.0 million, $x = 3$ should be excluded. Hence, $x = 2$ should be selected since NPV_2 = \$2.879 million is greater than NPV_1 = \$2.732 million.

Example 9.2

The cash flow profiles for the three mutually exclusive proposals (in dollars) together with various merit measures of these proposals are shown in Table 9.1. What is the implication for project selection if the budget constraint on the initial cost is (a) \$6,000 and (b) \$2,000?

TABLE 9.1 CASH FLOW PROFILES AND MERIT MEASURES

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−10,000	−6,000	−1,500
1–2 (each)	3,000	3,300	1,000
3–5 (each)	3,000	300	1,000
6	3,000	3,300	1,000
NPV (\$)	2,334	1,823	2,611
NFV (\$)	4,608	3,600	5,155
EUAV (\$)	568	443	635
ORR	15.99%	17.06%	32.50%
IRR	19.90%	24.92%	63.13%

Since the direct measures (NPV, NFV and EUAV) are highest for proposal 3, which has an initial cost below \$2,000, proposal 3 should be selected whether the budget constraint is \$6,000 or \$2,000. Hence, the budget constraint for this problem does not influence the decision on project selection. This example refutes the argument that the NPV decision criterion necessarily favors large projects since the selected project has the smallest initial cost.

On the other hand, if only proposals 1 and 2 are available and a budget constraint of \$6,000 is imposed, then the budget constraint makes a difference on project selection. Without the budget constraint, proposal 1 would be selected; with a budget constraint of \$6,000, proposal 2 should be selected. Thus, the concern for the bias of the NPV decision criterion in favor of large projects can be removed by the judicious imposition of a budget constraint reflecting realistically the conditions of project financing.

9.3 SELECTING A GROUP OF INDEPENDENT PROJECTS UNDER BUDGET CONSTRAINTS

When the decision maker is confronted with the problem of selecting a group of independent projects from a very large pool of profitable projects under budget constraints, the objective is to select the group that, taken together, will produce the maximum net benefit. Each project is size dependent and, hence, indivisible. This problem can be formulated by mathematical programming with the objective of maximizing the sum of net present values of all projects ultimately selected, subject to the budget constraint. Again, our consideration is limited to conventional cash flows in which the outlay occurs only at time $t = 0$ and net incomes are expected in subsequent periods.

Let B_x be the present value of benefit, C_x be the present value of cost, and N_x be the net present value of a project x. The budget constraint C_q is placed on the sum of the present values of costs for the projects selected. Furthermore, let T be the sum of the net present values of all projects finally selected, and let D_x be a binary variable that can take on the value of 0 or 1 only. Then, the mathematical programming problem of selecting a group of projects from m independent proposals can be stated as follows:

$$\text{Maximize } T = \sum_{x=1}^{m} N_x D_x \tag{9.3}$$

$$\text{subject to } \sum_{x=1}^{m} C_x D_x \leq C_q \tag{9.4}$$

where $D_x = \begin{cases} 0 \text{ if project } x \text{ is not selected} \\ 1 \text{ if project } x \text{ is selected} \end{cases}$

Since $N_x = B_x - C_x$, or $N_x = (B_x/C_x - 1)\, C_x$, Eq. (9.3) can be replaced by

$$\text{Maximize } T = \sum_{x=1}^{m} \left(\frac{B_x}{C_x} - 1\right) C_x D_x \tag{9.5}$$

Note that the benefit-cost ratio B_x/C_x is a merit measure of project x only, but T is the merit measure of the entire group of projects ultimately selected.

In general, the solution of Eqs. (9.3) and (9.4) based on mathematical programming techniques can be quite complicated and is beyond the scope of this book. However, a trial-and-error procedure may be used to obtain a solution on the basis of Eqs. (9.5) and (9.4). Consequently, the benefit-cost ratio plays a role in this procedure. Note that any project with $B_x/C_x < 1$ should be rejected since it will diminish the value of T in Eq. (9.5). In any case, the criterion for selecting a group of independent projects under the specified budget constraint is that the sum of the net present value of this group is maximum relative to other combinations of the available independent projects.

An approximate but practical approach is to rank all acceptable independent projects according to the benefit-cost ratios as computed by Eq. (7.1), using the specified MARR. If the budget constraint is reached *exactly* after a number of projects are selected according to the priority based on the highest benefit-cost ratios, then we have indeed accepted the group of projects which, taken together as a group, will produce the maximum net present value. However, if the budget constraint is not reached exactly after selecting a number of projects but it will be exceeded by the inclusion of the next highest ranked project, then it is not necessarily true that the group of projects under the budget constraint is the best selection. The reason for this outcome is that, given the possibility of not accepting one or more of the higher ranking projects, it is conceivable to find some combination of lower ranking projects to use up the budget completely such that, taken as a group, the new combination produces a larger net present value.

The indivisibility of acceptable projects is an important consideration in project selection under budget constraints. In general, if a relatively large number of independent projects of approximately the same size can be selected under a budget constraint, it is likely that selecting according to the priority based on the highest benefit-cost ratios leads to the best result, even though the actual investment is slightly under the budget constraint. On the other hand, if only a relatively small number of independent projects of different sizes can be selected under the budget constraint, then trial solutions may be necessary. These situations are depicted schematically in Figs. 9.1(a) and (b).

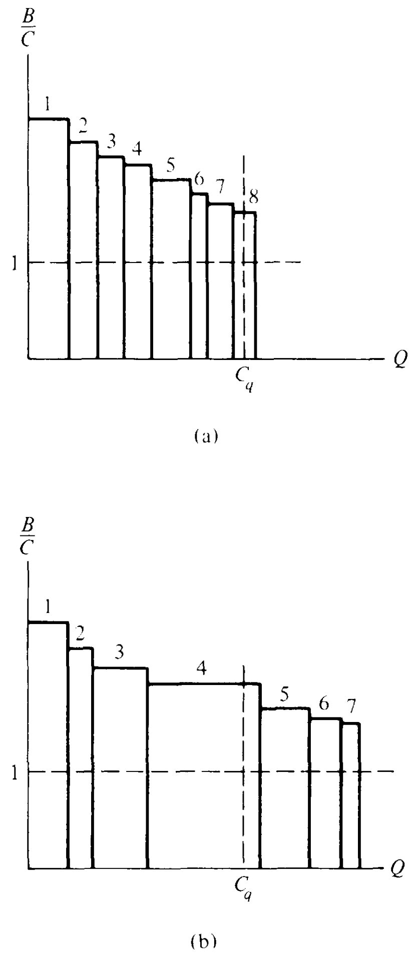

Figure 9.1 Ranking independent projects under budget constraint.

If the actual investment is slightly under the budget constraint due to indivisibility of the remaining available projects, the unused amount will be assumed to be invested at the MARR. Then, the net present value from the future return of this unused amount is the amount itself. The unused amount does not contribute any positive gain to the total net present value resulting from investing in the group of projects selected under the budget constraint and can therefore be ignored.

Example 9.3

A public agency is considering six independent projects that have different useful lives. The cash flow profile for each project x consists of an initial cost $C_{0,x}$ and a stream of net uniform annual benefits U_x for years $t = 1, 2, \ldots, N_x$ where N_x is the useful life of project x. The salvage value at $t = N_x$ for project x is $S_{N,x}$. There will be no replacement when the useful life of a project expires. The numerical data for these projects are given in Table 9.2. The agency intends to adopt a planning horizon of 10 years and specifies a MARR of 10%. Which projects should be selected if the present value of the budget constraint is (a) C_q = \$50 million and (b) C_q = \$100 million?

TABLE 9.2 CASH FLOW PROFILES OF SIX INDEPENDENT PROJECTS (IN \$ MILLION)

Project x	Life N_x	Initial cost $C_{0,x}$	Net uniform annual benefits U_x at $t = 1, 2, \ldots, N_x$	Salvage value $S_{N,x}$
1	10	50	19.460	14
2	10	20	9.451	5
3	4	10	4.733	0
4	10	40	13.040	10
5	6	30	18.250	8
6	6	30	11.620	6

The present values of benefits and costs (BPV_x and CPV_x, respectively) of these projects can be computed as follows:

$$\text{CPV}_x = C_{0,x}$$

$$\text{BPV}_x = U_x(P \mid U, 10\%, N_x) + S_{N,x}(P \mid F, 10\%, N_x)$$

Furthermore,

$$\text{NPV}_x = \text{BPV}_x - \text{CPV}_x$$

Using the priority based on the highest benefit-cost ratios in selecting these projects, we therefore compute first the benefit-cost ratios B_x/C_x. The results of the computation are shown in Table 9.3.

Under the given budget constraints, we select as many projects as possible according to the ranking of the benefit-cost ratios of these projects.

(a) For C_q = 50, we can select $x = 2$ and $x = 5$ according to the rank

TABLE 9.3 BENEFITS AND COSTS OF PROJECTS (IN $ MILLION)

x	BPV_x	CPV_x	NPV_x	B_x/C_x	Rank
1	125	50	75	2.5	3
2	60	20	40	3.0	1
3	15	10	5	1.5	6
4	84	40	44	2.1	4
5	84	30	54	2.8	2
6	54	30	24	1.8	5

based on benefit-cost ratios. Thus, for this group of projects,

$$\text{Total CPV} = 20 + 30 = 50$$

$$\text{Total NPV} = 40 + 54 = 94 \quad \text{(maximum possible)}$$

(b) For $C_q = 100$, we can select $x = 2$, $x = 5$, and $x = 1$ according to the rank based on the benefit-cost ratios. Then, for this group of projects,

$$\text{Total CPV} = 20 + 30 + 50 = 100$$

$$\text{Total NPV} = 40 + 54 + 75 = 169 \quad \text{(maximum possible)}$$

Example 9.4

Rework Example 9.3 if the budget constraint is (a) C_q = \$80 million and (b) C_q = \$90 million.

For both constraints in this problem, we cannot find a group of independent projects with the highest ranking of benefit-cost ratios to invest up to the exact limit of the budget. Consequently, several trials are made for each case, following the order of ranking whenever possible, but switching to a heuristic search if necessary. For the relatively small number of projects in this problem, the optimal solution becomes obvious after several trials.

(a) For $C_q = 80$, we seek the combinations whose total cost is closer to the budget constraint than 50 which results from selecting $x = 2$ and $x = 5$ only. (If $x = 1$ were also accepted, the total cost would exceed $C_q = 80$.)

Trial 1: $x = 2$, $x = 5$, and $x = 6$

$$\text{Total CPV} = 20 + 30 + 30 = 80$$

$$\text{Total NPV} = 40 + 54 + 24 = 118$$

Trial 2: $x = 2$, $x = 1$, and $x = 3$

$$\text{Total CPV} = 20 + 50 + 10 = 80$$

$$\text{Total NPV} = 40 + 75 + 5 = 120$$

Trial 3: $x = 5$ and $x = 1$

Total CPV = 30 + 50 = 80

Total NPV = 54 + 75 = 129 (observed maximum)

(b) For $C_q = 90$, we seek other combinations on the same basis.

Trial 1: $x = 5$, $x = 1$, and $x = 3$

Total CPV = 30 + 50 + 10 = 90

Total NPV = 54 + 75 + 5 = 134

Trial 2: $x = 2$, $x = 5$, and $x = 4$

Total CPV = 20 + 30 + 40 = 90

Total NPV = 40 + 54 + 44 = 138 (observed maximum)

9.4 RANKING INDEPENDENT PROJECTS UNDER BUDGET CONSTRAINTS ON THE BASIS OF OVERALL RATES OF RETURN

It is often stated in the literature that the internal rates of return on independent projects can be used to rank such projects when there is a budget constraint. Even if no multiple values of IRR exist for any of the projects, this statement is generally incorrect. The reason that the IRR should not be used as a basis for ranking independent projects is that the reinvestment rate for the early year returns from all projects will be based on the MARR of the organization rather than the internal rate of return for each of the individual projects. Hence, the overall rate of return (ORR) over the planning horizon rather than the IRR is the appropriate measure for ranking the profit potentials for a group of independent projects.

Let us illustrate this point by considering a group of m independent projects ($x = 1, 2, \ldots, m$), each of which has a net cash flow profile consisting of an initial cost $C_{0,x}$, a net uniform annual return U_x over its useful life of N_x years, and a salvage value of $S_{N,x}$ at the end of N_x as shown in Fig. 9.2(a). The useful life N_x of some or all projects may be identical to the planning horizon of n years. Since there is only one sign change in each cash flow profile, there is a single internal rate of return i_x for each project. However, the early returns from all projects will be reinvested at MARR $= i^*$ specified by the organization until the end of n years, not at the IRR $= i_x$ for each individual project x. Since no replacement is made for project x when it expires at $N_x < n$, it is *incorrect* to assume that the early returns from a project x will be reinvested at its IRR $= i_x$ if $N_x < n$.

On the other hand, the overall rate of return ORR$_x = i_x^0$ for a project x can be computed on the basis of a single equivalent sum $-P_x$ at $t = 0$ and a single equivalent sum F_x at $t = n$ as shown in Fig. 9.2(b). The procedure of computation has

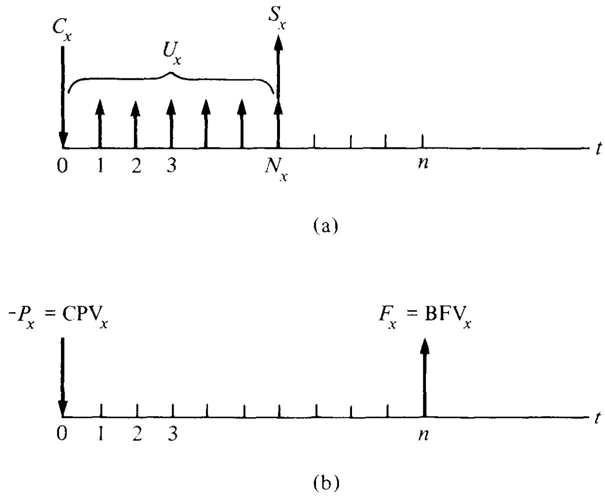

Figure 9.2 Net cash flow profiles for independent projects.

been explained in detail in Chapter 4. It is sufficient to point out that the ranking of m independent projects ($x = 1, 2, \ldots, m$) based on ORR$_x$ represents the correct measure of the profit potentials of the projects from the viewpoint of an organization. Thus, when there is no budget constraint, all projects for which ORR$_x$ is greater than the specified MARR should be accepted.

When there is a budget constraint, the ranking of independent projects becomes very important since usually only some of the economically feasible projects will be selected. If the budget constraint is reached exactly after a number of projects are selected according to the priority based on the highest overall rates of return, then we have indeed selected the group of projects which, taken together as a group, will produce the maximum net present value. However, if the budget constraint is not reached exactly after a number of projects, but it will be exceeded by the inclusion of the next highest ranked project, then it is not necessarily true that the group of projects under the budget constraint is the best selection.

Example 9.5

For the six independent projects in Example 9.3, compute the internal rate of return for each project and compare the rank of these projects based on IRR$_x$ with the rank based on B_x/C_x computed for Example 9.3.

Since the computation of the internal rate of return has been explained in detail in Chapter 4, it is sufficient to mention briefly the procedure of computation here. For any project x whose cash flow profile appears in Table 9.2, the net present value is given by

$$\text{NPV}_x = -C_{0,x} + U_x(P \mid U, i, N_x) + S_{N,x}(P \mid F, i, N_x)$$

The internal rate of return, IRR$_x = i_x$, can be obtained by letting NPV$_x = 0$.

Hence,

$$-C_{0,x} + U_x(P \mid U, i_x, N_x) + S_{N,x}(P \mid F, i_x, N_x) = 0$$

This equation can be solved either by interpolation from the discrete compound interest tables in Appendix A or by finding a numerical solution with the aid of an electronic spreadsheet.

The results of the IRR_x for the six independent projects are tabulated in Table 9.4. In comparing the rank of these results with the rank of the projects based on B_x/C_x, it is obvious that they are different.

TABLE 9.4 RANKS OF PROJECTS ACCORDING TO *B*, *C*, AND IRR

Project x	B_x/C_x	Rank of B_x/C_x	IRR_x	Rank of $IRR_x = i_x$
1	2.5	3	38.1%	3
2	3.0	1	46.9%	2
3	1.5	6	31.7%	5
4	2.2	4	31.1%	6
5	2.8	2	58.5%	1
6	1.8	5	33.0%	4

Example 9.6

Assuming that the reinvestment rate is the same as MARR = 10%, determine the overall rate of return (ORR) for each of the six independent projects in Example 9.3.

The procedure for computing the ORR has been explained in detail in Chapter 4. It is sufficient to note that in this problem the single sum $-P_x$ at $t = 0$ is CPV_x, and the equivalent single sum F_x at $t = n$ is BFV_x as shown in Fig. 9.2. Part (a) of the figure represents the given cash flow profile of the project x while part (b) shows the equivalent sums for computing ORR_x as follows:

$$-P_x = CPV_x = C_{0,x}$$

$$F_x = BFV_x = (BPV_x)(F \mid P, 10\%, 10)$$

where BFV_x can be conveniently computed from BPV_x of the cash flow profile. The $ORR_x = i_x^0$ can be obtained from the solution of the following equation:

$$NPV_x = -CPV_x + (BFV_x)(P \mid F, i_x^0, 10) = 0$$

or

$$(P \mid F, i_x^0, 10) = \frac{CPV_x}{BFV_x} = -\frac{P_x}{F_x}$$

Hence, the value of i_x^0 can be obtained by interpolation from the discrete compound interest tables in Appendix A. The results of $\text{ORR}_x = i^0$ for all six independent projects are tabulated in Table 9.5.

TABLE 9.5 RANKS OF PROJECTS ACCORDING TO ORR

Project x	BPV_x	CPV_x	BFV_x	$\frac{\text{CPV}_x}{\text{BFV}_x}$	ORR_x	Rank of $\text{ORR}_x = i_x^0$
1	125	50	324.213	0.1542	20.7%	3
2	60	20	155.622	0.1285	23.1%	1
3	15	10	38.906	0.2570	14.6%	6
4	84	40	217.871	0.1836	18.7%	4
5	84	30	217.871	0.1377	22.2%	2
6	54	30	140.060	0.2142	16.9%	5

Note that $(P \mid F, i^0, 10)$ is a monotonically decreasing function of i_x^0. That is, the value of $(P \mid F, i^0, 10)$ decreases as the value of i^0 increases. Conversely,

$$(F \mid P, i_x^0, 10) = \frac{1}{(P \mid F, i_x^0, 10)} = \frac{\text{BFV}_x}{\text{CPV}_x} = -\frac{F_x}{P_x}$$

is a monotonically increasing function of i_x^0. (See Fig. 3.17 in Chapter 3.) Since $\text{BFV}_x/\text{CPV}_x$ differs from $B_x/C_x = \text{BPV}_x/\text{CPV}_x$ by a constant factor of $(F \mid P, 10\%, 10)$, the rank of i_x^0 for the independent projects, as shown in Table 9.5, should be the same as the rank of B_x/C_x shown in Table 9.4.

9.5 PROJECT FINANCING

When internal funds of an organization are in short supply, it can raise money by increasing its equity or its debt. Each action has its own cost. For a public agency, an increase in all-equity financed projects under such circumstances means an increase in taxes, which may be politically unpopular. For a private corporation, the increase would require the issuance of new stocks, which may dilute the control of the current stockholders. For both types of organizations, an increase in debt may impact its credit rating in the financial community. The change in the mix of equity and debt in the long run will also influence the cost of capital of an organization. Such factors must be considered by an organization in financing capital projects.

If an organization decides to borrow from outside sources to finance its investment projects, it may enter into agreement with lenders regarding a schedule of repayments of the principal plus the accrued interest. The stream of cash flows representing the loan and repayments over time is referred to as the *borrowing* or *financing* cash flow profile. In contrast to the *investment* or *production* cash flow profile of a project, which is generated by acquiring and operating the physical asset, the borrowing cash flow profile is independent of the operation of the physical asset. Instead, it is influenced by the financial markets from which funds are borrowed.

So far, we have avoided the consideration of uncertainty and risk in the operation of physical assets. The use of debt financing may further increase the risk of an investment project and/or its sponsoring organization. In reality, the MARR specified for a project often takes into consideration the risk factor as well as the time preference of the decision maker. The treatment of the risk factor will be introduced in Chapter 15, with applications to private and public sectors in Chapters 16 and 17, respectively. In this chapter, we continue to make the assumptions of certainty for future cash flows and a perfect capital market in dealing with project financing.

To illustrate the application for discounting the operation and financing cash flow profiles over a planning horizon of n periods (for $t = 0, 1, 2, \ldots, n$), let us introduce the following notation:

B_t = total revenues and other benefits from operation
C_t = total expenses for operation
A_t = net cash flow for operation
$\bar{B}_t$ = cash receipt (from loan) for financing
$\bar{C}_t$ = cash disbursement (repayment of principal and interest on loan) for financing
$\bar{A}_t$ = net cash flow for financing
$\hat{B}_t$ = total benefit from all sources
$\hat{C}_t$ = total cost from all sources
$\hat{A}_t$ = net cash flow from all sources
NPV = net present value of the operating net cash flows A_t
FPV = net present value of the financing net cash flows $\bar{A}_t$
APV = adjusted present value reflecting the combined effects of the operation and financing net cash flows
i^* = the MARR used for discounting both the operation and financing net cash flow

Then, for the time stream $t = 0, 1, 2, \ldots, n$,

$$A_t = B_t - C_t$$

$$\bar{A}_t = \bar{B}_t - \bar{C}_t$$

$$\hat{A}_t = \hat{B}_t - \hat{C}_t$$

For investment decisions under the assumption of certainty for future cash flows and a perfect capital market,

$$\text{APV} = [\text{NPV}]_{i^*} + [\text{FPV}]_{i^*} \tag{9.6}$$

Alternatively, the APV can be obtained directly by discounting the combined cash flow given by

$$\hat{A}_t = A_t + \bar{A}_t \tag{9.7}$$

If the investment is financed by equity only, Eq. (9.6) becomes

$$\text{APV} = [\text{NPV}]_{i^*} \tag{9.8}$$

Example 9.7

Samluck County purchases a snow removal vehicle that is expected to generate a net benefit equivalent to $8,000 per year over the next 8 years. The purchase price is $50,000 with a salvage value of $5,000 at the end of 8 years. The county borrows $30,000 from a bank at 6% interest compounded annually, and the loan is to be repaid with equal annual payments including interest over a 5-year period. The remaining $20,000 is taken from internal funds. If the county uses a MARR of 8% for both the operation and financing cash flows, is the purchase worthwhile?

Since the amount borrowed at $t = 0$ is $30,000, which is to be repaid with uniform annual payments at $t = 1, 2, \ldots, 5$, the uniform annual amount at 6% annual interest rate is given by

$$(30{,}000)(U \mid P, 6\%, 5) = 7{,}122$$

The cash flow profiles for operation, financing and combined effects are shown in Table 9.6. Using separate operation and financing cash flow profiles and a MARR of $i^* = 8\%$, we get

$$[\text{NPV}]_{8\%} = -50{,}000 + (8{,}000)(P \mid U, 8\%, 8) + (5{,}000)(P \mid F, 8\%, 8)$$
$$= -1{,}325$$

$$[\text{FPV}]_{8\%} = +30{,}000 - (7{,}122)(P \mid U, 8\%, 5) = 1{,}564$$

$$\text{APV} = [\text{NPV}]_{8\%} + [\text{FPV}]_{8\%} = -1{,}325 + 1{,}564 = +239$$

The APV can also be obtained directly from the combined cash flow profile as follows:

$$[\text{APV}]_{8\%} = -20{,}000 + (878)\,(P \mid U, 8\%, 5)$$
$$+ (8000)\,(P \mid U, 8\%, 3)(P \mid F, 8\%, 5) + (5{,}000)(P \mid F, 8\%, 8)$$
$$= +239$$

TABLE 9.6 CASH FLOW PROFILES FOR OPERATION AND FINANCING

t	A_t	$\bar{A}_t$	$\hat{A}_t = A_t + \bar{A}_t$
0	−50,000	+30,000	−20,000
1–5 (each)	+8,000	− 7,122	878
6–8 (each)	+8,000	0	8,000
8 only	+5,000	0	5,000

When the operation and financing cash flows are discounted separately, we can see the contribution of financing to the project. In this case, the project would have been rejected if the project were financed by equity only. By borrowing externally at an interest rate of 6%, which is below the MARR of 8%, the debt financing makes a positive contribution to the project. Hence, the purchase is worthwhile.

Example 9.8

The Fulton Company plans to purchase a compressor that is expected to produce a savings of $5,000 per year over the next 10 years. The purchase price is $30,000 with no salvage value at the end. The company borrows $25,000 from the bank at 8% interest compounded annually, and the loan is to be repaid with equal annual payments including interest over a 6-year period beginning from the end of the first year. The MARR specified for both the operation and financing net cash flows is 10%. Is the purchase worthwhile?

For the financing cash flows from borrowing, the uniform annual payments for $t = 1, 2, \ldots, 6$ at 8% annual interest rate is

$$(25{,}000)(U \mid P, 8\%, 6) = 5{,}408$$

The cash flow profiles for operation and financing are given in Table 9.7.

TABLE 9.7 CASH FLOW PROFILES FOR OPERATION AND FINANCING

t	A_t	$\bar{A}_t$
0	−30,000	+25,000
1–6 (each)	+5,000	−5,408
7–10 (each)	+5,000	0

Using separate cash flow profiles and a MARR of 10% for both, we get

$$[\text{NPV}]_{10\%} = -30{,}000 + (5{,}000)(P \mid U, 10\%, 10) = +723$$

$$[\text{FPV}]_{10\%} = +25{,}000 - (5{,}408)(P \mid U, 10\%, 6) = +1{,}447$$

Hence,

$$\text{APV} = [\text{NPV}]_{10\%} + [\text{FPV}]_{10\%} = 723 + 1{,}447 = 2{,}170$$

In this case, the debt financing has produced a positive effect to the project since the borrowing interest rate at 8% is lower than the MARR of 10% for discounting the financing cash flows. Hence, the purchase is worthwhile.

9.6 EFFECTS OF DIFFERENT FINANCING PLANS

In undertaking an investment project through debt financing, an organization is confronted with many competing borrowing opportunities. The decision maker can shop around for the lowest interest rate, convenient timing of repayment, and other favorable terms. Consequently, the choice of a financing plan is not unlike that of selecting the best among mutually exclusive proposals for investment opportunities.

Since there are so many available instruments in the financial markets, public finance and corporate finance for investments are best left to the experts in those fields. It suffices to note that different financing plans will produce different effects on an investment, and that the available financing opportunities may affect the choice of investment opportunities. Although most organizations have separate departments for handling cost estimation for engineering projects and for seeking financing packages for their implementation, their possible interactions should not be ignored.

It should be noted that while individual financial instruments may be size independent, a project financing proposal reflecting a specific combination of financial instruments is often indivisible. Furthermore, a corporation may incur long-term debts through issuing bonds, but at the same time invest its idle funds in short-term commercial papers. Consequently, different financing plans for an investment project can be treated as indivisible mutually exclusive proposals. The objective of choosing a financing plan among these proposals is to minimize the present value of financing costs and thus to maximize the adjusted present value of the investment project. Hence, the net present value decision criterion is equally applicable to the choice of financing plans.

If a budget constraint is imposed, it does not mean that there can be no borrowing, but that there is a limit to borrowing. Thus, different financing plans may still be considered in order to produce the maximum desirable effect.

Example 9.9

The operation cash flow profile for acquiring a tractor is given in Table 9.8. The initial cost in year 0 is $22,000 and the salvage value at year 5 is $2,000. The uniform annual net income generated by the tractor is $6,000. There are four possible financing plans, each charging a borrowing rate of 8%. At a MARR of 8%, determine the adjusted present value based on each of these four financing plans.

The four financing plans shown in Table 9.8 illustrate some common methods of repayment:

(a) Repayment of principal and interest at the end of the planning horizon
(b) Annual repayment of interest only, plus the repayment of the principal at the end
(c) Uniform annual repayment covering both principal and interest

TABLE 9.8 CASH FLOWS FOR OPERATION AND FINANCING

t	A_t (Operation)	$\bar{A}_t$ (Financing) (a)	(b)	(c)	(d)
0	−22,000	+10,000	+10,000	+10,000	+10,000
1	+6,000	0	−800	−2,505	−2,800
2	+6,000	0	−800	−2,505	−2,640
3	+6,000	0	−800	−2,505	−2,480
4	+6,000	0	−800	−2,505	−2,320
5	{+6,000 +2,000	−14,693	−10,800	−2,505	−2,160

(d) Annual repayment of a constant fraction of principal and all accrued interest

Note that for each financing plan, the net present value discounted at the borrowing interest rate of 8% is zero.

The net present value of the cash flow profile for operation, discounted at MARR of 8%, is

$$\text{NPV} = -22{,}000 + (6{,}000)(P \mid U, 8\%, 5) + (2{,}000)(P \mid F, 8\%, 5) = +3{,}317$$

The cash flow profile combining operation and financing for each financing plan can be computed by Eq. (9.8) and the results for the four financing plans are shown in Table 9.9. For MARR = 8%, we find

(a) $[\text{APV}]_{8\%} = -12{,}000 + (6{,}000)(P \mid U, 8\%, 4) + (-8{,}693 + 2{,}000)(P \mid F, 8\%, 5) = +3{,}317$

(b) $[\text{APV}]_{8\%} = -12{,}000 + (5{,}200)(P \mid U, 8\%, 4) + (-4{,}800 + 2{,}000)(P \mid F, 8\%, 5) = +3{,}317$

TABLE 9.9 CASH FLOW PROFILES WITH DIFFERENT FINANCING PLANS

t	$\hat{A}_t = A_t = \bar{A}$ (combined) (a)	(b)	(c)	(d)
0	−12,000	−12,000	−12,000	−12,000
1	+6,000	+5,200	+3,495	+3,200
2	+6,000	+5,200	+3,495	+3,360
3	+6,000	+5,200	+3,495	+3,520
4	+6,000	+5,200	+3,495	+3,680
5	{−8,693 +2,000	{−4,800 +2,000	{+3,495 +2,000	{+3,840 +2,000

(c) $[\text{APV}]_{8\%} = -12{,}000 + (3{,}495)(P \mid U, 8\%, 5) + (2{,}000)(P \mid F, 8\%, 5)$
$= +3{,}317$

(d) $[\text{APV}]_{8\%} = -12{,}000 + (3{,}200)(P \mid U, 8\%, 5) + (160)(P \mid G, 8\%, 5)$
$+ (2{,}000)(P \mid F, 8\%, 5) = +3{,}317$

Since the MARR equals the borrowing interest rate for this problem, each financing plan contributes nothing (a net value of zero) to the adjusted present value of the cash flow profile combining operation and financing. Consequently, the adjusted present values for all four cases are identical.

It is noteworthy that the investment cash flow profile for operation has only one sign change, but two of the four combined cash flow profiles have multiple sign changes as indicated by cases (a) and (b) in Table 9.9. However, it can be shown that the combined cash flows for cases (a) and (b) are pure investment cash flows in the sense that each has a unique value of IRR (see Chapter 4). While the internal rate of return (IRR) for the cash flow profile for operation is found to be 11.3%, the values of IRR corresponding to the four combined cash flow profiles are as follows: (a) 20.3%, (b) 17.4%, (c) 14.0%, and (d) 17.0%. These values of IRR reflect the timing and intensities on the use of borrowed money in different financing plans. However, since the adjusted present values (APV) of these alternatives are equal for a MARR of 8%, the four financing plans are equally attractive.

Example 9.10

For the same investment project and financing plans in Example 9.9, determine the effect of each of the financing plans if (1) the specified MARR is 10%, and (2) the specified MARR is 6%.

In case 1, the specified MARR of 10% is greater than the borrowing interest rate of 8%. The net present value of the cash flow profile for operation, discounted at MARR = 10%, is

$$[\text{NPV}]_{10\%} = -22{,}000 + (6{,}000)(P \mid U, 10\%, 5) + (2{,}000)(P \mid F, 10\%, 5) = +1{,}987$$

The adjusted present value for each cash flow profile combining operation and a financing plan shown in Table 9.9 can be obtained as follows:

(a) $[\text{APV}]_{10\%} = -12{,}000 + (6{,}000)(P \mid U, 10\%, 4)$
$+ (-8{,}693 + 2{,}000)(P \mid F, 10\%, 5) = +2{,}864$

(b) $[\text{APV}]_{10\%} = -12{,}000 + (5{,}200)(P \mid U, 10\%, 4)$
$+ (-4{,}800 + 2{,}000)(P \mid F, 10\%, 5) = +2{,}745$

(c) $[\text{APV}]_{10\%} = -12{,}000 + (3{,}495)(P \mid U, 10\%, 5)$
$+ (2{,}000)(P \mid F, 10\%, 5) = +2{,}491$

(d) $[\text{APV}]_{10\%} = -12{,}000 + (3{,}200)(P \mid U, 10\%, 5)$
$+ (160)(P \mid G, 10\%, 5) + (2{,}000)(P \mid F, 10\%, 5)$
$= +2{,}470$

Note that for all four financing plans, the adjusted present value of the cash flow profile including debt financing is higher than that for operation alone. Thus, any of these four financing plans is more attractive than internal funding since the internal funds can be diverted to other opportunities to earn a return of 10%, which is higher than the borrowing interest rate of 8%. Furthermore, among the four financing plans, the one that makes the most use of the borrowed fund during the 5-year period is most desirable, as indicated by the highest APV for plan (a).

In case 2, the specified MARR of 6% is less than the borrowing interest rate of 8%. The net present value of the cash flow profile for operation, discounted at MARR = 6%, is

$$\begin{aligned}[\text{NPV}]_{6\%} &= -22{,}000 + (6{,}000)(P \mid U, 6\%, 5) + (2{,}000)(P \mid F, 6\%, 5)\\ &= +4{,}767\end{aligned}$$

The adjusted present value for each cash flow profile combining operation and a financing plan shown in Table 9.9 can also be obtained:

$$\text{(a)}\ [\text{APV}]_{6\%} = -12{,}000 + (6{,}000)(P \mid U, 6\%, 4) + (-8{,}693 + 2{,}000)(P \mid F, 6\%, 5) = +3{,}789$$

$$\text{(b)}\ [\text{APV}]_{6\%} = -12{,}000 + (5{,}200)(P \mid U, 6\%, 4) + (-4{,}800 + 2{,}000)(P \mid F, 6\%, 5) = +3{,}926$$

$$\text{(c)}\ [\text{APV}]_{6\%} = -12{,}000 + (3{,}495)(P \mid U, 6\%, 5) + (2{,}000)(P \mid F, 6\%, 5) = +4{,}217$$

$$\text{(d)}\ [\text{APV}]_{6\%} = -12{,}000 + (3{,}200)(P \mid U, 6\%, 5) + (160)(P \mid G, 6\%, 5) + (2{,}000)(P \mid F, 6\%, 5) = +4{,}244$$

Note that for this case, the net present value of each cash flow profile including debt financing is lower than that for operation alone, regardless of which financing plan is used. Thus, debt financing is not as attractive as internal funding, since any internal funds diverted to other opportunities can earn a return of only 6%, which is lower than the borrowing interest rate of 8%. As a result, the financing plan which keeps the largest amount of the loan over the borrowing period is least desirable, as indicated by the lowest APV for plan (a).

Example 9.11

A public agency is considering two mutually exclusive designs for different production capacities. The net cash flow profile of operation for both design alternatives over a planning horizon of 5 years is shown in Table 9.10. For each design alternative, the project can be financed through one of the two financing plans for that project shown in Table 9.10. The public agency has specified a MARR of 10% for discounting both the operation and financing cash flow profiles for this project. Determine the best combination of design and financing plan if (a) a design is selected before financing plans are consid-

TABLE 9.10 DESIGN AND FINANCING ALTERNATIVES (IN $ 1,000)

	Design proposal 1			Design proposal 2		
t	Operating cash flow	Financing plan A	Financing plan B	Operating cash flow	Financing plan C	Financing plan D
0	−1,550	+1,800	+3,800	−3,000	+3,000	$4,250
1	−2,500	+2,000	−400	−1,500	+1,500	−440
2	+1,500	−1,400	−400	+1,600	−1,500	−440
3	+1,500	−1,400	−400	+1,600	−1,500	−440
4	+1,500	−1,400	−400	+1,600	−1,500	−440
5	+1,500	−1,200	−4,400	+1,600	−2,250	−4,840

ered, or (b) the choice is made simultaneously rather than sequentially for operation and financing.

Using a MARR of 10% for both the operation and financing cash flow profiles, the resulting adjusted present values (in $ thousand) for various combinations of operating and financing cash flows are obtained and tabulated in Table 9.11. It is important to note that if the design proposals are evaluated first independent of the financing plans, the decision maker would have rejected design proposal 2 and considered only the alternative financing plans A and B for design proposal 1. Then, the best choice would be the combination of design proposal 1 and financing plan B, leading to APV = 300. However, if the design alternatives and financing plans are evaluated simultaneously, the best choice is the combination of design proposal 2 and financing plan C, leading to APV = 330.

TABLE 9.11 RESULTS FOR DESIGN AND FINANCING ALTERNATIVES (IN $ 1,000)

	Design proposal 1			Design proposal 2		
Item	Operating cash flow	Financing plan A	Financing plan B	Operating cash flow	Financing plan C	Financing plan D
NPV	+500			+338		
FPV		−292	−200		−8	−150
APV		+208	+300		+330	+188

Example 9.12

You are a small business person with an amount of $10,000 in a savings account which is not needed for business expansion until a year later. This savings account pays interest at an annual rate of 5%, which is your minimum attractive rate of return. Because of a sudden surge in the market interest rates for U.S. Treasury bills and the certificates of deposit offered by commercial banks, you are confronted with two unusual opportunities. The U.S. Treasury bill quoted at an annual interest rate of 14% requires an immediate payment of \$8,771.93 with a total return in the amount of \$10,000 at maturity 1 year later. On the other hand, a certificate of deposit quoted at an annual interest rate of 13.5% requires an immediate payment of \$10,000 with a total return of \$11,350 in principal and interest 1 year later. Assume that you cannot borrow more money and you can only deposit the money in the bank if you do not purchase either the U.S. Treasury bill or certificate of deposit. What is the total return on the \$10,000 at the end of 1 year as a result of making either of these investments?

This simple example in financial investment is used to illustrate the effects of the budget constraint on the use of the net future value, the net present value, the internal rate of return, and the overall rate of return for investment evaluation when the specified budget constraint is \$10,000. The minimum attractive rate of return is 5%, which is the interest rate from the savings account.

In buying the U.S. Treasury bill, you invest \$8,771.93 now at 14%, but 10,000 − 8,771.93 = 1,228.07 will be left in the savings account. Therefore, at the end of 1 year, you have

$$(8{,}771.93)(1 + 0.14) = \$10{,}000$$

$$(1{,}228.07)(1 + 0.05) = \$1{,}289.47$$

Hence, the total return is 10,000 + 1,289.47 = \$11,289.47. The total return from the certificate of deposit is simply (10,000)(1 + 0.135) = \$11,350.

The profit for buying the U.S. Treasury bill as expressed in the net future value is

$$\text{NFV} = 11{,}289.47 - (10{,}000)(1 + 0.05) = \$789.47$$

and that for buying the certificate of deposit is

$$\text{NFV} = 11{,}350.00 - (10{,}000)(1 + 0.05) = \$850.00$$

In other words, by buying the U.S. Treasury bill instead of leaving all \$10,000 in the bank, you will be \$789.47 ahead at the end of 1 year; but by buying the certificate of deposit instead of leaving the \$10,000 in the bank, you will be \$850.00 ahead at the end of 1 year. Consequently, the purchase of the certificate of deposit yields a higher net future value than that of the U.S. Treasury bill.

Using the net present value as a profit measure, we find for the former

$$\text{NPV} = (11{,}289.47)(1 + 0.05)^{-1} - 10{,}000 = \$751.88$$

and for the latter

$$\text{NPV} = (11{,}350)(1 + 0.05)^{-1} - 10{,}000 = \$809.52$$

These values also indicate that the purchase of the certificate of deposit is preferable. Note that if each of these present sums is deposited at the bank to accrue an annual interest of 5%, you will obtain at the end of the year the following:

$$(751.88)(1 + 0.05) = \$789.47$$

$$(809.52)(1 + 0.05) = \$850.00$$

This example also shows that the internal rate of return of an investment proposal per se is not necessarily the same as the overall rate of return to the investor. Note that the internal rate of return of the U.S. Treasury bill is 14% and the internal rate of return of the certificate of deposit is 13.5%. However, the overall rate of return from the purchase of the U.S. Treasury bill is

$$i^0 = \frac{11{,}289.47 - 10{,}000}{10{,}000} = 0.1289 = 12.89\%$$

For the certificate of deposit, the overall rate of return is simply 13.5%. Hence, the certificate of deposit is preferable in spite of the fact that the U.S. Treasury bill has a higher internal rate of return.

9.7 TRANSACTION COSTS IN FINANCING

In the discussion of project financing in Section 9.5, we introduce the concept of net cash flows in financing, which covers only the interests and principal repayments on loans. In reality, there are also transaction costs associated with project financing, and the net present value of financing cash flows (FPV) should include such costs.

For example, to obtain a mortgage loan of amount Q_0, the borrower may be required to pay k percentage points of the loan up front as a condition for lending; i.e., the borrower will pay an amount of kQ_0 at period $t = 0$ as well as interest I_t in subsequent periods $t = 1, 2, \ldots, n$ as stipulated in the loan agreement. In general, the higher the interest rate charged by the lender, the smaller will be the required percentage points up front. Let the amount representing the payment up front at $t = 0$ be denoted by PVQ and the present value of the cash flow representing the loan and repayments including interest over the life of the loan be denoted by PVT. Then, for this case

$$\text{PVQ} = kQ_0 \qquad (9.9)$$

Hence,

$$\text{FPV} = \text{PVT} - \text{PVQ} \qquad (9.10)$$

Similarly, if a corporation issues new bonds or new stocks through an investment bank, k percentage points will be charged on the basis of the total amount Q raised from the bonds or stocks. Then, the net amount Q_0 available will be $Q_0 = (1 - k)Q$. The transaction cost or origination fee is given by

$$\text{PVQ} = Q - Q_0 = \frac{Q_0}{1 - k} - Q_0 = \frac{kQ_0}{1 - k} \tag{9.11}$$

Hence, the FPV can also be obtained by using Eq. (9.10).

Example 9.13

Ernest Stone is trying to borrow \$90,000 for the purchase of a new home. A bank offers him a 30-year mortgage at 10% annual interest repayable in 30 uniform annual payments on the condition that he will pay 3% of the loan up front. A second bank offers him a 30-year mortgage at 11% annual interest repayable in 30 uniform annual payments but no payment up front. If his minimum attractive rate of return is 8%, which financing plan should he choose?

The annual uniform payments for each of the mortgages offered by the two banks are:

$$U_1 = (90{,}000)(U \mid P, 10\%, 30) = (90{,}000)(0.1061) = 9{,}549$$

$$U_2 = (90{,}000)(U \mid P, 11\%, 30) = (90{,}000)(0.1150) = 10{,}350$$

For MARR = 8%, the net present values of these financing plans are respectively

$$\begin{aligned} \text{FPV}_1 &= -(0.03)(90{,}000) - (9{,}549)(P \mid U, 8\%, 30) \\ &= -2{,}700 - (9{,}549)(11.2578) = -110{,}200 \\ \text{FPV}_2 &= -(10{,}350)(P \mid U, 8\%, 30) = -116{,}518 \end{aligned}$$

Hence, the mortgage offered by the first bank is preferable since the net present value of the financing plan is less negative.

If Ernest had enough money in a bank that pays annual interest at 8%, he could designate a specific amount that, together with the accrued interest, would be sufficient to pay off the mortgage eventually. Under the arrangement offered by the first bank, the designated amount would be \$110,200 while under the second arrangement, the designated amount would require \$116,518. Hence, the first financing plan is preferable.

Example 9.14

Jasper Corporation plans to raise \$30 million through issuing new stocks. The origination fee charged by the investment bank is 5%. What is the value of the stocks to be issued and what is the origination fee?

The value of stocks to be issued to raise \$30 million is

$$Q = \frac{Q_0}{1 - k} = \frac{30}{1 - 0.05} = \$31.579 \text{ million}$$

The origination fee is

$$\text{PVQ} = Q - Q_0 = 31.579 - 30 = \$1.579 \text{ million}$$

9.8 LEASING AS A FINANCING INSTRUMENT

A lease is a contract for the use of a physical asset over a specific period of time without acquiring its ownership. Under a lease contract, the user of the asset is referred to as the *lessee,* and the owner or provider is called the *lessor*.

From the viewpoint of the lessee, there are two major categories of lease contracts: the operating lease and the capital lease. An operating lease refers to a short-term contract whereby the lessee is entitled to use the asset for a fraction of its useful life, while the lessor may provide services such as maintenance and insurance. A capital lease refers to a noncancellable contract with the intention of transferring the ownership of the property to the lessee at or near the end of its useful life either through prior agreement or through a bargain purchase option in the contract.

If a physical asset can only be leased but not purchased, the decision whether to lease such an asset is an investment decision. Other than this exception, a leasing decision is a financing decision. In other words, the decision whether to acquire a physical asset for production or operation against other alternatives is an investment decision. Once a decision is made to acquire this asset, the decision whether to buy or lease is a financial decision. Consequently, leasing can be viewed as an instrument of financing in capital budgeting.

Renting is another means for allowing the use of a physical asset without owning it. In this context, it can be treated as an operating lease even though the two may have different legal and/or accounting implications. Both serve the purpose of reducing capital expenditures while retaining the use of a physical asset under capital rationing.

Example 9.15

The owner of a small business plans to lease a personal computer for 5 years. In talking to a sales representative, she learned that a new model with the same performance characteristics but lower price may be available every one or two years. She was offered three different options representing the changing prices of the newer models that may be provided when they become available. The cash flow profiles representing the costs of these noncancellable lease options are shown in Table 9.12. Assuming that the level of performance is identical for all options and using a MARR of 10%, which option should be selected?

The present values of costs of these options are computed as follows:

$$[\text{CPV}_1]_{10\%} = 24{,}000 + 16{,}000(P \mid U, 10\%, 4) = \$74{,}718$$

$$[\text{CPV}_2]_{10\%} = 22{,}000 + 11{,}000(P \mid U, 10\%, 2) + 11{,}000(P \mid U, 10\%, 4) = \$75{,}959$$

$$[\text{CPV}_3]_{10\%} = 18{,}000 + 18{,}000(P \mid U, 10\%, 4) = \$75{,}058$$

The first option has the minimum present value and is preferable.

TABLE 9.12 CASH FLOW PROFILES FOR THREE OPTIONS

t	$C_{t,1}$	$C_{t,2}$	$C_{t,3}$
0	24,000	22,000	18,000
1	16,000	22,000	18,000
2	16,000	22,000	18,000
3	16,000	11,000	18,000
4	16,000	11,000	18,000

Example 9.16

The Public Works Department of Alomos County plans either to rent or buy a special grader for repairing its access roads. The rental fee for the grader is \$60 per day. The purchase price of the grader is \$12,000 with an estimated useful life of 4 years and an estimated salvage value of \$2,000 at the end of 4 years. The annual maintenance cost of the grader, if purchased, is estimated to be \$1,200. All other benefits and expenses are otherwise identical. Based on MARR = 12%, how many working days per year must this grader be used in order to make the purchase worthwhile in comparison with renting?

Let W be the number of working days that the grader will be used during a year. Then, the cash flow profiles over 4 years for purchase (alternative x) and renting (alternative y) are given by $A_{t,x}$ and $A_{t,y}$ in Table 9.13, from which the incremental cash flow profile $A_{t,x-y}$ is computed. Hence,

$$\begin{aligned}\Delta\text{NPV}_{x-y} &= -12{,}000 + (60W - 1{,}200)(P \mid U, 12\%, 4) \\ &\qquad + (2{,}000)(P \mid F, 12\%, 4) \\ &= -12{,}000 + (60W - 1{,}200)(3.0373) + (2{,}000)(0.6355) \\ &= -14{,}374 + 182.24W\end{aligned}$$

For $\Delta\text{NPV}_{x-y} = 0$, we get $W = 79$. In other words, the alternative of purchasing the grader is preferred if it will be used for 79 days or more during each year.

TABLE 9.13 COMPUTATION OF THE INCREMENTAL CASH FLOW PROFILE

t	$\bar{A}_{t,x}$	$\bar{A}_{t,y}$	$\Delta\bar{A}_{t,x-y} = \bar{A}_{t,x} - \bar{A}_{t,y}$
0	−12,000	0	−12,000
1–4 (each)	−1,200	−60W	−1,200 + 60W
4	+2,000	0	+2,000

9.9 SUMMARY AND STUDY GUIDE

This chapter has presented the basic concepts of capital budgeting for allocating resources for investment projects in an organization. It discusses the problems of capital rationing when an organization does not wish to borrow to finance its investment opportunities, and covers the effects of financing when the organization borrows in the capital market to take up more investment opportunities.

The consideration of budget constraints has been limited to a conventional investment cash flow profile with outlay at time $t = 0$ only. The objective of project evaluation remains the maximization of net present value subject to the constraint imposed on the initial costs of available investment opportunities. For mutually exclusive proposals, the process that imposes the budget constraint is quite simple since only one of the proposals will eventually be selected. For independent projects, a number of proposals may be selected and a trial-and-error procedure is suggested in lieu of mathematical programming, which is beyond the scope of this book.

The choice of a financing option is not unlike that of selecting the best among mutually exclusive proposals for investment opportunities. Different financing plans will produce different effects on an investment. Furthermore, the available financing opportunities may affect the choice of investment opportunities; hence, their possible interaction should not be ignored.

Leasing may be viewed as an instrument of financing under budget constraint. Renting is another means for allowing the use of a physical asset without owning it. Both serve the purpose of reducing capital expenditures and the decisions on their use are financing decisions.

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PROBLEMS

P9.1 The cash flow profiles (in $1,000) of three mutually exclusive proposals have a specified MARR of 3%, the overall rates of return (ORR) of the projects are also computed. If the budget constraint on the present value of costs is $650,000, determine which alternatives should be selected on the basis of (a) net present values, and (b) the overall rates of return.

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−205	−350	−295
1	+50	+100	+70
2	+50	+90	+70
3	+50	+80	+70
4	+50	+70	+70
5	+50	+60	+70
ORR	5.3%	4.1%	4.7%

P9.2 Mr. Thorndike has $28,000 in a savings account that pays an annual interest rate of 5%. He plans to improve the income from this fund by investing either in U.S. Treasury notes or certificates of deposit. Each U.S. Treasury note has a face value of $10,000 at maturity 1 year later and is currently sold at a discount rate of 13%. Each certificate of deposit costs $10,000 now and is promised an annual interest rate of 12.5%. Assuming that Mr. Thorndike cannot borrow more money but can always deposit his money in the savings account, what is the overall rate of return on his $28,000 at the end of 1 year as a result of making either one of these two investments? Using the net present value as a measure of profit potential, which investment is better?

P9.3 Ms. Jordan has $12,000 in a savings account that pays an annual interest of 6%. She plans to purchase a U.S. Treasury note which has a face value of $10,000 at maturity 1 year later and is currently sold at a discount rate of 13.5%. Assuming that she cannot borrow more money but can always deposit her money in the savings account, what is the overall rate of return on her $12,000 at the end of 1 year if she decides to buy the U.S. Treasury note?

P9.4 Seven proposed independent projects with the present value of benefits (BPV) and the present value of costs (CPV) in thousands of dollars are shown below. Determine which projects should be selected if the budget constraint of the total present value of cost is: (a) $980,000, (b) $750,000, (c) $560,000, and (d) $370,000.

x	BPV_x	CPV_x
1	180	150
2	150	120
3	165	140
4	210	180
5	150	160
6	120	100
7	170	130

P9.5 Eight proposed independent projects with the present value of benefits (BPV) and the present value of costs (CPV) in million dollars are shown below. Determine which projects should be selected to maximize the total net benefit if the budget constraint on the total present value of costs is (a) $270 million and (b) $210 million.

x	BPV_x	CPV_x
1	85	75
2	80	65
3	70	45
4	50	30
5	95	80
6	35	25
7	32	35
8	84	60

P9.6 The present values of benefits and costs in millions of dollars for six independent projects ($x = 1, 2, \ldots, 6$) are found to be as follows:

x	BPV_x	CPV_x
1	1.700	0.500
2	2.250	1.200
3	6.000	3.100
4	3.300	1.400
5	2.800	1.000
6	3.750	1.900

If the budget constraint on the total present value of costs is \$3 million, which projects should be selected in order to maximize the net benefit? Use the trial method to obtain the most satisfactory answer.

P9.7 Rework Problem P9.6 if the budget constraint on the total present value of costs is (a) \$5 million and (b) \$6 million.

P9.8 The cash flow profiles of four *independent* projects (in millions of dollars) are given below. For a specified MARR of 8%, the overall rates of return (ORR) of the projects are also computed. Determine which projects should be selected if the budget constraint on the total present value of costs is (a) \$170 million and (b) \$250 million.

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$	$A_{t,4}$
0	−77.0	−77.0	−75.3	0
1	0	+38.0	+28.0	−28.0
2	0	+32.0	+28.0	−28.0
3	0	+26.0	+28.0	−28.0
4	0	+20.0	+28.0	−28.0
5	+235.0	+14.0	+28.0	+201.9
ORR	25.0%	15.5%	16.8%	16.9%

P9.9 Compute the benefit-cost ratios in Problem P9.8 to verify that the ranking of the projects based on B/C is identical to that based on ORR.

P9.10 The stream of cash flows for the acquisition and operation of an earth excavator is given by A_t. There are three financing plans, each charging a borrowing rate of 10%

but having a different method of repayment as represented by the stream of cash flows $\bar{A}_t$ for each financing plan. The values of A_t and $\bar{A}_t$ for $t = 0, 1, 2, 3, 4$ are given below. The MARR specified is also 10%. Find the three cash flow profiles combining operation and financing, and determine the adjusted present value for each of these three cash flow profiles.

		$\bar{A}_t$ (financing)		
t	A_t	(a)	(b)	(c)
0	−56,000	+20,000	+20,000	+20,000
1	+15,775	0	−6,310	−2,000
2	+15,775	0	−6,310	−2,000
3	+15,775	0	−6,310	−2,000
4	{+15,775 + 6,000	−29,282	−6,310	−22,000

P9.11 Jean Trudal is offered two alternatives to borrow $80,000 from a bank to finance the purchase of a new house. Both plans are based on a 20-year mortgage to be repaid by 20 uniform annual payments. The first plan charges 12% annual interest but no "points" up front. The second plan calls for 11% interest plus 4 points (4%) up front. Which borrowing plan should she choose if a MARR of 10% is specified?

P9.12 Wellington Group plans to raise $20 million through issuing new bonds. The origination fee charged by the investment bank is 6%. What is the origination fee and what is the value of the bonds to be issued?

P9.13 Crafton Corporation is considering the options of buying or leasing a machine for the next 5 years. The purchase price of the machine is $24,000 with a salvage value of $4,000 at the end of 5 years. The terms for leasing the machine require payments of $7,500, $6,500, $5,500, $4,500 and $4,000 at the end of years 0, 1, 2, 3, and 4, respectively. The net uniform annual income at the end of years 1, 2, 3, 4, and 5 is expected to be $6,500. The corporation has a MARR of 10% and does not plan to borrow money for the purchase. Determine whether it should buy or lease.

P9.14 A construction contractor has the option of leasing or renting a crane that she uses infrequently in some jobs. She can lease and maintain the equipment herself for the entire year at a cost of $3,200, and the operating cost is $80 per day when it is in use. She can also rent the equipment at $160 per day including operating cost if she does not want to keep the equipment when it is not in use. What is the number of days in use in a year that she will be indifferent to leasing or renting?

CHAPTER 10 Price Level Changes

10.1 INFLATION AND DEFLATION

Inflation refers to the general increase in prices and deflation refers to the general decrease in prices of goods and service with time. As the price level rises or falls to reflect the dynamic nature of the economy, the value of a dollar changes from year to year. The underlying causes of inflation or deflation are complex, but an immediate cause is the change in money supply. If there is an abundance of money available for the purchase of the same amount of goods and services, the prices will move up; conversely, if there is a scarcity of money, the prices will go down. The change in money supply results from the amount of money printed or made available to banks by the federal government, and from the velocity of circulation of money in the national economy.

While the prices of goods and services generally tend to move up or down together, they do not change at exactly the same rate. The general inflation rate or deflation rate reflects the trend of price level changes for all goods and services in the economy. The inflation rate or deflation rate for a specific item may be higher or lower than the general rate of change.

To measure inflation or deflation over a period of time, we keep track of the prices of a predefined package of goods and services over the years. We can select a specific year as the baseline of measurement. That year is referred to as the *base year* and the price of that package in the base year is referred to as the *base-year price or constant price*. A *price index* is the ratio of the price of the predefined package at a given year to the price of the same package in the base year. It is a dimensionless quantity, with the index of the base year set at a value of 100.

The major issues concerning the use of price indices in measuring inflation are the change of the package and the change of quality of goods and services over time. If the predefined package of goods and services is not changed from time to time, some items in the package may no longer be relevant while other relevant items are excluded from consideration in the package. Then, the index may be inadequate for the intended purpose. On the other hand, if the package is changed too frequently, the index will lose the continuity that is required. A compromise approach is to change the items in the package gradually and infrequently. The problem is that the change in the price level may be due either to improved quality or to inflation. It is not easy to sort out the causes. In spite of these imperfections, price indices are useful in measuring inflation or deflation in the economy.

10.2 GENERAL PRICE INDICES

The price indices most commonly used to measure inflation and deflation are the *consumer price index* (CPI) and the *gross national product implicit price deflator* (GNP deflator). The former is compiled and published by the Bureau of Labor Statistics, Department of Labor; the latter is produced by the Bureau of Economic Analysis of the Department of Commerce.

The consumer price index is a measure of the average change in prices over time in a fixed "market basket" of goods and services purchased by all urban consumers. The current CPI index is based on prices of food, clothing, shelter, fuels, transportation fares, charges for doctors' and dentists' services, drugs, etc. purchased for day-to-day living. All CPI indices previously expressed on a base of 100 in 1967 have been readjusted as 100 for the average of values in 1982, 1983, and 1984 to coincide with the updated expenditure weights applied to data from the Consumer Expenditure Surveys for those years.

The gross national product (GNP) is the total national output of goods and services valued at market prices. The goods and services included are largely those bought for final use in the market economy, including gross private domestic investment and net export of goods and services. The GNP deflator is derived as the ratio of current year to base year prices of the GNP multiplied by 100. The current GNP deflator is expressed on a base of 100 in 1982. Changes in the GNP deflator reflect not only changes in price but also changes in the composition of GNP.

The CPI and the GNP deflator and their corresponding annual percent changes from 1976 through 1987 are given in Table 10.1.[1]

[1] See Ref. 10.1 for the sources of data.

TABLE 10.1 GENERAL PRICE INDICES

	Consumer price index			GNP deflator	
Year	Base = 100 in 1967	Base = 100 in 1982–84	Annual % change	Base = 100 in 1982	Annual % change
1976	170.3	56.9	5.8	63.1	6.4
1977	181.4	60.6	6.5	67.3	6.7
1978	195.2	65.1	7.6	72.2	7.3
1979	217.3	72.6	11.3	78.6	8.9
1980	246.6	82.4	13.5	85.7	9.0
1981	272.0	90.9	10.3	94.0	9.7
1982	288.9	96.5	6.2	100.0	6.4
1983	298.1	99.6	3.2	103.9	3.9
1984	310.9	103.9	4.3	107.7	3.7
1985	322.1	107.6	3.6	110.9	3.0
1986	328.2	109.6	1.9	113.9	2.7
1987	340.0	113.6	3.6	117.7	3.3

10.3 ANNUAL PERCENT CHANGES IN PRICE INDICES

The annual percent change in a price index can be computed from the difference of its values in two consecutive years. Conversely, the index in any given year can be derived from its value in the previous year if the annual percent change in that year is known.

Let $I_0 = 100$ be a price index of a predefined package at the base year $t = 0$. Let I_t be the price index for the same package in year t, and I_{t+1} be the price index in the following year $t + 1$. Then, the percent change of price index in year $t + 1$ is given by

$$j_{t+1} = \frac{I_{t+1} - I_t}{I_t} (100\%) \tag{10.1}$$

where $t = 0, 1, 2, \ldots, n$, corresponding to the year from which the rate of price change is measured. If j_{t+1} is positive, it represents the inflation rate in year $t + 1$ over year t, and if it is negative, it represents the deflation rate. Conversely,

$$I_{t+1} = I_t(1 + j_{t+1}) \tag{10.2}$$

If the price index I_0 at the base year $t = 0$ is set at a value of 100, then, the price indices $I_1, I_2, \ldots, I_m$ for subsequent years $1, 2, \ldots, m$ can be computed successively by Eq. (10.2). For example,

$$I_m = I_0(1 + j_1)(1 + j_2) \cdots (1 + j_m) \tag{10.3}$$

where $j_1, j_2, \ldots, j_m$ are the annual percent change of prices in years $1, 2, \ldots, m$, respectively.

A set of price indices expressed on a base of 100 in year $t = 0$ can be rebased as 100 in year $t = m$, and vice versa. From Eq. (10.3), we obtain

$$\frac{I_m}{I_0} = (1 + j_1)(1 + j_2) \cdots (1 + j_{m-1})(1 + j_m) \tag{10.4}$$

If the set of price indices based on a value of 100 in year $t = 0$ is multiplied one by one by a factor of I_m/I_0, the resulting set of indices will be rebased on a value of 100 in year $t = m$. Conversely, if the set of price indices based on a value of 100 in year $t = 0$ is multiplied by a factor of I_0/I_m, the resulting set of indices will be rebased on the value of 100 in year $t = 0$.

Example 10.1

Determine the annual percent changes of the GNP deflator in 1986 and 1987 from the indices for 1985, 1986, and 1987 in Table 10.1.

$$j(1986) = \frac{113.9 - 110.9}{110.9} = 0.027 = 2.7\%$$

$$j(1987) + \frac{117.7 - 113.9}{113.9} = 0.033 = 3.3\%$$

Example 10.2

Suppose that we wish to convert the base of the GNP deflator from 1982 to 1987 by setting the index in 1987 at 100. What will be the GNP deflators for 1982 through 1987?

From Table 10.1, we note that $I(1982) = 100.0$ and $I(1987) = 117.7$. Then, the factor to be used for conversion is

$$\frac{I(1982)}{I(1987)} = \frac{100.0}{117.7} = 0.8496$$

Then, the GNP deflators for 1982 through 1987 are obtained by multiplying the corresponding indices in Table 10.1 by this factor. Hence, $I(1982) = 85.0$, $I(1983) = 88.3$, $I(1984) = 91.5$, $I(1985) = 111.0$, $I(1986) = 96.8$, and $I(1987) = 100$.

Example 10.3

Suppose that we wish to convert the base of the CPI from 1967 to a base using the average in 1982, 1983, and 1984 as 100. What conversion factor should be used?

From Table 10.1, we note that the average of the CPI indices in 1982, 1983, and 1984 based on 1967 value as 100 is given by

$$\frac{288.9 + 298.1 + 310.9}{3} = 299.3$$

Thus, the conversion factor from the base 1967 to this new base is

$$\frac{100}{299.3} = 0.3341$$

Using this new base, we obtain

$$I(1982) = (288.9)(0.3341) = 96.5$$

$$I(1983) = (298.1)(0.3341) = 99.6$$

$$I(1984) = (310.9)(0.3341) = 103.9$$

The average of these new indices is indeed 100.

10.4 SPECIAL PRICE INDICES

The need for special price indices has been highlighted by the fact that the price changes in some components of an investment project may outpace disproportionately those of the remaining components. For example, the outlays for plants and facilities often require new construction, the costs of which are affected by the prices of special materials and labor. Two special cost indices are compiled and reported periodically in the *ENR* (*Engineering News-Record*).[2] The construction cost index (CCI) was designed as a general-purpose construction cost index, which is a weighted aggregate measure of constant quantities of structural steel, portland cement, lumber, and common labor. The building cost index (BCI) was introduced to weigh the impact of skilled labor on construction cost by replacing the common labor component with the average wages of the carpenter, bricklayer, and structural iron workers. Both indices measure the effects of wage rate and material price trends, but they are not adjusted for productivity, efficiency, competitive conditions, or automation changes.

Very often, we may be interested in the price changes in a local market rather than a national market. Consequently, regional or local price indices are developed for that purpose. For examples, the *ENR* provides both the BCI and CCI for 20 cities in the United States: Atlanta, Baltimore, Birmingham, Boston, Chicago, Cincinnati, Cleveland, Dallas, Denver, Detroit, Kansas City, Los Angeles, Minneapolis, New Orleans, New York, Philadelphia, Pittsburgh, St. Louis, San Francisco, and Seattle. These local indices measure only the trend in an individual city. Differentials between cities may reflect differences in labor productivity and building codes, or different quoting bases for some materials.

On the other hand, special price indices derived from comparative international statistics are of great importance in comparing the economy of the United States and that of the world as a whole. For example, the annual percent changes in

[2] See *ENR* (*Engineering News-Record*), The McGraw-Hill Construction Weekly, March 29, 1990, pp. 68–77.

consumer prices in the United States and member countries of the Organization for Economic Cooperation and Development (OECD) are available for comparison.[3] The member countries include Australia, Canada, Japan, New Zealand, Austria, Belgium, Denmark, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, Turkey, the United Kingdom, and the United States. It should be noted that the quality and comparability of international statistics are affected by many factors including method of data collection and estimation, extent of coverage, and precision of definition.

10.5 FORECASTING GENERAL PRICE CHANGES

It can be seen from the price indices in Table 10.1 that the rate of price change varies from year to year. To convert an amount of cash expressed in base-year dollars to an equivalent amount expressed in the value of dollars in a later year, we must apply Eq. (10.2) successively. Let A_t be the cash flow in year t expressed in terms of base-year (year 0) dollars, and A_t' be the cash flow in year t expressed in terms of then-current dollars as shown in Fig. 10.1. Then,

$$A_t' = A_t(1 + j_1)(1 + j_2) \cdots (1 + j_{t-1})(1 + j_t) \tag{10.5}$$

Conversely,

$$A_t = A_t'(1 + j_t)^{-1}(1 + j_{t-1})^{-1} \cdots (1 + j_2)^{-1}(1 + j_1)^{-1} \tag{10.6}$$

For the purpose of economic evaluation, it is often sufficient to project the trend of future prices by using a constant rate of price change j for each year. Then,

$$A_t' = A_t(1 + j)^t \tag{10.7}$$

and

$$A_t = A_t'(1 + j)^{-t} \tag{10.8}$$

[3] See p. 472 of Ref. 10.2.

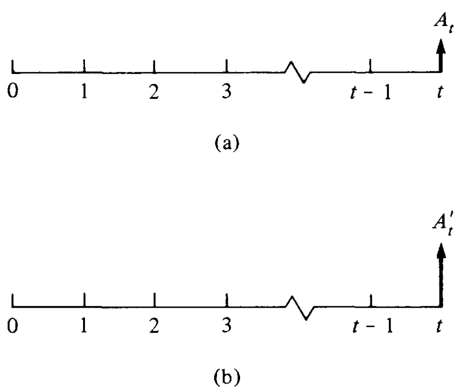


Figure 10.1 Constant price and current price at future time.

Note that the discrete compound interest tables in Appendix A may be used to evaluate the factors $(1 + j)^t$ and $(1 + j)^{-t}$ in Eqs. (10.7) and (10.8) by treating j as if it were a compound interest rate. In reality, of course, j is the compound rate of price change.

Although the rate of general price change j may be positive or negative, representing the general inflation rate or deflation rate, respectively, we will consider primarily the case of inflation since this has been the general trend of price level changes in the last four decades. Thus, a positive j (in percent) denotes the *general inflation rate* which may be used to convert an amount A_t in year t, expressed in terms of base-year dollars, to the inflated amount A'_t in year t, expressed in terms of then-current dollars according to Eq. (10.7). Conversely, A'_t may be converted to A_t by means of Eq. (10.8).

Example 10.4

Suppose that the annual inflation rate in the next 3 years is expected to be 10% and that in the following 2 years it slows to 8%. If a stream of cash flows for the next 5 years consists of a uniform annual amount of $10,000 expressed in terms of base-year dollars, what will they be when expressed in terms of then-current dollars?

For the first 3 years, then-current values can be computed directly by Eq. (10.7). For the last 2 years, however, we must consider the change of inflation rate. Hence,

$$A'_1 = (10{,}000)(1 + 0.10) = 11{,}000$$

$$A'_2 = (10{,}000)(1 + 0.10)^2 = 12{,}100$$

$$A'_3 = (10{,}000)(1 + 0.10)^3 = 13{,}310$$

$$A'_4 = (10{,}000)(1 + 0.10)^3(1 + 0.08) = 14{,}375$$

$$A'_5 = (10{,}000)(1 + 0.10)^3(1 + 0.08)^2 = 15{,}525$$

10.6 EFFECTS OF PRICE CHANGES ON ECONOMIC EVALUATION

In the economic evaluation of investment proposals, two approaches may be used to offset the effects of future price changes. The differences between the two approaches can be succinctly stated as follows:

1. The *constant price* approach. The cash flows are expressed in terms of base-year dollars, and a discount rate excluding general inflation is used in computing the net present value.
2. The *current price* approach. The cash flows are expressed in terms of then-current dollars, and a discount rate including general inflation is used in computing the net present value.

Let i be the discount rate excluding inflation and i' be the discount rate including the inflation. Let j be the average rate of price change for each year. Then, by definition, it may appear that

$$i' = i + j \tag{10.9}$$

or

$$i = i' - j \tag{10.10}$$

However, to make the two approaches produce identical results in discounting, it is necessary to redefine i' more precisely such as that when the current price approach is used. That is,

$$1 + i' = (1 + i)(1 + j) \tag{10.11}$$

Hence,

$$i' = i + j + ij \tag{10.12}$$

Noting that $i' - j = i(1 + j)$ in Eq. (10.12), we find

$$i = \frac{i' - j}{1 + j} \tag{10.13}$$

If i and j are small in comparison to 1, the cross-product term ij in Eq. (10.12) is negligible and the denominator $(1 + j)$ in Eq. (10.13) can be approximated by 1. Then, Eqs. (10.9) and (10.10) can be regarded as the approximation of Eqs. (10.12) and (10.13), respectively.

Let A_t be the cash flow in year t expressed in terms of base-year dollars and A_t' be the cash flow in year t expressed in terms of then-current dollars. Then, for the constant price approach, the net present value of a cash flow A_t in year t is the amount discounted at a rate i to year 0 as follows:

$$\text{NPV} = A_t(1 + i)^{-t} \tag{10.14}$$

In view of Eq. (10.8), we get

$$\text{NPV} = A_t'(1 + j)^{-t}(1 + i)^{-t}$$

Hence,

$$\begin{aligned}\text{NPV} &= A_t'[(1 + j)(1 + i)]^{-t}\\ &= A_t'(1 + i + j + ij)^{-t}\end{aligned}$$

Then,

$$\text{NPV} = A_t'(1 + i')^{-t} \tag{10.15}$$

For the current price approach, the net present value of a cash flow A_t' in year t is the amount discounted at a rate i' to year 0 as shown in Eq. (10.15). In view of Eq. (10.7), we get

$$\text{NPV} = A_t(1 + j)^t(1 + i')^{-t} \tag{10.16}$$

Hence,

$$\text{NPV} = A_t\left(\frac{1 + i'}{1 + j}\right)^{-t} = A_t\left(1 + \frac{i' - j}{1 + j}\right)^{-t}$$

Then,

$$\text{NPV} = A_t(1 + i)^{-t}$$

For a series of cash flows at $t = 0, 1, 2, \ldots, n$ over the planning horizon n as shown in Fig. 10.2, the NPV obtained by using the constant price approach is

$$\text{NPV} = A_0 + \sum_{t=1}^{n} A_t(1 + i)^{-t} \tag{10.17}$$

Similarly, the NPV obtained by using the current price approach is

$$\text{NPV} = A_0 + \sum_{t=1}^{n} A'_t(1 + i')^{-t} \tag{10.18}$$

In both cases, A_0 (at $t = 0$) is not affected by the inflation. It can easily be verified that as long as Eqs. (10.14) and (10.15) hold, the results obtained from Eqs. (10.17) and (10.18) are identical.

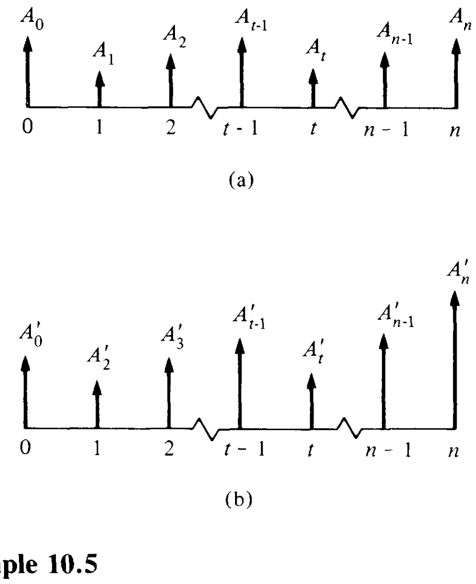


Figure 10.2 Constant price and current price cash flow profiles.

Example 10.5

Mary Benson plans to invest \$1,200 for 5 years with the expectation of a minimum attractive rate of return of 4% excluding inflation. If the inflation rate will be 9% per year, what is the return (in then-current dollars) at the end of the 5 years that will make it worthwhile? What is the rate of return including inflation?

For the investment to be worthwhile, the NPV should at least be zero. Let A'_5 be the return in then-current dollars; then $A_5 = A'_5(1 + j)^{-5}$. Using Eq. (10.14),

$$\text{NPV} = -1{,}200 + A'_5(1 + j)^{-5}(1 + i)^{-5} = 0$$

Hence, for $i = 4\%$, $j = 9\%$,

$$A'_5 = (1{,}200)(1 + 0.09)^5(1 + 0.04)^5 = 2{,}246$$

Also, from Eq. (10.12),

$$\begin{aligned} i' &= i + j + ij = 0.04 + 0.09 + (0.04)(0.09) \\ &= 0.1336 = 13.36\% \end{aligned}$$

Example 10.6

David Hearn deposited \$1,000 in a bank 5 years ago and received a total return of \$1,403 today. During this period, the annual inflation rate was 3%. What is the annual rate of return excluding inflation?

The interest rate i' paid by the bank which includes inflation can be obtained by Eq. (10.15):

$$\text{NPV} = -1{,}000 + 1{,}403(1 + i')^{-5} = 0$$

Hence,

$$(1 + i')^5 = \frac{1{,}403}{1{,}000} = 1.403 \qquad \text{or} \qquad (F \mid P, i', 5) = 1.403$$

from which we get $i' = 7\%$. Then, for $j = 3$, Eq. (10.13) yields

$$i = \frac{i' - j}{1 + j} = \frac{0.07 - 0.03}{1 + 0.03} = 3.88\%$$

Example 10.7

Harry Brown has recently inherited \$30,000 and wants to set aside the equivalent of \$20,000 in today's dollars to be available 10 years later when he will start his retirement. He expects the inflation rate to be 8% per year over the next 10 years. He wishes to consider only the opportunities available to him that are risk-free and has found the following two alternatives:

1. Depositing the money in a savings account in a bank that pays 6% interest compounded annually
2. Purchasing a 10-year certificate of deposit that pays 10% interest compounded annually

How much money must he now set aside to accomplish his goal for each of the two alternatives?

In both cases, the interest rate implicitly includes a component for inflation since no additional compensation to that effect will be paid by the bank if inflation does occur at a rate of $j = 8\%$ as anticipated by Harry. Consequently, he should first compute the cash flow A'_t at $t = 10$ in terms of then-current dollars, which is equivalent to $20,000 of base-year dollars. Thus,

$$A'_{10} = A_{10}(1 + j)^{10} = (20{,}000)(1 + 0.08)^{10} = 43{,}178$$

For case 1, $i' = 6\%$, the present value of cost is

$$[\text{CPV}]_{6\%} = A'_{10}(1 + i')^{-10} = (43{,}178)(1 + 0.06)^{-10} = 24{,}110$$

For case 2, $i' = 10\%$, the present value of cost is

$$[\text{CPV}]_{10\%} = A'_{10}(1 + i')^{-10} = (43{,}178)(1 + 0.10)^{-10} = 16{,}647$$

Although Harry can afford to choose either plan, it is obvious that the second plan is superior. In the first plan, the interest cannot even keep up with the inflation; consequently, he has to set aside more than $20,000 now to maintain the same purchasing power 10 years from now.

10.7 THE CONSTANT PRICE APPROACH

The constant price approach has the distinct advantage of estimating future benefits and costs in base-year dollars without speculating on the general inflation rate. The simplicity of this approach is particularly appealing to tax-exempt organizations and public agencies. For private corporations that pay taxes on the basis of then-current dollars, this approach does not offer the same degree of simplification.

The Office of Management and Budget (OMB) requires that, except for certain exemptions, federal agencies must use this approach for evaluating public investment projects.[4] The OMB also specifies that, with certain exceptions, federal agencies should use a discount rate of 10% excluding inflation to evaluate investment projects under its jurisdiction.

Example 10.8

A piece of office equipment costing $10,000 with no salvage value after 5 years is expected to produce a uniform annual net benefit of $3,000 in terms of the base-year dollars over the next 5 years. The MARR excluding inflation is specified to be 8% per year. Determine whether the investment is worthwhile if the average annual inflation rate is 5%.

Since the costs and benefits of this investment are given in terms of the base-year dollars, the annual inflation rate can be ignored by using the constant price approach. Thus,

[4] OMB Circular A-94, "Discount Rates to Be Used in Evaluating Time-Distributed Costs and Benefits," Executive Office of the President, March 27, 1972.

$$[\text{NPV}]_{8\%} = -10{,}000 + (3{,}000)(P \mid U, 8\%, 5)$$
$$= -10{,}000 + (3{,}000)(3.9927) = 1{,}978$$

Hence, the investment is worthwhile.

If we choose to use the current price approach, we can obtain the same result in a roundabout way. That is, we first find the annual cash flows in then-current dollars:

$$A'_1 = (3{,}000)(1.05)^1 = 3{,}150$$
$$A'_2 = (3{,}000)(1.05)^2 = 3{,}308$$
$$A'_3 = (3{,}000)(1.05)^3 = 3{,}473$$
$$A'_4 = (3{,}000)(1.05)^4 = 3{,}647$$
$$A'_5 = (3{,}000)(1.05)^5 = 3{,}829$$

The MARR including inflation is given by

$$i' = 0.08 + 0.05 + (0.08)(0.05) = 0.134$$

Then,

$$[\text{NPV}]_{13.4\%} = -10{,}000 + \frac{3{,}150}{1.134} + \frac{3{,}308}{(1.134)^2} + \frac{3{,}473}{(1.134)^3} + \frac{3{,}647}{(1.134)^4} + \frac{3{,}829}{(1.134)^5} = 1{,}979$$

10.8 THE CURRENT PRICE APPROACH

In some situations, the minimum attractive rate of return for an investor is restricted by a fixed lending or borrowing interest rate set for a relatively long period of time, irrespective of the general inflation trend during this period. For example, the interest rate on a mortgage loan for the purchase of real estate may remain constant for the duration of the loan, which is usually 20 or 30 years. Under such circumstances, it is more convenient to use a discount rate which includes a component accounting for the inflation rate. In that case, the current price approach can be used to offset the effects of inflation on the economic evaluation of investment proposals.

Furthermore, even if the annual cash flows are given in base-year dollars, they must be converted to then-current dollars to estimate tax liability in the economic evaluation of projects undertaken by private corporations. Consequently, the current price approach is more convenient under certain circumstances.

Example 10.9

A machine costing \$20,000 with no salvage value after 4 years is expected to generate a uniform net benefit of \$7,000 in terms of the base-year dollars over the next 4 years. The MARR including inflation is specified to be 10% per year. Determine whether this investment is worthwhile if the average annual inflation rate is 5%.

For the given $i' = 10\%$ including inflation, we can use the current price approach by noting that for $t = 1, 2, 3$, and 4,

$$A_t' = A_t(1 + j)^t = (7{,}000)(F \mid P, 5\%, t)$$

Then

$$[\text{NPV}]_{10\%} = -20{,}000 + \sum_{t=1}^{4} (7{,}000)(F \mid P, 5\%, t)(P \mid F, 10\%, t) = 4{,}960$$

If the constant price approach is used, we first find the MARR excluding inflation. From Eq. (10.12),

$$i = \frac{0.10 - 0.05}{1 + 0.05} = 0.0476$$

To find the NPV, we use Eq. (3.16) in Chapter 3:

$$P = U\frac{(1 + i)^n - 1}{i(1 + i)^n} = (7{,}000)\frac{(1.0476)^4 - 1}{(0.0476)(1.0476)^4} = 24{,}960$$

Hence, $[\text{NPV}]_{10\%} = -20{,}000 + 24{,}900 = 4{,}960$.

10.9 DIFFERENTIAL PRICE CHANGE RATES

In some situations, the prices of certain key items affecting the estimates of future benefits and costs are expected to escalate faster than the general price levels. Then, it becomes necessary to consider the differential price changes over and above the general inflation in assessing the cash flows. Let k (in percent) denote the *differential inflation rate* for a specific item over and above the general inflation rate. If the general inflation rate is j and the total inflation rate of that specific item is q, then by definition, it may appear that

$$q = j + k \tag{10.19}$$

However, to simplify the consideration of general price change and differential price change in two steps, it is necessary to redefine q more precisely such that

$$1 + q = (1 + j)(1 + k) \tag{10.20}$$

from which

$$q = j + k + jk \tag{10.21}$$

If j and k are small in comparison with 1, the cross-product term jk in Eq. (10.21) is negligible. Then Eq. (10.19) can be regarded as the approximation of Eq. (10.21).

To consider the effects on differential inflation of a special item above and beyond the general inflation in two steps, let us introduce the following notation:

A_t = the cash flow pertaining to a specific item in year t, expressed in base-year dollars (i.e., $j = 0$ and $k = 0$).
A_t^0 = the cash flow pertaining to that specific item in year t relative to the general price in year t, expressed in base-year dollars (i.e., $j = 0$ but $k \neq 0$).
A_t' = the cash flow pertaining to that specific item in year t expressed in then-current dollars (i.e., $j \neq 0$ and $k \neq 0$).

Then,

$$A_t^0 = A_t(1 + k)^t \tag{10.22}$$

$$A_t' = A_t^0(1 + j)^t \tag{10.23}$$

Combining Eqs. (10.22) and (10.23)

$$A_t' = A_t(1 + k)^t(1 + j)^t \tag{10.24}$$

In view of Eq. (10.20),

$$A_t' = A_t(1 + q)^t \tag{10.25}$$

In dealing with the differential price change of a specific item, two approaches may be used for discounting the cash flows:

1. *The constant general price approach.* Since this specific item has a differential price change relative to the general price level, the cash flow A_t^0 expressed in base-year dollars instead of A_t is pertinent in reflecting the price change of that item as if there were no general inflation when a discounting rate i excluding general inflation is applied.
2. *The current general price approach.* The cash flow A_t' expressed in then-current dollars is discounted at a rate i' which includes the general inflation but not the differential price change of the specific item. The inclusion of the general inflation only in i' provides the contrast of the differential price change of the specific item above the general price changes of all items considered.

Using the constant general price approach, the net present value of cash flow pertinent to this specific item is given by

$$[\text{NPV}]_i = A_t^0(1 + i)^{-t} \tag{10.26}$$

Substituting Eq. (10.22) into Eq. (10.26) gives

$$[\text{NPV}]_i = A_t(1 + k)^t(1 + i)^{-t} \tag{10.27}$$

Using the current general price approach, we find

$$[\text{NPV}]_{i'} = A_t'(1 + i')^{-t} \tag{10.28}$$

Substituting Eqs. (10.24) into Eq. (10.28) yields

$$[\text{NPV}]_{i'} = A_t(1 + k)^t(1 + j)^t(1 + i')^{-t} \tag{10.29}$$

In view of Eq. (10.11),

$$[\text{NPV}]_{i'} = A_t(1 + k)^t(1 + i)^{-t} \tag{10.30}$$

In other words, by using the definition of q in Eq. (10.20) as well as the definition of i' in Eq. (10.11), we can demonstrate that both approaches lead to identical results. Hence, the general inflation rate j need not enter into the computation if we are only interested in finding the effect of the differential inflation rate k.

For a cash flow profile of A_t for $t = 0, 1, 2, \ldots, n$, the relationship between A_t and A_t^0, and that between A_t^0 and A_t', are schematically illustrated in Fig. 10.3. In general, the effect of the differential inflation rate on economic evaluation of the given profile is indicated by the net present value using a discount rate i as follows:

$$[\text{NPV}]_i = A_0 + \sum_{t=1}^{n} A_t(1 + k)^t(1 + i)^{-t} \tag{10.31}$$

Alternatively, using a discount rate i'

$$[\text{NPV}]_{i'} = A_0 + \sum_{t=1}^{n} A_t(1 + k)^t(1 + j)^t(1 + i')^{-t} \tag{10.32}$$

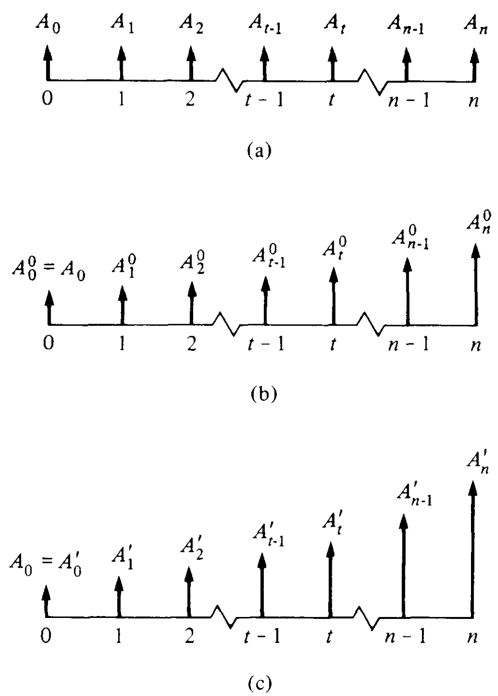


Figure 10.3 Differential price changes.

Example 10.10

In considering the investment in an energy conservation project for an existing building, it is expected that the extra insulation will save the same amount of energy each year over the next 15 years. However, the cost of energy is expected to increase over and above the general inflation rate by 3%. If the energy saved each year costs \$60,000 at today's prices, what would be the *relative* cost of the same amount of energy in year t, compared to the then-current general price level but expressed in base-year dollars?

Since the differential inflation rate k is 3%, the relative cost of the energy in year t is obtained by Eq. (10.22) for $n = 1, 2, \ldots, 15$,

$$A_t^0 = A_t(1 + k)^t = (60{,}000)(1 + 0.03)^t$$

The value A_t can be obtained with the aid of the discrete compound interest tables in Appendix A since A_t can be expressed in the functional notation:

$$A_t^0 = (60{,}000)(F \mid P, 3\%, t)$$

Example 10.11

An office building owned by the federal government is being retrofitted at a cost of \$500,000 for the purpose of energy conservation. It is expected that as a result of retrofitting, the consumption of electricity for heating and cooling the building will be reduced each year for the next 15 years by an amount that costs \$60,000 at today's prices. If the cost of electricity is expected to escalate 3% per year over and above the general inflation rate, determine whether the retrofitting project is worthwhile. The MARR excluding the general inflation rate is specified to be 10%.

The cash flow A_t^0 for $t = 1, 2, \ldots, 15$ representing the savings in base-year dollars including additional savings from the differential rate of $k = 3\%$ is given in Example 10.10 as $A_t^0 = (60{,}000)(1.03)^t$. Then, at the specified MARR of 10%, we have

$$\begin{aligned}[\text{NPV}]_{10\%} &= -500{,}000 + \sum_{t=1}^{15} (60{,}000)(1.03)^t(1.10)^{-t} \\ &= -500{,}000 + (60{,}000) \sum_{t=1}^{15} (F \mid P, 3\%, t)(P \mid F, 10\%, t) \\ &= -500{,}000 + (60{,}000)(9.2263) = +53{,}578\end{aligned}$$

Hence, the project is worthwhile.

It is easy to show that if there were no *differential* price change between the rate increase in the cost of electricity and the general inflation rate, the project would not be worthwhile. That is, with $A_t = 60{,}000$ for $t = 1, 2, \ldots, 15$, the net present value discounted at MARR = 10% is

$$[\text{NPV}]_{10\%} = -500{,}000 + (60{,}000)(P \mid U, 10\%, 15) = -43{,}634$$

Example 10.12

In anticipation of drastic increases in energy prices in the next five years, a publicly owned power plant installs an energy saving device which costs \$350,000 in base-year price and is expected to produce savings of \$100,000 per year in base-year prices of energy for the next five years. The device will last for 5 years with no salvage value at the end. The general inflation rate is expected to be 6% per year over the next 5 years, and the differential inflation rate for energy to be 2% per year above the general inflation rate. The power plant has a MARR of 8% excluding inflation.

(a) Find the NPV using the constant general price approach.
(b) Find the NPV using the current general price approach after expressing the cash flows in the then-current dollars.

Since both approaches should lead to identical results, the first approach is definitely preferred. The second approach is introduced only to demonstrate the solution.

(a) For the constant price approach, $k = 2\%$, and $i = 8\%$:

$$[\text{NPV}]_{8\%} = 350{,}000 + \sum_{t=1}^{5} (100{,}000)(1 + 0.02)^t(1 + 0.08)^{-t}$$

$$= -350{,}000 + (10{,}200)(0.9259) + (104{,}040)(0.8573)$$

$$+ (106{,}121)(0.7938) + (108{,}243)(0.7350) + (110{,}408)(0.6806)$$

$$= 72{,}584$$

(b) For the current price approach, $k = 2\%$, $i = 8\%$, and $j = 6\%$:

$$i' = (1 + 0.08)(1 + 0.06) - 1 = 0.1448 = 14.48\%$$

$$[\text{NPV}]_{14.48\%} = -350{,}000 + \sum_{t=1}^{5} (100{,}000)(1 + 0.02)^t(1 + 0.06)^t(1 + 0.1448)^{-t}$$

$$= 350{,}000 + \frac{108{,}120}{1.1448} + \frac{116{,}892}{(1.1448)^2} + \frac{126{,}392}{(1.1448)^3} + \frac{136{,}655}{(1.1448)^4} + \frac{147{,}751}{(1.1448)^5}$$

$$= 72{,}582$$

10.10 OBLIGATED CASH FLOWS IN BUDGETING FOR CAPITAL PROJECTS

It was noted in Section 9.1 that the estimates of costs and benefits in economic evaluation are not obligated amounts for budgeting in future years. However, for the purpose of capital budgeting, both private corporations and public agencies must apply inflation factors in projecting revenues (or benefits) and costs for proposed capital projects. Hence, accurate forecasting of inflation rates for specific labor, material, and energy costs as well as anticipated sales prices of goods and services (or user charges) becomes very important. The differential inflation rates of these factors may be decisive in both the evaluation and financing of capital projects.

Because the accounting costs for a project are recorded on an "as spent" and "as received" basis, the high inflation rates in the 1970s and 1980s have had an extremely important effect on the obligated amounts in budgeting for a capital project. An accurate projection of the starting date and completion date of a project is essential in maintaining the budgeted amounts of the then-current dollars. Any delay in a project, whether it is the delay of the starting date or the extension of the project duration, will automatically increase the accounting costs because of inflation.

Consequently, we must be fully aware of the distinction between the analysis of inflation in economic evaluation of capital projects and that in budgeting obligated cash flows for their implementation.

10.11 SUMMARY AND STUDY GUIDE

In this chapter, we discussed the concept of price level changes and price indices which reflect inflation or deflation. The consumer price index (CPI) and the gross national product implicit price deflator (GNP deflator) were used as examples of general price indices. The computation of annual percent changes in price indices and the conversion from one base year to another for such indices were also explained. Some special price indices were also cited for illustration.

We emphasized two approaches that may be used in economic evaluation to offset the effects of future price changes. In the *constant price approach,* the investor wants a specified minimum attractive rate of return excluding inflation. Consequently, the cash flows should be expressed in terms of base-year dollars, and a discount rate *excluding* inflation should be used in computing the net present value. On the other hand, in the *current price approach,* the investor includes an inflation component in the specified minimum attractive rate of return. Hence, the cash flows should be expressed in terms of then-current dollars, and a discount rate *including* inflation should be used in computing the net present value. If these approaches are applied correctly, they should lead to identical results.

The equivalence of these two approaches was demonstrated first by deriving the equations needed for treating the effects of price changes and then by tracing the similar effects resulting from the two approaches. Numerical examples were given to illustrate the application of both approaches. The *constant price approach* has the distinct advantage of simplicity when no income tax is involved and is favored by the public agencies. For private firms that pay income taxes on the basis of then-current value, the *current price approach* is often used.

If differential rates must be applied to some critical items in evaluating investment projects, then such modifications can be included in the evaluation regardless of which approach is used.

REFERENCES

10.1 *ENR (Engineering News-Record)*, New York: McGraw-Hill (weekly publication).

10.2 *Statistical Abstracts of the United States*, 109th ed. Washington, DC: U.S. Department of Commerce, Bureau of Census, 1989.

PROBLEMS

P10.1 Determine the annual percent changes of the CPI in 1980, 1982, and 1984 from CPI data in Table 10.1 with the base of 100 in (a) 1967 and (b) 1982–84.

P10.2 Suppose that we wish to convert the base of GNP deflator from 1982 to 1977 with a base of 100 in 1977. What will be the GNP deflators for 1977 through 1982?

P10.3 A hydraulic machine for laying pipes valued at $25,000 now must be replaced 7 years later. Suppose that the annual inflation rate during the next 4 years is expected to be 9% and that in the following 3 years it is to be reduced to 5%. The real cost of this machine would remain unchanged if there were no inflation. What is the then-current value of the machine at the end of 7 years?

P10.4 A corporation plans to set aside the equivalent of $100,000 in today's dollars for its chief executive when she retires 10 years from today. The inflation rate is expected to be 10% per year during the next 4 years and 8% per year thereafter. If the corporation can invest the money at 6% rate of return excluding inflation,

(a) How much should it invest now to obtain the equivalent of $100,000 in today's dollars 10 years from now?

(b) How many dollars (in the value of dollars 10 years later) will the executive receive 10 years from now?

P10.5 An investment of $80,000 in a new underwater bulldozer is expected to produce a net uniform revenue of $30,000 per year in base-year dollars over the next 5 years with no salvage value. Using a MARR of 12% including inflation, determine whether the investment is worthwhile if the average annual inflation rate is 6% during the next 5 years.

P10.6 An investment of $60,000 in a heavy truck is expected to produce a net uniform revenue of $20,000 per year in then-current dollars during the next 5 years with no salvage value. Using a MARR of 10% excluding inflation, determine whether the investment is worthwhile if the average annual inflation rate is 10% during the next 3 years and 8% during the subsequent 2 years.

P10.7 Joyce McNeil has just invested $10,000 in a 6-year certificate of deposit which pays an annual compound interest rate of 12%. If the annual inflation rate during this period is expected to be 8%, what is the annual rate of return excluding inflation? What is the value in base-year dollars for the amount she gets back when she cashes in the certificate at the end of 6 years?

P10.8 Edward Johnson has agreed to loan a sum of $1,000 to a friend for 4 years at a 5% annual compound interest rate excluding inflation. If the annual inflation rate during the next 4 years is 10%, what is the annual compound interest rate including inflation? What will be the sum in then-current dollars that will be repaid to him at the end of 4 years?

P10.9 In an investment proposal for a 4-year project, the cash flows expressed in base-year dollars are $A_0 = -\$20{,}000$, $A_1 = +\$4{,}000$, $A_2 = +\$6{,}000$, $A_3 = +\$8{,}000$, and $A_4 = +\$12{,}000$. The MARR specified for the project is 20% irrespective of whether there will be inflation. Determine whether the project is worthwhile assuming that

(a) There will be no inflation.

(b) The inflation rate will be 5% per year.

P10.10 An investment of $260,000 in a tower crane will generate a net annual uniform revenue of $60,000 in base-year dollars over the next 5 years with a salvage value of $20,000 at the end of 5 years. Using a MARR of 6% including inflation, determine whether the project is worthwhile if the average annual inflation rate is 3% during the next 5 years.

P10.11 A university invests $40,000 in an automated switch to turn off lights in classrooms and expects to save $8,000 per year in base-year dollars for the next 10 years with no salvage value at the end. The average annual general inflation rate is expected to be 4% in the next 10 years.The cost of the electricity is expected to have a differential inflation rate of 2% above the general inflation rate for the next 4 years and 1% in the subsequent 6 years. A MARR of 7% excluding inflation is specified.

(a) Find the NPV using the constant general price approach.

(b) Find the NPV using the current general price approach after expressing the cash flows in the then-current dollars.

P10.12 The energy cost of conventional home heating is expected to have an inflation rate of 6% per year above the general inflation rate during the next 10 years and 3% above the general inflation rate for the following 15 years. If a solar heating system is installed, a typical home costs $3,000 more for additional equipment with no salvage value at the end of the 25 years of service. The annual saving in energy cost in terms of base-year price levels and expressed in base-year dollars is $500 per year for 25 years to the owner. Determine whether this investment is worthwhile if the MARR excluding inflation is 10%.

P10.13 A new model of electric clothes dryer is advertised as saving 5% of electricity compared to an existing model which is sold for $100 less. Both dryers last for 5 years with no salvage value at the end. A family that spends an estimated $80 per year at the base-year value on electricity for drying clothes with the existing model needs to replace its dryer and is comparing the new and the existing models. The annual differential inflation rate of electricity is expected to be 2% above the general inflation. If the MARR of this family excluding inflation is 6%, which model is a better choice?

P10.14 A public agency plans to install an energy-saving device for a new building at $600,000. This device is expected to save $90,000 per year in base-year dollars for the next 25 years with no salvage value. The average annual general inflation rate is expected to be 3% in the next 25 years. The energy price is expected to have a differential infla-tion rate of 2% above the general inflation rate during the next 10 years and 4% above the general inflation rate for the following 15 years. The agency uses a MARR of 10% excluding inflation. Find the net present value of this investment to determine whether it is worthwhile.

CHAPTER

11 Depreciation

11.1 DEPRECIATION AS A TAX DEDUCTION

The government levies taxes on corporations and individuals to pay the costs of governmental operations, to promote socially desired services such as defense and education, and to redistribute income among different segments of the society. The federal government also plays an active role to stimulate economic growth and to maintain stability, and the mix and amount of taxation are matters of public policy. The regulations on depreciation allowances are part of the taxation policy intended to encourage capital investments made by private corporations.

Depreciation is defined as the decline in value of physical assets, such as buildings and equipment, over their estimated useful lives. In the context of tax liability, depreciation is the amount allowed as a deduction in computing taxable income and, hence, income tax. It is a bookkeeping entry that does not involve an outlay of cash, and the amount of depreciation allowed by laws and regulations is referred to as the *depreciation allowance*.

It is important to differentiate between the *estimated useful life* used in depreciation computation and the *actual useful life* of a physical asset. The former is often an arbitrary length of time, specified in the regulations for computing federal income taxes, while the latter refers to the economic life of the physical asset which may be terminated because of deterioration or technologial obsolescence. The estimated useful life used in depreciation computation is also referred to as *depreciable life*.

To understand the depreciation allowance, it is necessary to retrace the cycle of the acquisition, use, and disposal of a physical asset. When the physical asset is first purchased, an expenditure is made, but the expenditure cannot be deducted as an expense in computing federal income taxes because no expense is incurred when one asset is exchanged for another, e.g., cash for a physical asset. The outflow of resources comes at the end of the useful life of the physical asset when it is disposed of. The depreciation allowance is a way of recognizing that this outflow did not happen all at once but was in the process of taking place over a period of years. So, the depreciation allowance is a systematic allocation of the cost of a physical asset between the time it is acquired and the time it is disposed of.

11.2 DEPRECIATION REGULATIONS

The methods of computing depreciation, as well as the estimated useful lives for various classes of equipment, are specified by government regulations which are a part of the federal tax code. Thus, depreciation regulations are subject to revision periodically. For the purpose of capital project planning and evaluation, it is more important to understand the rationale underlying the changes than the regulations themselves. By understanding the economic climates in which the changes were made in the past, the planner can anticipate possible future changes in making economic evaluations. While such regulations are important to accountants who are responsible for filing tax returns, they are not treated in detail here.

The federal government has established a set of guidelines for specifying the depreciable lives for various classes of assets, called the *asset depreciation range* (ADR), which is revised periodically in accordance with the prevailing tax laws. The most drastic change in such guidelines was embodied in the Economic Recovery Tax Act of 1981 which provided an *accelerated cost recovery system* (ACRS) to encourage capital investment.[1] However, the Tax Reform Act of 1986 modified the ACRS rules for tangible depreciable assets, which are referred to as recovery property in the tax code.[2] Under the modified ACRS, the classes of depreciable assets can be illustrated by examples in Table 11.1.

At the election of the taxpayer, depreciation for most tangible assets may be computed by the straight-line method, the sum-of-the-years'-digits method, the double declining balance method, and other "consistent methods." The straight-line method assumes that the depreciation is uniform during the useful life. Other methods are referred to as accelerated depreciation methods. However, the use of accelerated depreciation for real estate is limited. In any case, an asset may not be depreciated below a reasonable salvage value under any method of computing depreciation.

[1] The Economic Recovery Tax Act, which was signed into law on Aug. 13, 1981, provided much faster recovery of the costs of capital assets but some provisions were repealed in 1986.

[2] See Ref. 11.1 for highlights of revision in the Tax Reform Act, which was signed into law on October 22, 1986.

TABLE 11.1 CLASSES OF RECOVERY PROPERTY

Property class	Examples of depreciable assets
3-year	Special tools for manufacturing
5-year	Cars, light truck and qualified technological equipment
7-year	Railroad track and agriculture structures
10-year	Railroad tank cars and certain coal utilization property
15-year	Waste treatment plants and telephone distribution plants
20-year	Municipal sewers
27.5-year	Residential real property with 80% or more rental income from dwelling units
31.5-year	Nonresidential properties

11.3 ACCOUNTING CASH FLOW PROFILE

A systematic allocation of the cost of a physical asset over its estimated useful economic life, often referred to as the depreciable life, can be represented by an accounting cash flow profile consisting of annual depreciation allowances. The different methods of computing depreciation lead to different annual depreciation allowances and, hence, have different effects on the taxable income and the taxes paid.

The cost of an asset less the salvage value is called the *net depreciable value*. The *annual depreciation allowance* refers to the amount of depreciation allowed for a given year. The sum of all the annual depreciation allowances from the current and previous years is called *accumulated depreciation*. The *book vlaue* of an asset is the undepreciated value which is equal to the cost of the asset less accumulated depreciation up to the current year.

Let P be the cost of an asset, S its estimated salvage value, and N the estimated useful life (depreciable life) in years. Furthermore, let D_t denote the depreciation amount in year t, T_t denote the accumulated depreciation up to year t, and B_t denote the book value of the asset at the end of year t, where $t = 1, 2, \ldots,$ or N refers to the particular year under consideration. Then,

$$T_t = D_1 + D_2 + \cdots + D_t = \sum_{j=1}^{t} D_j \tag{11.1}$$

where j is a dummy index for summation from 1 to t. Also, by definition,

$$B_t = P - T_t \tag{11.2}$$

or

$$B_t = B_{t-1} - D_t \tag{11.3}$$

Since depreciation is calculated over the estimated useful economic life of a physical asset and not over the longer period of the asset's actual useful life, we should distinguish between the depreciable life N and the actual useful life n in economic evaluation. Generally, the book value of the asset at the end of n years is assumed to be S since no more depreciation is allowed after year N. Suppose that the actual useful life n is equal to the depreciable life N. Then, the cost of a physical asset at the time of its acquisition and the residual value at the time of its disposal shown in part (a) of Fig. 11.1 can be replaced by the corresponding accounting cash flow profile of depreciation allowances in part (b) of Fig. 11.1. However, if the actual useful life n is greater than the depreciable life N, the cycle of acquisition and disposal over n years has the same effect as if the used asset were sold and repurchased at a price S at the end of year N with no change of salvage value for periods beyond N as shown in part (a) of Figure 11.2. Then, the corresponding accounting

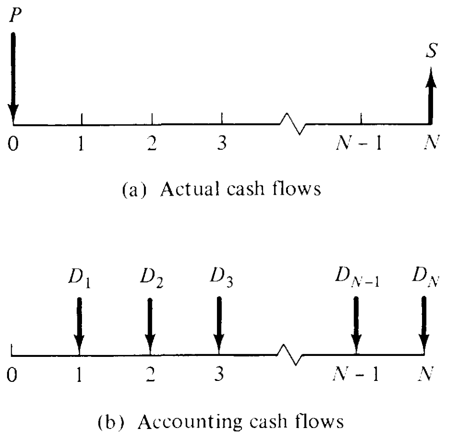


Figure 11.1 Accounting cash flows over the estimated useful life.

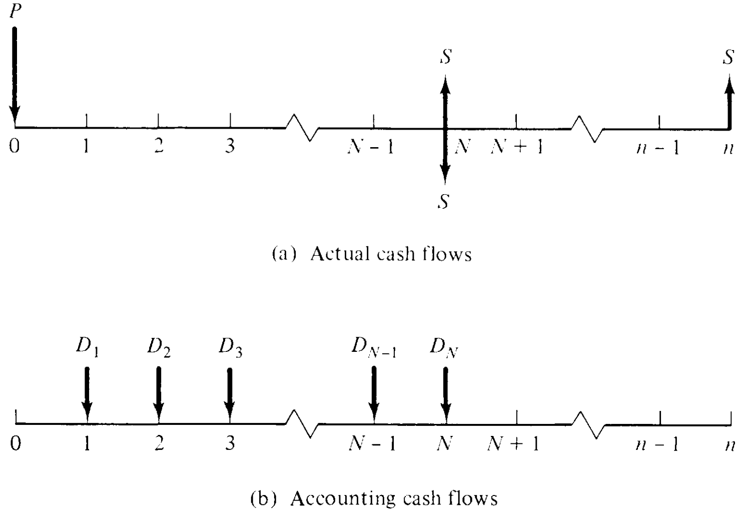


Figure 11.2 Accounting cash flows relative to actual life of asset.

cash flow stream can be represented by the depreciation allowances in part (b) of Fig. 11.2.

The annual depreciation allowances are deduced from the annual incomes of a corporation before taxes are computed. The larger the annual depreciation allowance, the smaller the taxable income for a given annual income before tax and hence the smaller the income tax for the year. Since the net depreciable value over the estimated useful life of an asset is constant, the total tax liability is the same, given everything else being equal. However, by using the depreciation methods which lead to accelerated depreciations in early years, a taxpayer can defer the tax liability and thus earn interest on the tax deferments.

11.4 STRAIGHT-LINE DEPRECIATION

The most common form of depreciation, and the one easiest to comprehend, is *straight-line depreciation*. Under straight-line (SL) depreciation, the cost of a physical asset, less its estimated salvage value, is allocated uniformly to each year of the estimated useful life. Thus, D_t is constant for all values of t and is given by

$$D_t = \frac{P - S}{N} \tag{11.4}$$

From Eq. (11.2), the book value at the end of year t is

$$B_t = P - \left(\frac{P - S}{N}\right)t \tag{11.5}$$

The relationship of various terms in Eqs. (11.4) and (11.5) for straight-line depreciation is illustrated in Fig. 11.3.

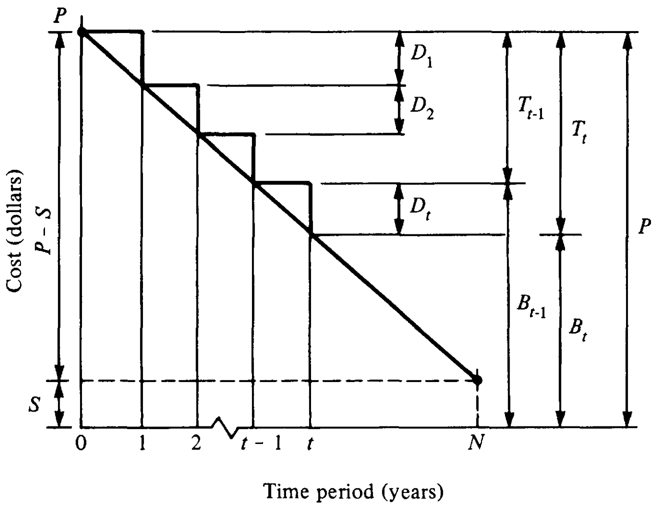


Figure 11.3 Annual depreciation allowances.

Example 11.1

A compressor was purchased at a cost of \$16,000 with an estimated salvage value of \$2,000 and an estimated useful life of 7 years. Determine the depreciation allowance for each year and the book value at the end of each year, using straight-line depreciation.

For $P = \$16{,}000$, $S = \$2{,}000$, and $N = 7$, we have for each year ($t = 1, 2, \ldots,$ or 7)

$$D_t = \frac{16{,}000 - 2{,}000}{7} = \$2{,}000$$

Furthermore,

$$B_t = 16{,}000 - 2{,}000t$$

Hence, $B_1 = \$14{,}000$, $B_2 = \$12{,}000$, $B_3 = \$10{,}000$, $B_4 = \$8{,}000$, $B_5 = \$6{,}000$, $B_6 = \$4{,}000$, and $B_7 = \$2{,}000$.

11.5 SUM-OF-THE-YEARS'-DIGITS DEPRECIATION

Several other methods of depreciation are known as *accelerated depreciation methods*. Under an accelerated depreciation method, the depreciation allowance for an asset will be greater in the earlier years of its life and less in the later years of its life than under the straight-line method. The total depreciation allowance would be the same under the accelerated method as under the straight-line method, and, hence, total taxable income would be the same for the whole period; however, greater allowances in the earlier years also mean less tax in the earlier years. When the time value of money is taken into consideration, there is a substantial advantage in having the smaller amounts of taxable income in earlier years and larger amounts of taxable income in later years. Thus, using a method of accelerated depreciation amounts to getting an interest-free loan from the government.

One form of accelerated depreciation is the *sum-of-the-years'-digits* (*SOYD*) *method*. Under the SOYD method, the annual depreciation allowance is obtained by multiplying the net depreciable value ($P - S$) by a fraction which has as its numerator the number of years of remaining useful life, and its denominator the sum of all the digits from 1 to N. Let Z be the sum of the years' digits such that

$$Z = 1 + 2 + \cdots + (N - 1) + N = \frac{N(N - 1)}{2}$$

Then, the depreciation allowance for any year t is

$$D_t = \frac{N - (t - 1)}{Z}(P - S) \qquad (11.6)$$

Thus, for $t = 1, 2, \ldots, (N - 1), N$, we have

$$D_1 = \frac{N}{Z}(P - S),\ D_2 = \frac{(N - 1)}{Z}(P - S),\ \ldots,\ D_{N-1} = \frac{2}{Z}(P - S),$$

$$D_N = \frac{1}{Z}(P - S)$$

Substituting the value of Z into Eq. (11.6) yields

$$D_t = \frac{N - (t - 1)}{N(N + 1)/2}(P - S) \tag{11.7}$$

The accumulated depreciation at the end of year t is

$$T_t = \frac{N + (N - 1) + \cdots + (N - t + 1)}{Z}(P - S)$$

$$= \frac{(2N - t + 1)t}{2Z}(P - S) = \frac{(2N - t + 1)t}{N(N + 1)}(P - S)$$

Hence, the book value at the end of year t is given by

$$B_t = P - \frac{(2N - t + 1)t}{N(N + 1)}(P - S) \tag{11.8}$$

or from Eq. (11.3), we find

$$B_t = B_{t-1} - \frac{N - (t - 1)}{Z}(P - S) \tag{11.9}$$

Example 11.2

Find the depreciation allowance for each year and the book value at the end of each year for the asset in Example 9.1, using the SOYD method.

In this problem, $Z = 1 + 2 + 3 + \cdots + 7 = 28$ and $P - S = 14{,}000$. From Eqs. (11.6) and (11.3), we get

$D_1 = \frac{7}{28}(14{,}000) = \$3{,}500$ $B_1 = 16{,}000 - 3{,}500 = \$12{,}500$

$D_2 = \frac{6}{28}(14{,}000) = \$3{,}000$ $B_2 = 12{,}500 - 3{,}000 = \$9{,}500$

$D_3 = \frac{5}{28}(14{,}000) = \$2{,}500$ $B_3 = 9{,}500 - 2{,}500 = \$7{,}000$

$D_4 = \frac{4}{28}(14{,}000) = \$2{,}000$ $B_4 = 7{,}000 - 2{,}000 = \$5{,}000$

$D_5 = \frac{3}{28}(14{,}000) = \$1{,}500$ $B_5 = 5{,}000 - 1{,}500 = \$3{,}500$

$D_6 = \frac{2}{28}(14{,}000) = \$1{,}000$ $B_6 = 3{,}500 - 1{,}000 = \$2{,}500$

$D_7 = \frac{1}{28}(14{,}000) = \$500$ $B_7 = 2{,}500 - 500 = \$2{,}000$

11.6 DECLINING BALANCE DEPRECIATION

The *declining balance depreciation* is another form of accelerated depreciation in which the depreciation allowance for any year t is obtained by multiplying the book value of the previous year $(t - 1)$ by a *constant depreciation rate* r. Since the book value at year 0 is the original cost of the asset, i.e., $B_0 = P$, we have

$$D_1 = B_0 r,\ D_2 = B_1 r,\ \ldots,\ D_{N-1} = B_{N-2} r,\ D_N = B_{N-1} r$$

Thus, for year t,

$$D_t = B_{t-1} r \tag{11.10}$$

The book value at the end of any year t can be obtained by noting that, from Eqs. (11.2) and (11.3),

$$B_1 = B_0 - D_1 = P - Pr = P(1 - r)$$

$$B_2 = B_1 - D_2 = B_1 - B_1 r = B_1(1 - r) = P(1 - r)^2$$

$$B_3 = B_2 - D_3 = B_2 - B_2 r = B_2(1 - r) = P(1 - r)^3$$

Hence, for year $t - 1$,

$$B_{t-1} = P(1 - r)^{t-1} \tag{11.11}$$

Substituting the value of B_{t-1} into Eq. (11.10), we get

$$D_t = Pr(1 - r)^{t-1} \tag{11.12}$$

Also, from Eq, (11.3),

$$B_t = B_{t-1} - D_t = P(1 - r)^t \tag{11.13}$$

The advantage of the declining balance depreciation method is that the salvage value S need not be estimated in the beginning years when the method is used for filing annual income tax returns. But there is no assurance that the accumulated depreciation at the end of year N will equal the net depreciable value $(P - S)$ if the constant depreciation factor r is arbitrarily chosen. In order to ensure that $T_N = P - S$ or $B_N = S$ for $S > 0$, we find the value of B_N by letting $t = N$ in Eq. (11.13) such that

$$P(1 - r)^N = S$$

from which we obtain

$$r = 1 - \sqrt[N]{\frac{S}{P}} \tag{11.14}$$

Under the long-standing guidelines issued by the U.S. government, a factor $r = 2/N$ is allowed for all new depreciable property other than real estate (which includes buildings but excludes land, which is not depreciable). A smaller factor

$r = 1.5/N$ is allowed for used machinery and equipment. Thus, Eq. (11.14) cannot be applied if the r thus obtained exceeds the specified limits.

On the other hand, when the specified r does not satisfy Eq. (11.14), the book value B_N at the end of N years will either be less than or greater than S. In the former case, no further depreciation will be allowed when the accumulated depreciation reaches the net depreciation value $(P - S)$ before year N. In the latter case, larger depreciation is permitted in the latter years so that the accumulated depreciation at the end of year N will be adjusted to the net depreciable value $(P - S)$. The conditions of $B_N < S$, $B_N = S$, and $B_N > S$ are shown in Fig. 11.4. If $B_N = S$ at the specified r, no adjustment need be made. If $B_N < S$, then no further depreciation is allowed after t_2, and the accumulated depreciation at t_2 is $(P - S)$. If $B_N > S$, the regulations allow the switching from the declining balance method to the straight-line method after t_1 when the latter method leads to larger annual allowances.

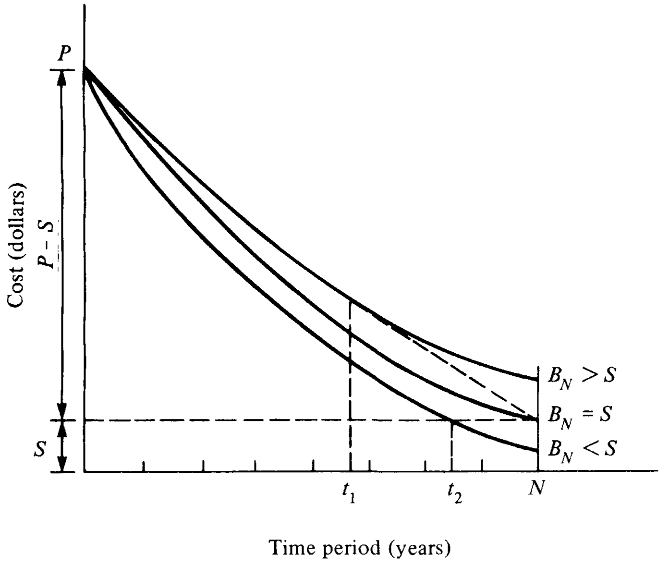


Figure 11.4 Declining balance depreciation.

Example 11.3

For the asset in Example 11.1, find the constant depreciation factor r such that $B_N = S$.

Using Eq. (11.14), we find that for $P = 16{,}000$, $S = 2{,}000$, and $N = 7$,

$$r = 1 - \sqrt[7]{\frac{2{,}000}{16{,}000}} = 1 - 0.743 = 0.257$$

Since $2/N = 0.286$, the value of $r = 0.257$ is below the specified factor for new equipment except real estate and is therefore acceptable. If $r = 0.257$ is used instead of $r = 2/N = 0.286$, we have not taken full advantage of the accelerated depreciation permitted by the regulations.

Example 11.4

If the salvage value of the asset in Example 11.1 is \$500 instead of \$2,000, find the constant depreciation factor of an asset r such that $B_N = S$.

In this case, we have $P = 16{,}000$, $S = 500$, and $N = 7$. Hence

$$r = 1 - \sqrt[7]{\frac{500}{16{,}000}} = 1 - 0.610 = 0.390$$

Since this value of r is greater than $2/N = 0.286$, it is not permitted under the regulations.

11.7 DOUBLE DECLINING BALANCE DEPRECIATION

Since the constant depreciation factor $r = 2/N$ is applicable to many classes of assets except real estate, the declining balance depreciation based on $r = 2/N$ is a common form of accelerated depreciation and is referred to as the *double declining balance* (DDB) *depreciation* since the factor $2/N$ is twice as great as the corresponding multiplier $1/N$ for the straight-line method.

Because $r = 2/N$ is arbitrarily chosen without using Eq. (11.4), the book value B_N at the end of year N is not expected to be equal to S. If $B_N < S$, we simply stop further depreciation at the end of the year during which the accumulated depreciation has reached the net depreciable value $(P - S)$. If $B_N > S$, we can switch to the straight-line method after the year when the latter method leads to larger annual depreciation allowances. The resulting annual depreciation allowances for these two cases are shown schematically in Fig. 11.5.

In converting the later years of double declining balance depreciation to straight-line depreciation, we note that the annual depreciation allowance at year t based on the double declining balance depreciation is

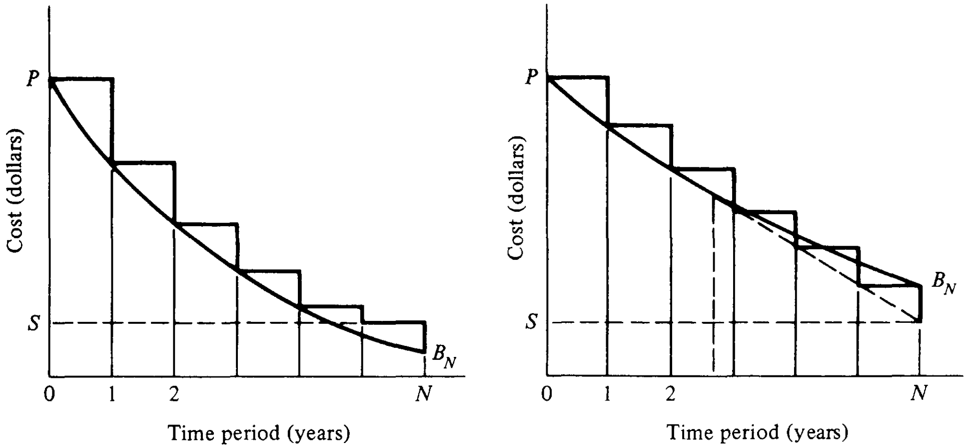
$$D_t = rB_{t-1} = \frac{2}{N} B_{t-1}$$

The annual depreciation allowance for year t and for each subsequent year if the depreciation is converted to the straight-line depreciation after year $(t - 1)$ is given by

$$D_t = \frac{1}{N - (t - 1)} (B_{t-1} - S)$$

Thus, we should switch to straight-line depreciation when the SL depreciation is greater than the DDB depreciation; i.e.,

$$\frac{B_{t-1} - S}{N - (t - 1)} > \frac{2}{N} B_{t-1} \tag{11.15}$$



(a) $B_N < S$ (b) $B_N > S$

Figure 11.5 Modification of double declining balance depreciation.

We can start with year N in the descending order of t, i.e., N, $(N - 1)$, $(N - 2)$, . . . , until we reach a value of t at which the relationship in Eq. (11.15) no longer holds. For example, for $t = N, (N - 1), (N - 2)$, Eq. (11.15) becomes, respectively,

$$\frac{B_{N-1} - S}{1} > \frac{2}{N} B_{N-1} \quad \text{(or } D_N \text{ based on DDB)}$$

$$\frac{B_{N-2} - S}{2} > \frac{2}{N} B_{N-2} \quad \text{(or } D_{N-1} \text{ based on DDB)}$$

$$\frac{B_{N-3} - S}{3} > \frac{2}{N} B_{N-3} \quad \text{(or } D_{N-2} \text{ based on DDB)}$$

These relationships can easily be verified for the given values of P, S, and N.

Example 11.5

For the asset in Example 11.1, using the DDB method, find the depreciation allowance for each year and the book value at the end of each year. The accumulated depreciation must not exceed the limit allowed by the U.S. Internal Revenue Service.

Since $r = 2/N = 2/7$, we can use Eqs. (11.10) through (11.13) to compute D_t and B_t alternately as follows:

$$D_t = B_{t-1}(\tfrac{2}{7})$$

$$B_t = B_{t-1} - D_t$$

Thus, using the DDB method and noting that $B_0 = 16{,}000$, we get

$$D_1 = \$4{,}571.43 \qquad B_1 = \$11{,}428.57$$

$$D_2 = \$3{,}265.31 \qquad B_2 = \$8{,}163.26$$

$$D_3 = \$2{,}332.36 \qquad B_3 = \$5{,}830.90$$

$$D_4 = \$1{,}665.97 \qquad B_4 = \$4{,}164.93$$

$$D_5 = \$1{,}189.98 \qquad B_5 = \$2{,}974.95$$

$$D_6 = \$849.99 \qquad B_6 = \$2{,}124.96$$

$$D_7 = \$607.13 \qquad B_7 = \$1{,}517.83$$

Since B_7 is less than the salvage value $S = 2{,}000$, the full depreciation for that year is not allowed. Therefore, we must adjust the annual depreciation allowance for $N = 7$ by the amount

$$S - B_7 = 2{,}000 - 1{,}517.83 = 482.17$$

Then the adjusted values of D_7 and B_7 are

$$D_7' = 607.13 - 482.17 = \$124.96$$

$$B_7' = \$2{,}000$$

Since B_6 is greater than $S = 2{,}000$, no adjustment is needed for $N \leq 6$.

Example 11.6

If the salvage value of the asset in Example 11.1 is \$500 instead of \$2,000, using the DDB method, find the depreciation allowance for each year and the book value at the end of each year. Convert to the SL method to take advantage of the full net depreciable value allowed by the U.S. Internal Revenue Service if appropriate.

Since $P = 16{,}000$, $S = 500$, and $N = 7$, the results of D_t and B_t using the DDB method are exactly the same as those computed for Example 11.5. However $B_7 = 1{,}517.83$ thus obtained is greater than $S = 500$. Hence, the conversion to the SL method is carried out by finding out the year at which the conversion should begin. By using Eq. (11.15), we find

$$t = 7 \quad \frac{2{,}124.96 - 500}{1} = 1{,}624.96 \quad (>607.13)$$

$$t = 6 \quad \frac{2{,}974.95 - 500}{2} = 1{,}237.48 \quad (>849.99)$$

$$t = 5 \quad \frac{4{,}164.93 - 500}{3} = 1{,}221.64 \quad (>1{,}189.98)$$

$$t = 4 \quad \frac{5{,}830.90 - 500}{4} = 1{,}332.73 \quad (<1665.97)$$

Consequently, we should convert to SL for $t = 5$ through $t = 7$. Then, the adjusted values of D_t and B_t for $t = 5$, 6, and 7 are as follows:

$$D_5' = \$1{,}221.64 \qquad B_5' = 4{,}164.93 - 1{,}221.64 = \$2{,}943.29$$

$$D_6' = \$1{,}221.64 \qquad B_6' = 2{,}943.29 - 1{,}221.64 = \$1{,}721.65$$

$$D_7' = \$1{,}221.64 \qquad B_7' = 1{,}721.65 - 1{,}221.64 = \$\ \ 500.01$$

It can be seen that if $D_5' > D_5$ based on DDB as shown in the computation for $t = 5$, it follows that $D_6' > D_6$ and $D_7' > D_7$, where D_6 and D_7 are also based on DDB, since D_5', D_6', and D_7' are equal for straight-line depreciation. Note also that $B_7' = 500.01$ is the same as the salvage value $S = 500$ except for the truncation error in the numerical computation.

11.8 UNITS OF PRODUCTION DEPRECIATION

When the depreciation of an asset is more closely related to its use than to the length of time of ownership, it is permissible to determine the depreciation allowance on that basis under certain restricted conditions. Essentially, the unit of production may be expressed in terms of one of the following measures:

1. Production output, such as the volume or weight of material handled by a piece of equipment in any given year compared to the respective total volume or weight that may be handled during the useful life of the equipment
2. Operating days, indicating the number of days used in any given year compared to the expected total number of days in the useful life of the equipment
3. Projected income, which estimates the rental income from a rented property in any given year, compared to the expected total rental income from the property during its useful life

Under the *units of production* (UP) *depreciation,* the depreciation allowance is based on the principle that equal depreciation is allowed for each unit of production regardless of the lapse of time. The unit of production may be expressed in terms of one of the measures just described. Let U_t be the number of units of production during year t and U be the total number of units of production during the useful life of the asset. Then, the annual depreciation allowance D_t is equal to the net depreciable amount $(P - S)$ multiplied by the ratio U_t/U. That is,

$$D_t = \frac{U_t}{U}(P - S) \tag{11.16}$$

Also, from Eq. (11.2), the book value at the end of year t is given by

$$B_t = P - \left(\frac{P - S}{U}\right)(U_1 + U_2 + \cdots + U_t) \tag{11.17}$$

Example 11.7

A stone crusher is purchased at a cost of $12,000 with an estimated salvage value of $2,000 at the end of its useful life of 5 years. This stone crusher will be used for providing crushed stones for the construction of a large concrete dam over the next 5 years. The construction schedule calls for the following units of production (cubic yards of crushed stones) in the 5-year period: 8,000, 12,000, 18,000, 8,000, and 4,000 for years 1, 2, 3, 4, and 5, respectively. Using the units of production method, determine the annual depreciation allowance and the book value at the end of each year.

The total number of units of production over the useful life of 5 years is

$$U = 8{,}000 + 12{,}000 + 18{,}000 + 8{,}000 + 4{,}000 = 50{,}000$$

The net depreciable value over 5 years is

$$P - F = 12{,}000 - 2{,}000 = 10{,}000$$

Consequently, the values D_t and B_t can be obtained from Eqs. (11.16) and (11.17) as follows:

$$D_1 = (\tfrac{8}{50})(10{,}000) = \$1{,}600 \qquad B_1 = 12{,}000 - 1{,}600 = \$10{,}400$$

$$D_2 = (\tfrac{12}{50})(10{,}000) = \$2{,}400 \qquad B_2 = 10{,}400 - 2{,}400 = \$8{,}000$$

$$D_3 = (\tfrac{18}{50})(10{,}000) = \$3{,}600 \qquad B_3 = 8{,}000 - 3{,}600 = \$4{,}400$$

$$D_4 = (\tfrac{8}{50})(10{,}000) = \$1{,}600 \qquad B_4 = 4{,}400 - 1{,}600 = \$2{,}800$$

$$D_5 = (\tfrac{4}{50})(10{,}000) = \$800 \qquad B_5 = 2{,}800 - 800 = \$2{,}000$$

11.9 COMPARISON OF DEPRECIATION METHODS

The depreciation methods for computing the depreciation allowance may be classified as follows:

1. Uniform depreciation
2. Accelerated depreciation
3. Units of production depreciation
4. Decelerated depreciation

The straight-line (SL) method leads to uniform depreciation and is the easiest to apply. Both the sum-of-the-years'-digits (SOYD) method and the double declining balance (DDB) method are accelerated methods of depreciation, with no decided advantage of either one over the other.

The units of production method may produce accelerated or decelerated depreciation in certain periods of the useful life of the asset, depending on the timing of the production. When it is used properly within the restricted conditions, it may produce an overall schedule of depreciation with accelerated characteristics.

There is one other form of depreciation, called *sinking fund depreciation,* which is essentially a decelerated method of depreciation. Although it was used until some years ago, it has now been totally abandoned since no firm is interested in paying more tax in the earlier years of the life of an asset, thus adding interest to its tax liability. Consequently, we have not included this method in this chapter, and we mention it here only to warn against its use.

A schematic view of the comparison of various depreciation methods discussed in this chapter is shown in Fig. 11.6.

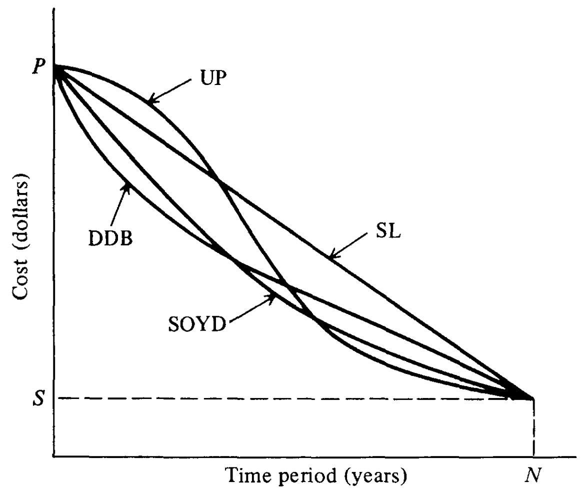


Figure 11.6 Comparison of depreciation methods.

11.10 MULTIPLE ASSET DEPRECIATION

In a large organization that owns many physical assets, it is permissible to combine a number of assets into a *multiple asset account* and to apply a single depreciation charge to the entire account.

The types of multiple asset accounts often used for depreciation computation are:

1. Group account, which contains similar assets of approximately the same estimated useful lives, such as trucks, copying machines, etc.
2. Classified account, which contains assets of similar characteristics but different useful lives, such as construction equipment or other categories of large special assets
3. Composite account, which includes assets of dissimilar characteristics and different useful lives

The use of a multiple asset account is based on the principle that the mean life expectancy and dispersion for a group of assets can be determined statistically from historical data. Consequently, the depreciation charge can be made against the entire group of assets instead of a single asset by using one of the depreciation methods discussed earlier in this chapter. Since the application of multiple asset accounting requires additional information on the groups of assets, it will not be discussed in detail here.

11.11 SUMMARY AND STUDY GUIDE

This chapter has covered various methods of depreciation and their applications. They provide the necessary background information for the computation of corporate taxes, which is discussed in the next chapter. Since depreciation regulations form a part of the federal income tax laws which are revised periodically, we emphasized the principles of computing depreciation allowances instead of the ever-changing guidelines which specify the allowable cost recovery periods for various categories of assets.

There are several methods for computing depreciation which are acceptable to the U.S. Internal Revenue Service, i.e., the straight-line method, the sum-of-the-years'-digits method, the double declining balance method, the units of production method, and others. The different methods of computing depreciation have different effects on the stream of taxable income and taxes paid. All these methods have been discussed in detail.

REFERENCES

11.1 *1987 U.S. Master Tax Guide*, 70th ed., Chicago, IL: Commerce Clearing House, 1986.

11.2 *Standard Federal Tax Reporter*, Chicago, IL: Commerce Clearing House (annual publication).

PROBLEMS

P11.1 A tractor costing $17,000 has a useful life of 5 years with a salvage value of $2,000 at the end of 5 years. Determine the annual depreciation allowance and the book value for each year by using

(a) The straight-line method

(b) The sum-of-the-years'-digits method

P11.2 A pressure vessel costing $70,000 has a useful life of 6 years with a salvage value of $10,000 at the end of 6 years. Determine the annual depreciation allowance and the book value for each year by using

(a) The straight-line method
(b) The sum-of-the-years'-digits method

P11.3 A concrete mixer costing $20,480 has a useful life of 6 years with a salvage value of $3,645 at the end of 6 years. Using a declining balance depreciation method with a constant depreciation rate $r = 1.5/N$, determine the annual depreciation allowance and the book value for each year.

P11.4 A remote-controlled small industrial robot costing $72,900 has a useful life of 6 years with a salvage value of $6,400 at the end of 6 years. Using the double declining balance depreciation method, determine the annual depreciation allowance and the book value for each year.

P11.5 A soldering machine costing $25,000 has a useful life of 5 years with a salvage value of $5,000. Using the double declining balance method, determine the annual depreciation allowance and the book value for each year until the maximum net depreciable value of $5,000.

P11.6 A pile-driving machine costing $250,000 has a useful life of 10 years with a salvage value of $25,000 at the end of 10 years. Using the double declining balance method with conversion to straight line, if necessary, determine the annual depreciation allowance and the book value for each year.

P11.7 A pump costing $2,400 has a useful life of 6 years with a salvage value of $300 at the end of 6 years. Determine the annual depreciation and the book value for each year by using the following methods:
(a) Straight-line method
(b) Sum-of-the-years'-digits method
(c) Double declining balance method, with the maximum net depreciable value not to exceed $2,100

Tabulate the results systematically for comparison.

P11.8 The cost of an off-shore installation for oil-drilling is $5 million. The useful life of the installation for depreciation is 8 years. Its salvage value at the end of the useful life is estimated to be $300,000.
(a) Using the double declining balance method, determine the annual depreciation and book value for each year.
(b) If the book value at the end of 8 years does not equal $300,000 exactly, adjust the values of the annual depreciation allowance and book value for the best tax advantage.

P11.9 A silo for grain storage costing $94,000 has an actual useful life of 10 years. According to the tax guideline, its depreciable life is 7 years with a salvage value of $10,000 at the end of 7 years.
(a) Find the annual depreciation allowance for each year by using the sum-of-the-years'-digits method.
(b) If a uniform annual revenue generated by the silo is A_t = $30,000 for the next 10 years, find $(A_t - D_t)$ for each of these 10 years.

P11.10 A greenhouse for an orchid nursery costing $256,000 has a depreciable life of 8 years with a salvage value of $30,000 at the end. It is intended to be used for 10 years during which a uniform annual income of A_t = $80,000 is expected. Using the double declining balance method, find the annual depreciation allowance for each year. Also find $(A_t - D_t)$ for each year.

P11.11 A railroad boxcar costing \$400,000 has a depreciable life of 8 years with a salvage value of \$20,000 at the end. It is intended to be placed in service for 8 years during which a uniform annual income of A_t = \$180,000 is expected. Using the double declining balance method with conversion to straight line if necessary, determine the annual depreciation allowance and $(A_t - D_t)$ for each year.

P11.12 A printing press costing \$18,000 has a depreciable life of 5 years with a salvage of \$3,000 at the end. Its actual useful life is expected to be 8 years during which a uniform annual income of A_t = \$6,000 will be generated. Using the straight-line method, find the annual depreciation allowance and the book value for each year. Also find $(A_t - D_t)$ for each year.

P11.13 A drilling machine costing \$60,000 has a useful life of 7 years with a salvage value of \$4,000 at the end of 7 years. Determine the annual depreciation allowance and the book value for each year by using the sum-of-the-years'-digits method.

P11.14 A tower crane costing \$729,000 has a depreciable life of 9 years with a salvage value of \$40,000 at the end. Using the double declining balance method with conversion to straight line, determine the annual depreciation allowance and book value for each year.

P11.15 The Confederate Coal Co. bought a tract of land which has an estimated reserve of 2 million tons of coal underground. The cost of the property including the mineral rights was \$22 million, of which \$2 million was the value of the land and \$20 million was the value of the coal reserve. During the first year of operation, 250,000 tons of coal were produced. What was the allowable depletion for the year on the basis of unit of production cost depletion method?

P11.16 A truck rental company purchases a heavy-duty truck at \$20,000 with the intention of renting it on a daily basis. The truck is expected to be used for 5 years with forecasted net annual revenues of \$14,000 in the first year, and \$2,000 less in each succeeding year (i.e., \$6,000 in the fifth year). The company intends to depreciate this truck on the basis of the unit of production method until the book value reaches zero. Determine the annual depreciation allowances.

CHAPTER

12 Corporate Taxation

12.1 FEDERAL TAX LAWS

Federal tax laws are complex and subject to change over time. There are many types of taxes imposed by the U.S. Congress which reflect public policy to encourage certain actions or to discourage other actions to be taken by corporations and individuals. The responsibility of enforcing these laws rests with the U.S. Internal Revenue Service.[1] Only the general concepts and basic principles of federal tax laws as applied to private corporations are introduced here to illustrate their application to the economic evaluation of capital investment projects.

For the purpose of capital project planning and evaluation, it is more important to understand the rationale underlying the promulgation and revisions of the tax laws than specific provisions at any given time. Capital projects are long-lived physical assets which are affected by adjustments resulting from the revision in tax laws including the transition provisions after each revision, or from the multiyear "carry back" or "carry forward" of profits and losses. An understanding of the trend of federal tax laws will allow intelligent and prudent planning even though some provisions and tax rates may change periodically.

The federal income tax is the single most important item, since profits from business operations are taxed annually at a substantial rate. Beginning July 1, 1987, the graduated tax rates on ordinary income of corporations are 15% on the first \$50,000, 25% on the next \$25,000 up to \$75,000, 34% on the next \$25,000 up to \$100,000, 39% on income between \$100,000 and \$335,000, and 34% on all income

[1] For sheer volume of the federal tax laws, see Refs. 12.1 and 12.2.

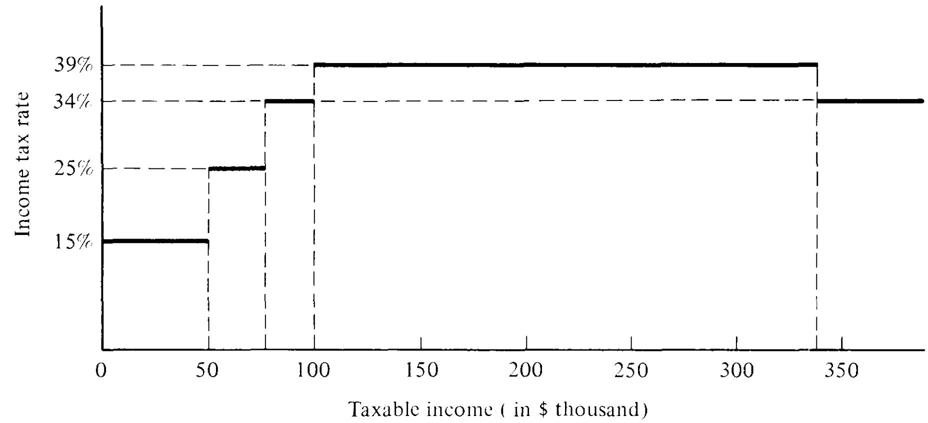


Figure 12.1 Federal income tax rates for corporations.

over $335,000. These graduated tax rates are illustrated in Fig. 12.1. The lower tax rates at the lower income brackets were made as a concession to provide some measure of relief to small businesses, while the "bubble" of tax rates for the income bracket between $100,000 and $335,000 was intended to remove this concession as the size of business increases. Thus, for businesses with ordinary income exceeding $335,000, the marginal or flat tax rate is 34%.

There are provisions in the federal income tax laws that offer special advantages to capital investment. For example, if an asset is sold at a price higher than the book value, the difference is known as the *capital gain;* if the sales price is lower, the difference is called a *capital loss*. Gains on sale of assets held for more than a required holding period are defined as long-term capital gains, which are taxed at a rate lower than profit from ordinary operations. Short-term capital gains are taxed at the rates that apply to ordinary income. However, if a corporation has both gains and losses on sales of capital assets during the same tax year, it must net all long-term capital gains against long-term capital losses and net all short-term capital gains against short-term capital losses. The excess of net long-term capital gain over net short-term capital loss is subject to the capital gain tax; the excess of net short-term capital gain over net long-term capital loss is taxed as ordinary income. The current required holding period for long-term capital gain is 6 months and the corresponding tax rate is 28% which is lower than the marginal tax rate for ordinary income.

Another example of offering incentives for capital investment is the *investment tax credit* (ITC), which allows a deduction from the income tax of a small percentage of the value of certain newly acquired eligible assets. This provision was first established in 1962, suspended briefly later, and terminated in 1969. It was restored in 1971 with a 7% rate, and the rate was raised to 10% in 1975 for qualified investments. In 1981, the classification of qualified investments was simplified and graduated rates up to 10% were introduced on the basis of the estimated useful life of the equipment. However, the investment tax credit for all types of assets was eliminated in 1986. On the other hand, a new category of tax credit for low-income housing was introduced in 1986 for new residential buildings completed before 1990. Since ITC provisions have periodically been in and out of the tax laws, one should check for new revisions when making capital investments.

12.2 STATE AND LOCAL TAXES

State and local governments also levy taxes for their operations. The common types of taxes adopted by a state are income tax, sales tax, and property transfer tax. Similarly, the most common types of taxes adopted by cities and other local entities are income tax, real property tax, and school tax. While the tax rates vary from state to state and from city to city, the same principle can be applied for the inclusion of state or local taxes in economic evaluation.

Generally, taxes can be deducted from gross income as a part of the expense of doing business. However, some of the state and local taxes are further deductible on the tax return for federal income tax to avoid double taxation. Since the corporate income tax is particularly significant at both the federal level and the state and local level, it deserves to be treated in greater detail.

Let X be the federal income tax rate, X' be the state income tax rate, and $\bar{X}$ be the effective combined rate for federal and state income taxes. To eliminate the effect of double taxation, the effective combined rate is given by

$$\bar{X} = X' + (1 - X')X \tag{12.1}$$

That is, for each dollar of income, the state income tax amounts to X' dollars, but the federal income tax is based on a taxable income of $(1 - X')$ resulting in an amount of $(1 - X')X$ for the federal income tax. Hence, the combined income tax from federal and state levels is $\bar{X}$ dollars for each dollar of income. Alternatively, $\bar{X}$ can be expressed as

$$\bar{X} = X + X' - XX' \tag{12.2}$$

Consequently, we can simplify the computation of income taxes by using a single income tax rate to cover both the federal and state income taxes. Since local income tax is generally not deductible in computing state income tax, X' in Eqs. (12.1) and (12.2) may represent the sum of state and local income tax rates if such a tax is imposed by a locality.

Example 12.1

Suppose that the federal income tax rate is 34%. Find the effective combined federal and state tax rate if the state income tax rate is (a) 3% and (b) 5%.

In case (a), the effective combined tax rate is given by

$$X = 0.03 + (0.34)(1 - 0.03) = 0.3598 = 36\%$$

and in case (b), the result is

$$\bar{X} = 0.05 + (0.34)(1 - 0.05) = 0.3796 = 38\%$$

12.3 CORPORATE TAX PLANNING

For private corporations, tax planning is an important element of capital investment consideration since the economic feasibility of an investment project is affected by the taxation of corporate profits. Tax accountants and counselors specializing in tax laws are available for consultation in filing tax returns and in tax planning for corporations and individuals engaged in business enterprises.

In making estimates of tax liability, several factors deserve special attention and appropriate assumptions should be made accordingly:

1. Number of years for which the asset will be retained
2. Depreciation method used
3. Size and timing of debt financing
4. Purchase versus lease
5. Capital gains upon the sale of assets
6. Effects of inflation

In general, physical assets may be kept longer than the estimated useful life for depreciation specified in the tax laws. If the net income generated by the asset after the estimated useful life is ignored in the economic evaluation, the result of the analysis is on the conservative side. Unless otherwise stated in the problem, the net income after the estimated useful life is assumed to be zero. Then the planning horizon is the same as the estimated useful life. On the other hand, there are physical assets, such as real estate property, that are often intended to be sold long before the expiration of the estimated useful life in order to realize a capital gain. In that case, the planning horizon should be shortened accordingly.

Except for real estate property for which the choice of depreciation methods is limited, accelerated depreciation methods generally offer the advantage of deferring tax liability to later years. Even for new enterprises that have small profits in early years, the use of an accelerated depreciation method will reduce cash payment at the time when the cash is needed the most. The net tax payments can usually be adjusted through multiyear "carry back" or "carry forward" of profits and losses.

Nearly all corporations borrow money for one purpose or another. However, there is a difference between general debts for operations, including the purchase of physical assets, and special debts dedicated only to the acquisition of a particular plant or equipment. In the former case, the debts and the associated tax deduction on interests are dealt with at the corporate level, and the project will be treated as if it were equity financed in economic evaluation at the project level. In the latter case, debt financing will be considered at the project level. As indicated in Section 9.6, the interest rate for borrowing relative to the opportunity cost of capital has a significant effect on the size and timing of debt financing. If the borrowing interest rate is low, it pays to borrow more and longer; if the opposite is true, borrow as little as needed to finance the acquisition.

In some cases, opportunities may arise for leasing the building or equipment if funds for purchase are not available or if the interest is too high. Very often, a lessor is willing not only to supply the needed building or equipment upon request, but also can make a reasonable return in the process. For a lessee who uses the building or equipment, it is necessary to analyze the comparative advantages between purchasing and leasing.

When a physical asset is sold, whether before or after the expiration date of the estimated useful life, long-term capital gains will be taxed at a rate below that for ordinary income. If the asset is expected to appreciate in value over time and be sold for a substantial profit, the projected capital gain and the associated tax should be considered accordingly.

Finally, the effects of inflation on economic evaluation should be properly considered. Since the taxes are based on then-current dollars in future years, the after-tax cash flows must reflect such tax payments whether the constant price approach or the current price approach is used in the economic evaluation.

12.4 AFTER-TAX CASH FLOWS

The net profits of a corporation over time are measured by the net cash flows after tax. Consequently, after-tax cash flows are needed for economic evaluation of a proposed project. For the sake of simplicity, only the marginal or flat tax rate for corporations with annual income over $335,000 is used in the analysis. Although a single income tax rate is used for all future years in economic evaluation, the tax rate can be changed from year to year for retrospective analysis.

Let us introduce the following notation which is applicable to any year t in a planning horizon of n years.

A_t = net cash flow from operation before tax
$\bar{A}_t$ = net cash flow from financing before tax
$\hat{A}_t = A_t + \bar{A}_t$ = combined net cash flow before tax
Y_t = net cash flow from operation after tax
$\bar{Y}_t$ = net cash flow from financing after tax
$\hat{Y}_t = Y_t + \bar{Y}_t$ = combined net cash flow after tax
D_t = annual depreciation allowance
I_t = annual interest on the unpaid balance of a loan
Q_t = annual payment to the lender for reducing the unpaid balance of a loan (negative for a payment and positive for a receipt from additional borrowing)

W_t = annual taxable income
X_t = income tax rate
T_t = annual income tax

For operation only, the taxable income and income tax in year $t = 1, 2, \ldots, n$ are given respectively by

$$W_t = A_t - D_t \tag{12.3}$$

$$T_t = X_t W_t \tag{12.4}$$

Thus, the after-tax net cash flow profile for operation ($t = 0, 1, 2, \ldots, n$) is

$$Y_t = A_t - X_t W_t \tag{12.5}$$

or

$$Y_t = A_t - X_t(A_t - D_t) \tag{12.6}$$

For financing only, the interest on a loan in year $t = 1, 2, \ldots, n$ is obtained from the relations in Eqs. (4.12) through (4.14) in Chapter 4. Replacing A_t in those equations by $\bar{A}_t$, we get

$$I_t = Q_t - \bar{A}_t \tag{12.7}$$

Hence, the after-tax cash flow profile for financing (for $t = 0, 1, 2, \ldots, n$) is

$$\bar{Y}_t = \bar{A}_t + X_t I_t \tag{12.8}$$

where $X_t I_t$ is referred to as the *tax shield*. That is, $X_t I_t$ may be regarded as a gain from debt financing because of the deductibility of interest in computing the income tax.

Alternatively, we can consider the combined net cash flow profile after tax by noting that both depreciation and interest are deductible. Then, for $t = 1, 2, \ldots, n$

$$W_t = A_t - D_t - I_t \tag{12.9}$$

Hence, the after-tax combined net cash flow profile (for $t = 0, 1, 2, \ldots, n$) is

$$\hat{Y}_t = \hat{A} - X_t W_t \tag{12.10}$$

or

$$\hat{Y}_t = A_t + \bar{A}_t - X_t(A_t - D_t - I_t) \tag{12.11}$$

Since $\hat{Y}_t = Y_t + \bar{Y}_t$, Eq. (12.11) can also be obtained by adding Eq. (12.8) and Eq. (12.6).

Example 12.2

A large firm received revenues of $500,000 and incurred expenses of $300,000 (not counting depreciation) in one of its divisions last year. The allowable depreciation on the equipment for the operation was $20,000. Assume that this division did not borrow money to finance the acquisition of the equipment or its operation and hence had no interest payment last year. For an income tax rate of 34%, what was the net cash inflow after tax from this division?

Assuming equity financing, the before-tax cash flow for last year was given by:

$$A_t = 500{,}000 - 300{,}000 = 200{,}000$$

From Eq. (12.6), we obtain the after-tax cash flow as follows:

$$Y_t = 200{,}000 - (34\%)(200{,}000 - 20{,}000) = \$138{,}800$$

Example 12.3

A loan of $200 million is obtained to finance an energy development project. The loan is subsidized by the federal government at an annual interest rate of 3%, repayable in a series of uniform annual payments of $16.76 million over 15 years. Find the principal and interest payments for each of these 15 years.

As a check, the uniform annual payment for the loan at an annual interest rate of 3% is computed as follows:

$$(200)(U \mid P, 3\%, 15) = (200)(0.0838) = 16.76$$

The portions of principal and interest payments for the 15 years as reflected in Eq. (12.7) and computed by Eqs. (4.12), (4.13), and (4.14) are shown in Table 12.1. Note that the loan of $200 million received in year 0 is positive and the uniform annual payments to retire the loan are negative. Following the sign conventions defined in Section 4.5 (Chapter 4), the interest I_t on the unpaid balance for each year t is positive and the repayment Q_t to the unpaid balance in each year t is negative.

TABLE 12.1 PRINCIPALS AND INTERESTS IN ANNUAL PAYMENTS

t	$\bar{A}_t$	I_t	Q_t	R_t
0	200	0	0	200
1	−16.76	6.00	−10.76	189.24
2	−16.76	5.68	−11.08	178.16
3	−16.76	5.34	−11.42	166.74
4	−16.76	5.00	−11.76	154.98
5	−16.76	4.65	−12.11	142.87
6	−16.76	4.29	−12.47	130.40
7	−16.76	3.91	−12.85	117.55
8	−16.76	3.53	−13.23	104.32
9	−16.76	3.13	−13.63	90.69
10	−16.76	2.72	−14.04	76.65
11	−16.76	2.30	−14.46	62.13
12	−16.76	1.87	−14.89	47.30
13	−16.76	1.42	−15.34	31.96
14	−16.76	0.96	−15.80	16.16
15	−16.76	0.48	−16.28	−0.12

12.5 EVALUATION OF AFTER-TAX CASH FLOWS

For private corporations, the financing of an investment project by equity or debt may influence the overall equity-debt mix of the entire corporation, depending on the size of the project and the risk involved. When a corporation decides to finance a particular investment project by borrowing, it incurs new obligations and risks, and thus affects its future capacity for additional borrowing. Hence, the internal funds of a corporation represent a common pool of resources from both equity and debt. If an investment project is financed through internal funds, it is difficult to determine the exact proportions of equity and debt for this particular investment project. Equity financing is only an idealized case for which the after-tax cash flow profile of an investment project is assumed to be independent of debt financing.

As stated in Section 9.6, we continue to make the assumptions of certainty in future cash flows and a perfect capital market. Consequently, the equations developed in Sections 9.6 and 9.7 will also be used in the evaluation of after-tax cash flows in this chapter. In analogy to Eq. (9.6),

$$\text{APV} = [\text{NPV}]_{i^*} + [\text{FPV}]_{i^*} \tag{12.12}$$

where i^* is the after-tax MARR specified. Since a single MARR is used for both the operation net cash flows Y_t and the financing net cash flows $\bar{Y}_t$, the APV may be obtained directly by applying i^* to discount the combined net cash flow $\hat{Y}_t$. If the project is financed by equity only, Eq. (12.12) becomes

$$\text{APV} = [\text{NPV}]_{i^*} \tag{12.13}$$

Example 12.4

The Rockwood Corporation is considering the installation of a mechanical device to facilitate the production of one of its products during the next 5 years. The device will cost \$12,000 and will have a salvage value of \$2,000 at the end of 5 years. The firm chooses to use the straight-line depreciation method for the device and pays an annual income tax for its profits at a rate of 36%. If the minimum attractive rate of return after tax is 6%, what is the minimum uniform annual net benefit before tax that must be generated by this device in order to justify its installation? Assume that the purchase is financed by equity.

Let the uniform annual net benefit before tax be denoted by $A_t = U$ (for $t = 1, 2, \ldots, 5$). The straight-line depreciation for each year is given by Eq. (11.4):

$$D_t = (1/5)(12{,}000 - 2{,}000) = 2{,}000$$

The after-tax cash flow for each year is obtained by Eq. (12.6):

$$\begin{aligned} Y_t &= U - (0.36)(U - 2{,}000) \\ &= 0.64U + 920 \end{aligned}$$

The net present value of the cash flow profile for the investment discounted at the specified after-tax MARR of 6% is

$$[\text{NPV}]_{6\%} = -12{,}000 + (0.64U + 920)(P \mid U, 6\%, 5) + 2{,}000(P \mid F, 6\%, 5)$$
$$= -12{,}000 + 2.696U + 3{,}875 + 1{,}495 = 2.696U - 6{,}630$$

For the investment to be acceptable, NPV must at least be zero. Hence,

$$2.696U - 6{,}630 = 0$$

from which we find the minimum uniform annual net benefit to be

$$U = \$2{,}459$$

Example 12.5

A pilot plant for coal gasification requires an initial investment of $800 million and is expected to generate a uniform after-tax cash income of $150 million per year for 15 years, and there is no salvage value at the end. The firm undertaking this project intends to secure a loan of $200 million, which will be repaid at the end of 15 years but the interest at a rate of 12% will be paid annually. The tax rate of this firm including federal, state and local taxes is 46%. The minimum attractive rate of return after tax is 8%. Determine whether this project is worthwhile if there is no transaction cost.

The after-tax operation cash flow for year t (for $t = 1, 2, \ldots, 15$) is given in the problem statement as

$$Y_t = A_t - X_t(A_t - D_t) = \$150 \text{ million}$$

Hence,

$$[\text{NPV}]_{8\%} = -800 + (150)(P \mid U, 8\%, 15)$$
$$= -800 + (150)(8.5595) = +\$438.9 \text{ million}$$

The after-tax financing cash flow in year t is

$$\bar{Y}_t = \bar{A}_t + X_t I_t$$

where

$$I_t = (200)(0.12) = \$24 \text{ million} \quad (\text{for } t = 1, 2, \ldots, 14)$$

For $t = 15$, the before-tax cash flow A_{15} includes both the interest payment and the repayment of the principal. Since the tax rate X_t is assumed to be 46% for all years,

$$X_t I_t = (0.46)(24) = \$11.04 \text{ million}$$

Hence, the after-tax financial stream $\bar{Y}_t$ can be obtained from the before-tax financial stream $\bar{A}_t$, as shown in Table 12.2.

The FPV of the after-tax financial profile $\bar{Y}_t$ (in $ million) is

$$[\text{FPV}]_{8\%} = +200 - (12.96)(P \mid U, 8\%, 15) - (200)(P \mid F, 8\%, 15)$$
$$= +200 - 110.9 - 63.0 = 26.1$$

Finally, the adjusted present value for the combined effects of the operation and financing cash flows is

$$\text{APV} = [\text{NPV}]_{8\%} + [\text{FPV}]_{8\%}$$
$$= 483.9 + 26.1 = \$510 \text{ million}$$

TABLE 12.2 AFTER-TAX FINANCIAL STREAM OF CASH FLOWS

t	$\bar{A}_t$	$X_t I_t$	$\bar{Y}_t = \bar{A}_t + X_t I_t$
0	+200	0	+200
1–14	−24	11.04	−12.96
15	{−24 −200	11.04	{−12.96 −200

Example 12.6

Suppose that the pilot plant for coal gasification in Example 12.5 is qualified for a low-interest loan from a federal government program for encouraging the development of synthetic fuel projects. The loan of \$200 million is subsidized at an interest rate of 3%, repayable in a series of uniform annual payments of \$16.76 million each year over 15 years, covering both principal and interest. Hence, the firm will no longer try to obtain a commercial loan at 12% in the capital markets. What is the adjusted net present value of this project?

The only difference between this example and Example 12.5 is the term FPV in the computation of the adjusted net present value APV because of the government loan of \$200 million at a subsidized interest rate of 3%. We must first determine the portions of principal and interest payments in each year from the uniform annual payments in order to assess the tax shield. However, these annual interest payments have been obtained in Example 12.3.

Therefore, the after-tax financial stream of cash flows can be obtained from Table 12.3 as follows:

$$\text{At } t = 0,\ \bar{Y}_0 = \bar{A}_0$$
$$\text{At } t = 1 \text{ to } 15,\ \bar{Y}_t = \bar{A}_t + X_t I_t$$
$$= (Q_t - I_t) + X_t I_t$$
$$= Q_t - (1 - X_t) I_t$$

where $X = 46\%$ is the income tax rate. Consequently,

$$[\text{FPV}]_{8\%} = \bar{A}_0 + \sum_{t=1}^{15} [Q_t - (1 - X_t) I_t](P \mid F, 8\%, t)$$

TABLE 12.3 COMPUTATION OF QUANTITIES LEADING TO FPV

t	$\bar{Y}_t = Q_t - (1 - X_t)I_t$	$(P \mid F, 8\%, t)$	$\bar{Y}_t(P \mid F, 8\%, t)$
0	+200.00	1.0000	+200.0000
1	−14.00	0.9259	−12.9626
2	−14.15	0.8573	−12.1308
3	−14.30	0.7938	−11.3513
4	−14.46	0.7350	−10.6281
5	−14.62	0.6806	−9.9503
6	−14.79	0.6302	−9.3207
7	−14.96	0.5835	−8.7292
8	−15.14	0.5403	−8.1801
9	−15.32	0.5002	−7.6631
10	−15.51	0.4632	−7.1842
11	−15.70	0.4289	−6.7337
12	−15.90	0.3971	−6.3139
13	−16.11	0.3677	−5.9236
14	−16.32	0.3405	−5.5570
15	−16.54	0.3152	−5.2134
			Total = +72.1580

The computation of FPV can be simplified by tabulating the results in Table 12.3. Then $[\text{FPV}]_{8\%}$ = \$72.2 million.

Finally, the value of NPV can be taken from Example 12.5. Then,

$$\begin{aligned} \text{APV} &= [\text{NPV}]_{8\%} + [\text{FPV}]_{8\%} \\ &= 483.9 + 72.2 = \$556.1 \text{ million} \end{aligned}$$

Hence, the project is worthwhile.

12.6 EFFECTS OF DEDUCTION FOR DEPRECIATION

The depreciation methods will produce different effects on the after-tax cash flow profile of an investment. The accelerated depreciation methods generate larger depreciation allowances during the early years and correspondingly smaller allowances in the later years of the "estimated useful life" of the asset. Hence, the tax liabilities in the early years will be lower and those in the later years will be higher. The total undiscounted amount of taxes on ordinary incomes throughout the life of an asset will be the same, regardless of the depreciation method used, if the asset is fully depreciated to its salvage value at the end of the estimated useful life. Consequently, the discounted value of the after-tax cash flow using any one of the accelerated depreciation methods is expected to be better than that obtained by using the straight-line method.

Strictly speaking, if the taxable income in any year is negative, the income tax is zero. However, because most corporations have a number of ongoing projects, profits from some projects can compensate losses in other projects. Furthermore, profits and losses can be carried back or carried forward to other years. Consequently, a negative income tax is usually assessed for a negative taxable income as indicated in the examples in this section and in subsequent sections.

Example 12.7

Roper Manufacturing Corporation considers purchasing new automatic equipment to improve its production. An expenditure of \$46,000 on the equipment is expected to produce a before-tax benefit of \$15,000 in the first year and \$2,000 less in each succeeding year for a total of 6 years (i.e., before-tax benefit of \$13,000 in the second year, \$11,000 in the third year, until \$5,000 in the sixth year). The salvage value of the equipment will be \$4,000 at the end of 6 years. Assume that the purchase is financed by equity and that the income rate for the firm is 34%. If the minimum attractive rate of return after tax is 6%, determine whether the purchase is worthwhile, using each of the following depreciation methods:

1. Sum-of-the-years'-digits (SOYD) depreciation
2. Straight-line (SL) depreciation

The annual depreciation allowances based on the SOYD method are obtained by Eq. (11.6). The computation of the after-tax cash flows using Eq. (12.6) is shown in part (a) of Table 12.4. Then, for the after-tax MARR of 6%, we get

$$[\text{NPV}]_{6\%} = -46{,}000 + (13{,}980)(P \mid U, 6\%, 6) - (2{,}000)(P \mid G, 6\%, 6) + (4{,}000)(P \mid F, 6\%, 6)$$
$$= +2{,}645$$

Similarly, the annual depreciation allowances based on the SL method are obtained by Eq. (11.4). The computation of the after-tax cash flows using Eq. (12.6) is shown in part (b) of Table 12.4. For after-tax MARR of 6%,

$$[\text{NPV}]_{6\%} = -46{,}000 + (12{,}280)(P \mid U, 6\%, 6) - (1{,}320)(P \mid G, 6\%, 6) + (4{,}000)(P \mid F, 6\%, 6)$$
$$= +2{,}378$$

The results of this example indicate that while both depreciation methods lead to positive NPV, the SOYD depreciation produces a higher NPV and is the preferred method.

Example 12.8

A private firm is interested in the purchase of a computer workstation through equity financing. The before-tax cash flow profile for the acquisition and oper-

TABLE 12.4 COMPUTATION OF AFTER-TAX CASH FLOWS

(a) SOYD DEPRECIATION

Year t	Before-tax cash flow A_t	Depreciation (SOYD) D_t	Taxable income W_t	Income tax T_t	After-tax cash flow Y_t
0	−46,000	0	0	0	−46,000
1	+15,000	12,000	3,000	1,020	+13,980
2	+13,000	10,000	3,000	1,020	+11,980
3	+11,000	8,000	3,000	1,020	+9,980
4	+9,000	6,000	3,000	1,020	+7,980
5	+7,000	4,000	3,000	1,020	+5,980
6	{ +5,000	2,000	3,000	1,020	{ +3,980
	+4,000				+4,000

(b) SL DEPRECIATION

Year t	Before-tax A_t	Depreciation (SL) D_t	Taxable income W_t	Income tax T_t	After-tax Y_t
0	−46,000	0	0	0	−46,000
1	+15,000	7,000	8,000	2,720	+12,280
2	+13,000	7,000	6,000	2,040	+10,960
3	+11,000	7,000	4,000	1,360	+9,640
4	+9,000	7,000	2,000	680	+8,320
5	+7,000	7,000	0	0	+7,000
6	{ +5,000	7,000	−2,000	−680	{ +5,680
	+4,000				+4,000

ation of this computer workstation is given by the values of A_t in Table 12.5. The income tax rate is 34% and the after-tax MARR specified by the firm is 8%. Determine whether the purchase is worthwhile for each of the following depreciation methods:

1. Double declining balance (DDB) depreciation up to the limit
2. Sum-of-the-years'-digits (SOYD) depreciation

The annual depreciation allowances based on the DDB method are obtained by Eqs. (11.10) through (11.13) and tabulated in part (a) of Table 12.5. However, for $t = 5$, the computed depreciation allowance $D_5 = 1{,}140$ leads to a book value $B_5 = 1{,}711$. Since B_5 cannot be less than the salvage value $S = 2{,}000$, the adjusted depreciation allowance becomes

$$D_5' = 1{,}140 - (2{,}000 - 1{,}711) = 851$$

The computation of the after-tax cash flows using Eq. (12.6) is also shown in part (a) of Table 12.5. Then, for the after-tax MARR of 8%, we have

TABLE 12.5 EFFECTS OF DEPRECIATION METHODS ON AFTER-TAX ANALYSIS

(a) DDB DEPRECIATION

Year t	Before-tax cash flow A_t	Depreciation (DDB) D_t	Taxable income W_t	Income tax T_t	After-tax cash flow Y_t
0	−22,000	0	0	0	−22,000
1	+6,000	8,800	−2,800	−952	+6,952
2	+6,000	5,280	720	245	+5,755
3	+6,000	3,168	2,832	983	+5,017
4	+6,000	1,901	4,099	1,394	+4,606
5	{ +6,000 +2,000	851	5,149	1,575	{ +4,249 +2,000

(b) SOYD DEPRECIATION

Year t	Before-tax cash flow A_t	Depreciation (SOYD) D_t	Taxable income W_t	Income tax T_t	After-tax cash flow Y_t
0	−22,000	0	0	0	−22,000
1	+6,000	6,667	−667	−226	+6,226
2	+6,000	5,333	667	227	+5,773
3	+6,000	4,000	2,000	680	+5,320
4	+6,000	2,667	3,333	1,133	+4,867
5	{ +6,000 +2,000	1,333	4,667	1,586	{ +4,414 +2,000

$$[\text{NPV}]_{8\%} = -22{,}000 + (6{,}952)(P \mid F, 8\%, 1) + (5{,}755)(P \mid F, 8\%, 2) + (5{,}017)(P \mid F, 8\%, 3) + (4{,}606)(P \mid F, 8\%, 4) + (4{,}249 + 2{,}000)(P \mid F, 8\%, 5)$$

$$= +992$$

Similarly, the annual depreciation allowances based on the SOYD method are obtained by Eq. (11.6). The computation of the after-tax cash flows using Eq. (12.6) is shown in part (b) of Table 12.5. For the after-tax MARR of 8%, we get

$$[\text{NPV}]_{8\%} = -22{,}000 + (6{,}226)(P \mid U, 8\%, 5) - (453)(P \mid G, 8\%, 5) + (2{,}000)(P \mid F, 8\%, 5)$$

$$= +880$$

Note that the purchase is worthwhile, regardless of which depreciation method is used, although the DDB method is more advantageous.

12.7 EFFECTS OF DEDUCTION FOR INTEREST

The interest on debt financing is deductible for income tax purposes. If a firm can borrow money at a borrowing interest rate equal to or less than its MARR, it is more profitable for the firm to finance an investment project by debt because of the allowable deduction for interest. However, this advantage is realized only if the borrowing does not alter the long-term financial position of the firm.

Financing plans with different timing of repayments on a loan will lead to different interest payments even though the same interest rate applies. Furthermore, the period of borrowing does not necessarily coincide with the planning horizon of the investment but depends on the needs of the borrower. Consequently, the present value of each financing plan should be computed for comparison.

Example 12.9

Suppose that the purchase of the computer workstation in Example 12.8 is financed partly by debt (for an amount of $10,000). Two financing plans are being considered, and the cash flow profiles for these financing plans are shown in the column $\bar{A}_t$ ($t = 0, 1, 2, \ldots, n$) in parts (a) and (b) of Table 12.6. The double declining balance (DDB) depreciation up to the limit of allowable deduction will be used in computing depreciation allowance. The income tax rate is 34% and the after-tax MARR specified by the firm is 8%. De-

TABLE 12.6 EFFECTS OF FINANCING PLANS ON AFTER-TAX CASH FLOWS
(a) FINANCING PLAN WITH A SINGLE REPAYMENT AT THE END

t	A_t	$\bar{A}_t$	D_t	I_t	W_t	T_t	$\hat{Y}_t$
0	−22,000	+10,000	0	0	0	0	−12,000
1	+6,000	−800	8,800	800	−3,600	−1,224	+6,424
2	+6,000	−800	5,280	800	−80	−27	+5,227
3	+6,000	−800	3,168	800	2,032	691	+4,509
4	+6,000	−800	1,901	800	3,299	1,122	+4,078
5	{ +6,000	−10,800	851	800	4,349	1,479	{ −6,279
	{ +2,000						{ +2,000

(b) FINANCING PLAN WITH UNIFORM ANNUAL REPAYMENTS

t	A_t	$\bar{A}_t$	D_t	I_t	W_t	T_t	$\hat{Y}_t$
0	−22,000	+10,000	0	0	0	0	−12,000
1	+6,000	−2,505	8,800	800	−3,600	−1,224	+4,719
2	+6,000	−2,505	5,280	664	56	19	+3,476
3	+6,000	−2,505	3,168	516	2,316	787	+2,708
4	+6,000	−2,505	1,901	357	3,742	1,272	+2,223
5	{ +6,000	−2,505	851	185	4,964	1,688	{ +1,807
	{ +2,000						{ +2,000

termine whether the purchase is worthwhile for each of the two financing plans.

The amount of interest I_t (for $t = 1, 2, \ldots, 5$) associated with each of the financing plans has been computed, and the results for the two financing plans are tabulated in the column I_t in parts (a) and (b) of Table 12.6. The computation of the after-tax cash flows in part (a) of Table 12.6 discounted at the after-tax MARR of 8% is given by

$$\begin{aligned}[\text{APV}]_{8\%} &= -12{,}000 + (6{,}424)(P \mid F, 8\%, 1) + (5{,}227)(P \mid F, 8\%, 2) \\ &\quad + (4{,}509)(P \mid F, 8\%, 3) + (4{,}078)(P \mid F, 8\%, 4) \\ &\quad + (-6{,}279 + 2{,}000)(P \mid F, 8\%, 5) \\ &= +2{,}093\end{aligned}$$

Similarly, by adopting the financing plan (b), the net present value for the combined after-tax cash flows in part (b) of Table 12.6 discounted at the after-tax MARR of 8% is

$$\begin{aligned}[\text{APV}]_{8\%} &= -12{,}000 + (4{,}719)(P \mid F, 8\%, 1) + (3{,}476)(P \mid F, 8\%, 2) \\ &\quad + (2{,}708)(P \mid F, 8\%, 3) + (2{,}223)(P \mid F, 8\%, 4) \\ &\quad + (1{,}807 + 2{,}000)(P \mid F, 8\%, 5) \\ &= +363\end{aligned}$$

Note that both financing plans will make the investment worthwhile, but plan (a) is superior to plan (b).

12.8 PURCHASE VERSUS LEASE

If a firm lacks the necessary funds to acquire a physical asset which is deemed desirable, it can often lease the asset by entering into a contract with another party which will obligate the firm legally to make payments for a well-defined period of time. The payments in leasing are expenses that can be deducted in full from the gross revenues in computing taxable income. On the other hand, if the firm borrows money to buy the asset, it can deduct the allowable depreciation and interest for financing. The decision to buy or to lease depends on a number of factors, such as the borrowing interest rate and the term of the lease.

For the leasing option, the after-tax cash flow at any year t (for $t = 1, 2, \ldots, 5$) can be obtained as follows:

$$A_t = B_t - C_t$$

$$Y_t = A_t - A_tX_t = A_t - T_t$$

We can compare the purchase-or-lease options by comparing the net present values of the cash flow profiles of these alternatives.

Example 12.10

Suppose that a firm is interested in acquiring the computer workstation in Example 12.8 but lacks the funds to purchase it through equity. However, it has two options: (1) borrowing the full amount at a borrowing rate of 8%, using a financing plan that requires a single repayment at the end of the 5 years; and (2) leasing the equipment with terms to pay $7,500, $6,500, $5,500, $4,500, and $3,500 at the end of years 1, 2, 3, 4, and 5, respectively. The income tax rate is 34% and the after-tax MARR specified by the firm is 8%. Should the firm buy or lease?

With the proposed financing plan and the DDB method of computing depreciation allowable, the computation of the after-tax cash flows combining operation and financing is obtained by Eq. (12.11) and the results are shown in part (a) of Table 12.7. The net present value discounted at the after-tax MARR of 8% is

$$
\begin{aligned}
[\text{APV}]_{8\%} = {} & (5{,}790)(P \mid F, 8\%, 1) + (4{,}594)(P \mid F, 8\%, 2) \\
& + (3{,}876)(P \mid F, 8\%, 3) + (3{,}445)(P \mid F, 8\%, 4) \\
& + (-18{,}912 + 2{,}000)(P \mid F, 8\%, 5) \\
= {} & +3{,}707
\end{aligned}
$$

The results for the leasing option are shown in part (b) of Table 12.7. Hence, the net present value discounted at the after-tax MARR of 8% is

TABLE 12.7 AFTER-TAX CASH FLOWS FOR BUY-OR-LEASE ALTERNATIVES

(a) THE BUY OPTION

t	A_t	$\bar{A}_t$	D_t	I_t	W_t	T_t	$\hat{Y}_t$
0	−22,000	+22,000	0	0	0	0	0
1	+6,000	−1,760	8,800	1,760	−4,560	−1,550	+5,790
2	+6,000	−1,760	5,280	1,760	−1,040	− 354	+4,594
3	+6,000	−1,760	3,168	1,760	1,072	364	+3,876
4	+6,000	−1,760	1,901	1,760	2,339	795	+3,445
5	{ +6,000	−23,760	851	1,760	3,389	1,152	{ −18,912
	+2,000						+2,000

(b) THE LEASE OPTION

t	B_t	C_t	A_t	T_t	Y_t
0	0	0	0	0	0
1	6,000	7,500	−1,500	−510	−990
2	6,000	6,500	−500	−170	−330
3	6,000	5,500	+500	170	+330
4	6,000	4,500	+1,500	510	+990
5	6,000	3,500	+2,500	850	+1,650

$$\text{APV} = [\text{NPV}]_{8\%} = -(990)(P \mid U, 8\%, 5) + (660)(P \mid G, 8\%, 5)$$
$$= +913$$

Thus, both options are acceptable, but the purchase option is preferable.

12.9 EFFECTS OF DEBT SIZE

It was pointed out earlier that substantial borrowing may alter the long-term financial position of a firm because of the charge of the debt-to-equity ratio of its financial structure. However, within limits, debt commitments such as the purchase of a piece of equipment through credit would not present a problem.

We can examine the effects of debt size by comparing the net present values of the cash flow profiles of various alternatives with varying debt sizes. Using the same operations cash flow profile for all alternatives, the effects of different debt sizes in the alternative financing plans can be assessed.

Example 12.11

The purchase of the computer workstation in Example 12.8 was financed by equity. The same workstation was purchased with debts of $10,000 and $22,000 in Examples 12.9 and 12.10, respectively. Considering case (a) in each problem, which uses the DDB method for depreciation, compare the net present value of these alternatives with different financing plans.

It is interesting to note from Examples 12.8, 12.9, and 12.10 that for the same conditions of depreciation and after-tax MARR, the investment plan coupled with different financing plans leads to different net present values as follows:

1. Equity financing APV = NPV = +992
2. Borrowing $10,000 APV = +2,093
3. Borrowing $22,000 APV = +3,707

Since the borrowing rate is equal to the after-tax MARR in this case, the allowable deduction for interest produces beneficial effects. However, if the borrowing rate were higher, debt financing would produce a less favorable or even negative effect on the investment proposal.

12.10 EFFECTS OF INFLATION ON EQUITY-FINANCED PROJECTS

For private firms, the taxable income in each future year will be based on the then-current value. Consequently, we must perform a double conversion of the cash flows if the discount rate excluding inflation rate is used in computing the net present value.

To simplify the discussion, we consider the case of equity financing in this section and defer the case of debt financing to the next section. We can modify the notation in Section 12.4 for inflation analysis, and introduce those terms related to the computation of income tax for the operation cash flow in any year t over a planning horizon of n years.

A_t = before-tax cash flow in terms of base-year dollars
A'_t = before-tax cash flow in terms of then-current dollars
D_t = depreciation allowance based on one of the acceptable methods
X_t = income tax rate
T'_t = income taxes in terms of then-current dollars
Y'_t = after-tax cash flow in terms of then-current dollars
Y_t = after-tax cash flow in terms of base-year dollars

The federal tax laws do not contain any provision for adjusting the annual depreciation allowance or the salvage value to account for inflation. Hence, while the net annual income A_t is adjusted for inflation, the annual depreciation allowance D_t is not adjusted. Although the salvage value S is set at time $t = 0$ on the basis of base-year dollars, it remains S for the book value in then-current dollars at $t = n$. There has been an attempt to change the laws so that the salvage value can be indexed to reflect inflation, but so far it has been unsuccessful. Consequently, the S value at $t = n$ is expressed in then-current dollars and should be converted to base-year dollars for discounting if necessary.

The equations for computing income tax of projects with equity financing are also modified for inflation consideration as follows:

$$A'_t = A_t(1 + j)^t \tag{12.14}$$

$$T'_t = X_t(A'_t - D_t) \tag{12.15}$$

$$Y'_t = A'_t - T'_t = A'_t - X_t(A'_t - D_t) \tag{12.16}$$

$$Y_t = Y'_t(1 + j)^{-t} \tag{12.17}$$

Equation (12.14) is identical to Eq. (10.7), and Eq. (12.17) is analogous to Eq. (10.8) except that the after-tax cash flow Y_t instead of the before-tax cash flow A_t is considered, as shown in Fig. 12.2. Consequently, Eq. (10.17) is applicable to after-tax computation if A_t is replaced by Y_t, i.e., using constant price approach,

$$\text{NPV} = Y_0 + \sum_{t=1}^{n} Y_t(1 + i)^{-t} \tag{12.18}$$

where i is the after-tax MARR excluding inflation.

Similarly, the after-tax cash flows can be obtained by using Eqs. (12.14) through (12.16). Equation (12.17) is not needed since Y'_t (not Y_t) is needed to replace A'_t in Eq. (10.18) for computing the net present value, i.e., using constant price approach,

$$\text{NPV} = Y_0 + \sum_{t=1}^{n} Y'_t(1 + i')^{-t} \tag{12.19}$$

where i' is the after-tax MARR including inflation.

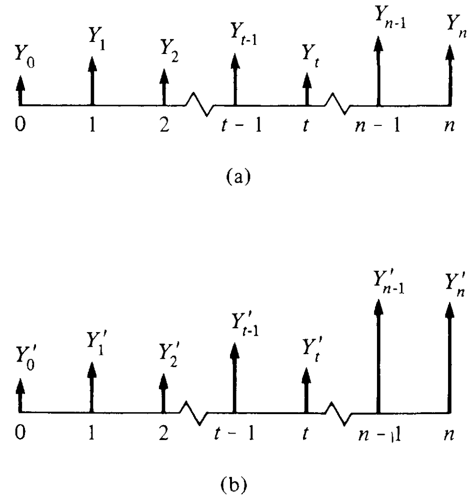


Figure 12.2 Constant price and current price after-tax cash flows.

Example 12.12

A piece of equipment costing \$10,000 is expected to produce a uniform annual net benefit before tax of \$3,000 in terms of the base-year dollars over the next 5 years. The equipment has no salvage value at the end of 5 years and the depreciation allowance is based on the straight-line depreciation method. The federal and state income tax rate is 38%, and the after-tax MARR specified by the firm is 8% excluding inflation. Considering each of the following two cases, determine whether the investment is worthwhile.

(a) There will be no inflation during the next 5 years.
(b) There will be an annual inflation rate of 5%.

The computations of the after-tax cash flows for both cases are shown in parts (a) and (b) of Table 12.8. For case (a), the net present value discounted at the after-tax MARR = 8% is

$$[\text{NPV}]_{8\%} = -10{,}000 + (2{,}620)(P \mid U, 8\%, 5) = +461$$

Consequently, without inflation, the investment is worthwhile.

For case (b), the before-tax cash flow A_t in terms of base-year dollars is first inflated at $j = 5\%$ to then-current dollars for the computation of the taxable income and income taxes. The resulting after-tax cash flow Y'_t in terms of then-current dollars is converted back to base-year dollars. Using Eqs. (12.14) through (12.17),

$$A'_t = A_t(1 + 0.05)^t$$

$$T'_t = (38\%)(A'_t - 2{,}000)$$

TABLE 12.8 AFTER-TAX CASH FLOWS FOR INVESTMENT IN THE EQUIPMENT

(a) NO INFLATION

Year t	Before-tax cash flow A_t	Straight-line depreciation D_t	Taxable income $A_t - D_t$	Income tax T_t	After-tax cash flow Y_t
0	−10,000	0	0	0	−10,000
1–5 (each)	+3,000	2,000	1,000	380	+2,620

(b) INFLATION RATE $j = 5\%$

t	A_t	A'_t	D_t	$A'_t - D_t$	T'_t	Y'_t	Y_t
0	−10,000	0	0	0	0	−10,000	−10,000
1	+3,000	+3,150	2,000	1,150	437	+2,713	+2,584
2	+3,000	+3,308	2,000	1,308	497	+2,811	+2,550
3	+3,000	+3,473	2,000	1,473	560	+2,913	+2,516
4	+3,000	+3,647	2,000	1,647	626	+3,021	+2,485
5	+3,000	+3,829	2,000	1,829	695	+3,134	+2,456

$$Y'_t = A'_t - (38\%)(A'_t - 2{,}000)$$

$$Y_t = Y'_t(1 + 0.05)^{-t}$$

The net present value discounted at the after-tax MARR = 8% excluding inflation is obtained from Eq. (12.18) as follows:

$$\begin{aligned}[\text{NPV}]_{8\%} &= -10{,}000 + (2{,}584)(P \mid F, 8\%, 1) + (2{,}550)(P \mid F, 8\%, 2) \\ &\quad + (2{,}516)(P \mid F, 8\%, 3) + (2{,}485)(P \mid F, 8\%, 4) \\ &\quad + (2{,}456)(P \mid F, 8\%, 5) \\ &= +74\end{aligned}$$

With 5% inflation, the investment has a lower NPV since higher income taxes are paid on the taxable incomes based on then-current dollars.

Example 12.13

Perrymont Corporation plans to purchase a computer workstation that costs \$22,000 and has a salvage value of \$2,000. It is expected to produce a uniform annual net benefit of \$7,000 before tax in base-year dollars in the next 4 years. The workstation will be depreciated by the straight-line method. The federal and state income tax rate is 38%, and the after-tax MARR specified by the firm is 10% per year including inflation. If the inflation rate is expected to be 5% per year in the next 4 years, determine whether the investment is worthwhile.

The inflation rate is 5% per year, but the after-tax MARR including inflation is 10%, which is fixed irrespective of the inflation rate. Thus the before-tax cash flows A_t (t = 1, 2, 3, and 4) in terms of base-year dollars are converted to then-current dollars for computing the taxable income and income taxes. The after-tax cash flows Y'_t (t = 1, 2, 3, and 4) in terms of then-current dollars are obtained and tabulated in Table 12.9. Note that the after-tax cash flows Y_t in base-year dollars are not needed, although they have also been tabulated in the last column of Table 12.9. The net present value of this investment is based on Y'_t (t = 1, 2, 3, 4) discounted at 10%, using Eq. (12.19). Hence, using current price approach,

$$\begin{aligned} \text{NPV} = -22{,}000 &+ (6{,}457)(P \mid F, 10\%, 1) + (6{,}685)(P \mid F, 10\%, 2) \\ &+ (6{,}924)(P \mid F, 10\%, 3) + (7{,}176)(P \mid F, 10\%, 4) \\ &+ (2{,}000)(P \mid F, 10\%, 4) \\ = +864 \end{aligned}$$

Consequently, the investment is worthwhile.

Alternatively, we can compute first the discount rate i excluding inflation by Eq. (10.13). Noting that i' = 10% and j = 5%, we find

$$i = \frac{i' - j}{1 + j} = \frac{0.10 - 0.05}{1 + 0.05} = 0.0476 = 4.76\%$$

Then, the after-tax cash flows Y_t in terms of base-year dollars in the last column of Table 12.9 can be discounted at the rate i by using Eq. (12.18). Then, using constant price approach,

$$\begin{aligned} [\text{NPV}]_{4.76\%} = -22{,}000 &+ \frac{6{,}150}{1.0476} + \frac{6{,}063}{(1.0476)^2} + \frac{5{,}981}{(1.0476)^3} \\ &+ \frac{5{,}904}{(1.0476)^4} + \frac{1{,}645}{(1.0476)^4} \\ = +864 \end{aligned}$$

This result is the same as that obtained previously.

TABLE 12.9 AFTER-TAX CASH FLOWS FOR INVESTMENT IN THE MACHINE

t	A_t	A'_t	D_t	$A'_t - D_t$	T'_t	Y'_t	Y_t
0	−22,000	−22,000	0	0	0	−22,000	−22,000
1	+7,000	7,350	5,000	2,350	893	+6,457	+6,150
2	+7,000	7,718	5,000	2,718	1,033	+6,685	+6,063
3	+7,000	8,103	5,000	3,103	1,179	+6,924	+5,981
4	+7,000	8,509	5,000	3,509	1,333	+7,176	+5,904
	+2,000					+2,000	+1,645

12.11 EFFECTS OF INFLATION ON DEBT-FINANCED PROJECTS

For projects financed by debts, we introduce additional terms from the notation in Section 12.4 for inflation analysis that includes the financing cash flow in year t over a planning horizon of n years.

$\bar{A}_t$ = before-tax financing cash flow in terms of base-year dollars
$\bar{A}'_t$ = before-tax financing cash flow in terms of then-current dollars
$\bar{Y}_t$ = after-tax financing cash flow in terms of base-year dollars
$\bar{Y}'_t$ = after-tax financing cash flow in terms of then-current dollars

To account for inflation on the financing cash flow, we need to modify Eqs. (12.7) and (12.8) as follows:

$$I'_t = Q'_t - \bar{A}'_t \tag{12.20}$$

$$\bar{Y}'_t = \bar{A}'_t + X_t I'_t \tag{12.21}$$

Then,

$$\text{FPV} = \bar{Y}_0 + \sum_{t=1}^{n} \bar{Y}'_t (1 + i')^{-t} \tag{12.22}$$

Similarly, using $\bar{Y}_t$ and i,

$$\text{FPV} = \bar{Y}_0 + \sum_{t=1}^{n} \bar{Y}_t (1 + i)^{-t} \tag{12.23}$$

If i' is used to discount the operation cash flow and the financing cash flow which are expressed in then-current dollars, then Eq. (12.12) becomes

$$\text{APV} = [\text{NPV}]_{i'} + [\text{FPV}]_{i'} \tag{12.24}$$

Example 12.14

The purchase of the computer workstation in Example 12.13 is financed by a debt of \$20,000 at $t = 0$. The lender agrees to an annual interest rate of 10% including inflation with a uniform annual payment of \$6,310 in then-current dollars over the next 4 years. All other information remains the same. Determine whether the purchase is worthwhile.

Since the interest from borrowing is deductible for tax purposes, we first compute the annual interest according to Eq. (12.20) and the associated equations in Chapter 4. The result of the computation is shown in Table 12.10.

TABLE 12.10 ANNUAL INTEREST AND PRINCIPAL BALANCE

t	$\bar{A}'_t$	R'_{t-1}	I'_t	Q'_t	R'_t
0	+20,000	0	0	0	+20,000
1	−6,310	+20,000	+2,000	−4,310	+15,690
2	−6,310	+15,690	+1,569	−4,741	+10,949
3	−6,310	+10,949	+1,095	−5,215	+5,734
4	−6,310	+5,734	+573	−5,737	−3

TABLE 12.11 AFTER-TAX FINANCIAL CASH FLOWS

t	$\bar{A}'_t$	I'_t	$X_t I'_t$	$\bar{Y}'_t$	$\bar{Y}_t$
0	+20,000	0	0	+20,000	+20,000
1	−6,310	2,000	1,000	−5,310	−5,057
2	−6,310	1,569	785	−5,525	−5,011
3	−6,310	1,095	548	−5,762	−4,977
4	−6,310	573	287	−6,023	−4,955

The before-tax financial flows for $t = 1, 2, 3$, and 4 are already expressed in then-current dollars. Hence, the tax shields and the after-tax financial cash flows can be computed by Eq. (12.21) and the results are shown in Table 12.11.

Using the financial cash flow including inflation, we obtain from Eq. (12.22)

$$\text{FPV} = +20{,}000 - (5{,}310)(P \mid F, 10\%, 1) - (5{,}525)(P \mid F, 10\%, 2) - (5{,}762)(P \mid F, 10\%, 3) - (6{,}023)(P \mid F, 10\%, 1)$$

$$= 20{,}000 - 4{,}827 - 4{,}566 - 4{,}329 - 4{,}114 = 2{,}164$$

Using the financing cash flows excluding inflation, we also obtain from Eq. (12.23)

$$\text{FPV} = 20{,}000 - \frac{5{,}057}{1.0476} - \frac{5{,}011}{(1.0476)^2} - \frac{4{,}977}{(1.0476)^3} - \frac{4{,}955}{(1.0476)^4}$$

$$= 20{,}000 - 4{,}827 - 4{,}566 - 4{,}329 - 4{,}114 = 2{,}164$$

Since NPV = 864 was obtained in Example 12.13, then for $i' = 10\%$,

$$\text{APV} = \text{NPV} + \text{FPV} = 864 + 2{,}164 = 3{,}028$$

12.12 EFFECTS OF CAPITAL GAINS

Capital gain is defined as the difference between the sale price and the book value of an asset. When the asset is held for more than a required holding period under the tax laws, the capital gain is regarded as long-term capital gain and is taxed at a rate lower than that applied to ordinary income of a corporation. In a period of inflation, the sales price of an asset as expressed in then-current dollars increases but the book value is not allowed to be indexed to reflect the change of price level. Consequently, capital gains tax increases with the surge in sale price resulting from inflation.

Let us introduce the following notation in the computation of capital gains tax upon the sale of an asset at year t:

B_t = book value of the asset in base-year dollars at end of year t
S_t = sale price in then-current dollars when the asset is sold at year t
G_t = capital gain when the asset is sold at year t
X_c = tax rate for capital gain
T_c = capital gain tax

Then,

$$G_t = S_t - B_t \tag{12.25}$$

and the capital gains tax is

$$T_c = X_c G_t = X_c(S_t - B_t) \tag{12.26}$$

Example 12.15

A tractor costing \$60,000 has a useful life of 7 years with a salvage value of \$4,000 at the end of 7 years. Using the straight-line depreciation method, the book value at the end of 3 years is known to be \$36,000.

(a) Suppose that this tractor is sold for \$40,000 at the end of 3 years, and the capital gain is taxed at a rate of 28%. What is the capital gains tax?
(b) Suppose that this same tractor is kept for 7 years and sold for \$10,000 at the end of 7 years. What is the capital gains tax?

For (a), $G_t = 40{,}000 - 36{,}000 = \$4{,}000$. Hence the capital gains tax is

$$T_c = X_c G_t = (0.28)(4{,}000) = \$1{,}120$$

For (b), $G_t = 10{,}000 - 4{,}000 = \$6{,}000$. Then,

$$T_c = X_c G_t = (0.28)(6{,}000) = \$1{,}680$$

Example 12.16

A developer bought a plot of land for \$100,000 and spent \$1,600,000 to construct an apartment building on the site for a total sum of \$1,700,000. The before-tax annual rental income after the deduction of maintenance expenses is expected to be \$300,000 in base-year dollars over the next 40 years. The developer plans to sell this building in 6 years when the property appreciates in value. According to the current tax laws, the cost of land cannot be depreciated and original land cost is treated as the salvage value, but the cost of construction can be depreciated entirely over 32 years based on the straight-line depreciation method. The developer specifies an after-tax minimum attractive rate of return of 10% excluding inflation. The income tax rate is 34% and the capital gains tax rate is 28%.

(a) If there is no inflation and the sales price of the property including the land is \$2,100,000 at the end of 6 years, find the net present value of this investment.

(b) If the inflation rate is 4% per year and the annual rental income keeps up with the inflation in the next 6 years, and the sales price of the property including the land is \$2,700,000 in then-current dollars at the end of 6 years, find the net present value of this investment.

Using the straight-line depreciation method, the annual depreciation allowance over 32 years is

$$D_t = \frac{1}{N}(P - S) = \frac{1}{32}(1{,}600{,}000) = 50{,}000$$

The book value of the property after 6 years is

$$B_6 = P - \frac{t}{N}(P - S) = 1{,}700{,}000 + -(6)(50{,}000) = 1{,}400{,}000$$

For (a), the after-tax net income in each year t in the next 6 years is given by

$$\begin{aligned} Y_t &= A_t - X_t(A_t - D_t) \\ &= 300{,}000 - (34\%)(300{,}000 - 50{,}000) = 215{,}000 \end{aligned}$$

The capital gains tax for the sale of the property at the end of 6 years is

$$T_c = X_c G_6 = (28\%)(2{,}100{,}000 - 1{,}400{,}000) = 196{,}000$$

Then

$$\begin{aligned} [\text{NPV}]_{10\%} &= -1{,}700{,}000 + (215{,}000)(P \mid U, 10\%, 10) \\ &\qquad + (2{,}100{,}000 - 196{,}000)(P \mid F, 10\%, 10) \\ &= -1{,}700{,}000 + (215{,}000)(3.7908) + (1{,}904{,}000)(0.6209) \\ &= +2{,}997{,}216 \end{aligned}$$

For (b), the after-tax net income in each year t in the next 6 years for an annual inflation rate of 4% is given in Table 12.12. Since the property will be sold after 6 years, the cash flows beyond year 6 need not be considered.

The capital gains tax for the sale of the property expressed in then-current dollars at the end of 6 years is

$$T'_c = X_c G'_6 = (28\%)(2{,}700{,}000 - 1{,}400{,}000) = 364{,}000$$

Since the after-tax MARR is 10% excluding inflation, the capital gains less tax should be converted to base-year dollars as follows:

$$(G'_6 - X_c G'_6)(1.04)^{-6} = (2{,}700{,}000 - 364{,}000)(0.7903) = 1{,}846{,}141$$

TABLE 12.12 AFTER-TAX CASH FLOWS (IN $ THOUSAND)

t	A_t	A'_t	D_t	$A'_t - D_t$	T'_t	Y'_t	Y_t
0	−1,700	−1,700	0	0	0	−1,700	−1,700
1	300	312.00	50	262.00	89.08	222.92	214.34
2	300	324.48	50	274.48	93.32	231.16	213.73
3	300	337.47	50	287.47	97.74	239.73	213.12
4	300	350.97	50	300.97	102.33	248.64	212.54
5	300	365.01	50	315.01	107.10	257.91	211.98
6	300	379.59	50	329.59	112.06	267.53	211.43

Then, using the constant price approach,

$$\begin{aligned}[\text{NPV}]_{10\%} = -1{,}700{,}000 &+ (214{,}300)(P \mid F, 10\%, 1) \\ &+ (213{,}730)(P \mid F, 10\%, 2) + (213{,}120)(P \mid F, 10\%, 3) \\ &+ (212{,}540)(P \mid F, 10\%, 4) + (211{,}980)(P \mid F, 10\%, 5) \\ &+ (211{,}430)(P \mid F, 10\%, 6) + (1{,}846{,}141)(P \mid F, 10\%, 6) \\ = 1{,}969{,}881 &\end{aligned}$$

12.13 SUMMARY AND STUDY GUIDE

This chapter has covered some aspects of the federal tax laws as background information for the computation of after-tax cash flows. It emphasizes the principles of analyzing the effects of corporate taxes on the economic evaluation of investment proposals instead of specific provisions of current tax rates which are subject to change over time. It also deals with the issue of deductibility of state and local taxes from taxable income in order to avoid double taxation.

The procedure of computing after-tax cash flows is developed for projects financed by equity and those financed by debt. The effects of deduction for depreciation, interest, and debt size are discussed and illustrated by examples including the comparison of purchase versus lease. The effects of inflation on the after-tax cash flows and the treatment of such cash flows in economic evaluation are also included. The effects of capital gains are also discussed.

REFERENCES

12.1 *Federal Tax Guide*. Englewood Cliffs, NJ: Prentice-Hall (annual publication).

12.2 *Standard Federal Tax Reporter*. Chicago, IL: Commerce Clearing House (annual publication).

12.3 *The Price Waterhouse Guide to the New Tax Law*. New York: Bantam, 1986.

PROBLEMS

P12.1 An automatic copier will cost $10,000 and have no salvage value at the end of 4 years. The sum-of-the-years'-digits method will be used for computing the depreciation allowance. The firm has an income tax rate of 34% and an after-tax MARR of 6%. What should be the before-tax uniform annual benefit generated by this copier to make the purchase worthwhile?

P12.2 A heavy-duty lifting machine costing $70,000 will be depreciated to no salvage value at the end of 5 years by the straight-line method. The machine will produce a gross income of $45,000 per year for 5 years and the operating expenses for years 1 through 5 are $15,000, $16,000, $17,000, $18,000, and $19,000, respectively. The income tax rate of the firm is 34%, and the after-tax MARR is 8%. Determine the net present value of the after-tax cash flows for the project.

P12.3 An investment in an automatically controlled printing press will cost $50,000 and have no salvage value at the end of 5 years when it will be retired. The press will generate a gross income of $15,000 per year, but its operating cost will be $3,000 during the first year, increasing by $500 per year until it reaches $5,000 in the fifth year. The straight-line depreciation method is used. The tax rate is 34% and the after-tax MARR is 10%. Determine the net present value of the proposal.

P12.4 The Acron Construction Company is considering the purchase of a diesel power shovel to improve its productivity. The shovel, which costs $77,000, is expected to produce a before-tax benefit of $38,000 in the first year, and $6,000 less in each succeeding year for a total of 5 years (i.e., before-tax benefit of $32,000 in the second year, $26,000 in the third year, until $14,000 in the fifth year). The salvage value of the equipment will be $5,000 at the end of 5 years. The firm chooses to use the sum-of-the-years'-digits depreciation for the equipment and pays an annual income tax for its profits at a rate of 34%. If the minimum attractive rate of return after tax is 10%, is the purchase worthwhile?

P12.5 The Sherman Corporation is considering the purchase of a number of pipe-laying machines to facilitate the operation in a new pipeline project which is expected to last for 6 years. Each machine will cost $26,000 and will have a salvage value of $2,000 when the production under this contract is completed at the end of 6 years. The firm chooses to use the straight-line depreciation method for the machine and pays annual federal and state income taxes for its profits at a rate of 34%. If the minimum attractive rate of return after tax is 10%, what is the minimum uniform annual benefit before tax that must be generated by this device to justify its installation?

P12.6 The Myron Corporation is planning to purchase a metal-forming machine which has an estimated useful life of 8 years. The machine costs $50,000 with no salvage value and is expected to generate a gross income of $12,000 per year before tax. The tax rate is 34% and the after-tax MARR is 10%. The straight-line method is used in computing depreciation. Determine whether the investment is worthwhile if

(a) The depreciation is based on the useful life of 8 years.

(b) The machine is depreciated in 4 years with no salvage at the end of 4 years even though the machine can still be used to generate income for the remaining years. The company has other profitable projects to offset unused deductions for taxable income from this project.

P12.7 A branch office of Telesafe Bank is considering the installation of an automated "night-depositor" to increase its profits by \$10,000 per year over the next 10 years. This installation will have no salvage value at the end of 10 years and is depreciated over the 10-year period using the straight-line method. Based on a rate of 34% for federal and state income taxes and an after-tax MARR of 10%, what is the maximum amount that should be spent for the equipment to justify the investment?

P12.8 The Cantor Company is considering either leasing or buying a computer for the next 5 years. The company pays federal and state income taxes at a rate of 34% and uses an after-tax MARR of 8%. The details of the two options are as follows:

(a) Under the lease agreement, the company will pay the lessor a uniform fee at the *beginning* of each of the 5 years, plus a cost of \$10,000 per year at the *end* of each year under a maintenance contract which accompanies the lease agreement. Assume that the company does not receive any benefit from depreciation or investment tax credit.

(b) Under the purchase plan, the purchase price is \$200,000, but the company can apply 10% of the purchase price for an investment tax credit in the year of the purchase (year 0). The computer will be depreciated by the straight-line method with no salvage value at the end of 5 years. The maintenance cost is \$20,000 per year charged at the end of each year.

Determine the uniform fee paid to the lessor at the beginning of each year which will make the lease agreement as attractive as the purchase plan.

P12.9 Sandhurst Electric Corporation is considering four mutually exclusive alternatives for a major capital investment project. All alternatives have a useful life of 10 years with no salvage value at the end. Straight-line depreciation will be used. The corporation pays federal and state tax at a rate of 34%, and expects an after-tax MARR of 12%. Determine which alternative should be selected, using the NPV method.

Alternative	Initial cost (\$ million)	Before-tax uniform annual net benefits (\$ million)
1	4.0	1.5
2	3.5	1.1
3	3.0	1.0
4	3.7	1.3

P12.10 An investment of \$40,000 in a vibratory hammer is expected to produce a net uniform annual cash flow of \$15,000 per year in base-year dollars before tax. The depreciation allowance is based on the straight-line method with no salvage value at the end of 5 years. A MARR of 12% after tax excluding inflation is specified. The income tax rate is 34%. Determine whether the project is worthwhile assuming that

(a) There will be no inflation.

(b) The inflation rate will be 6% per year.

P12.11 An investment of \$40,000 in a delivery truck is expected to produce net annual benefits in base-year dollars for the next 3 years as follows: +18,000 for year 1, +16,000 for year 2, +14,000 for year 3. The equipment is depreciated by the straight-line method with no salvage value at the end. The inflation rate is expected to be 10% per year. The specified MARR excluding inflation is 8%. The corporate income tax rate is 34%. Is this investment worthwhile?

P12.12 The Ternadem Corporation plans to purchase a demolition and wrecking machine to save labor costs. The machine costs $50,000 and has a salvage value of $10,800 at the end of 5 years. The machine is expected to be in operation for 5 years, and it will be depreciated by the double declining balance method up to the salvage value. The corporation specifies an after-tax MARR excluding inflation of 20% and has an income tax rate of 34%. The annual inflation rate is expected to be 10% during the next 5 years. If the uniform annual net benefit before tax in terms of base-year dollars for the next 5 years is $20,000, is the investment worthwhile?

P12.13 Macmillion Chemical Company plans to invest $2 million in a new plant which is expected to produce a uniform annual net benefit before tax of $600,000 in terms of the base-year dollars over the next 6 years. The plant has a salvage value of $250,000 at the end of 6 years and the depreciation allowance is based on the SOYD depreciation method. The corporate tax rate is 34%, and the after-tax MARR specified by the firm is 12% excluding inflation. If the annual inflation rate during the next 6 years is expected to be 8%, determine whether the investment is worthwhile.

P12.14 A company is considering investing in a water purification system. Three alternatives are under consideration and their before-tax cash flow profiles in base-year dollars are given, and all three have no salvage values after 4 years.

Year	System A	System B	System C
0	−80,000	−120,000	−100,000
1	+40,000	+60,000	+35,000
2	+35,000	+50,000	+35,000
3	+30,000	+40,000	+35,000
4	+25,000	+30,000	+35,000

The depreciation allowance is based on the straight-line method. The general inflation rate is expected to be 6% per year during the next 4 years. The tax rate is 34%, and the after-tax MARR excluding inflation is 8%. Determine which alternative, if any, should be adopted.

P12.15 A heavy truck was purchased by a construction company 4 years ago at a cost of $75,000 with a useful life of 8 years and a salvage value of $3,000 at the end of 8 years. The SOYD method of depreciation was used. If the truck was sold at $35,000 4 years after its purchase, find the capital gains tax if the tax rate for capital gain is 20%.

P12.16 An office building is purchased by a real estate company at a cost of $3,600,000 with a useful life of 32 years and a salvage value of $400,000 (for the land value at purchase) at the end of 32 years. The straight-line depreciation method is used and the capital gains tax rate is 28%. If the company expects to sell this building after 5 years at a price of $5,000,000, determine the capital gains tax at the time of sale.

P12.17 An apartment building costs $4,400,000 including the land cost of $800,000. The corporation that owns the building pays cash for the building and expects to receive an annual net revenue of $620,000 before tax for the next 8 years. The building will be sold after 8 years at an expected price of $5,200,000 including land. The corporate income tax rate is 36% and capital gains tax rate is 28%. The after-tax MARR of the corporation is 8%. The straight-line method will be used for depreciation regardless of the depreciable life. It is assumed that there is no inflation. Find the net present value of the investment for each of the following cases:

(a) The depreciable life of the building specified by the IRS is 32 years.

(b) The depreciable life of the building specified by the IRS is 15 years.

P12.18 Terrytown Corporation plans to finance a machine of $67,000 from internal funds. The machine has a salvage value of $7,000 at the end of its useful life of 5 years, and is expected to be sold for a current price of $20,000 at the end of 5 years. Using the SOYD depreciation method, determine the annual depreciation allowance and book values for each year. The machine is expected to generate an annual income of $25,000 in base-year dollars for 5 years. The after-tax MARR including inflation is 8%, and the inflation rate per year is expected to be 4%. The corporate income tax rate is 34% and the capital gains tax rate is 28%. Find the net present value of this investment if there is no borrowing.

P12.19 A pile-driving machine costing $729,000 has a useful life of 9 years and a salvage book value of $40,000 at the end of 9 years. The machine is expected to be sold for an actual value of $100,000 after 9 years. Using the double declining balance method with conversion to the straight-line method if necessary, determine the annual depreciation allowance and book value for each year. This machine will generate a uniform annual income of $220,000 before tax for the next 9 years. The company that owns this machine has an after-tax MARR of 8%. The corporate income tax rate is 36%, and the capital gains tax rate is 28%. Find the net present value of this investment if there is no inflation and no debt financing is involved.

P12.20 A machine purchased at a price of $200,000 is expected to produce a uniform annual revenue of $70,000 (in base-year dollars) before tax in the next 4 years. The depreciation is based on the straight-line method with a salvage value of $20,000 at the end of the 4 years. Half of the purchase price (i.e., $100,000) is borrowed at annual interest of 10%, and the borrower agrees to pay only annual interest on the loan but to repay the entire principal at the end of 4 years, all in then-current dollars. The inflation rate is expected to be 4% per year. The company has an after-tax MARR of 8% excluding inflation and the corporate tax rate is 34%. Find the net present value of the investment and debt financing by the following approaches:

(a) The constant price approach

(b) The current price approach

CHAPTER

13 Production Cost and Benefit Estimation

13.1 COSTS AND BENEFITS IN PRODUCTION

In Chapter 9, we characterized the investment cash flow profile as the production cash flow profile, as opposed to the financing cash flow profile resulting from the financing plan for the implementation of the investment proposal. By production, we mean the process of converting resources such as capital, labor, and materials to desirable goods and services through the use of appropriate technologies available. Thus, production is used in a broad sense to refer to activities in manufacturing, construction, chemical, mining, and other industries.

The costs and benefits in a production process vary over the years and are dependent on the scale of initial investment, future expansion, operation and maintenance, and the market economy. To some extent, the accounting system of an organization is an important source of historical data on costs and benefits for similar projects. The difficulties in estimating costs and benefits in production lie not only in the method and reliability of measurement, but in the knowledge of pertinent technologies for production.

For various industries, the returns on capital investments are dependent on the technological, operating, and pricing strategies adopted by the management. To maximize profits, a firm must decide what is the best capacity of a facility based on available technologies used in production, and once a facility is selected, what is the best way to operate the facility and price its products. These are familiar questions of economic efficiency addressed in microeconomics. Essentially, the relationship between costs and output for a particular facility provides a convenient tool for the analysis of operations over the short run, during which no substantial change can be made in facility capacity, while the relationship among costs, output, and facility capacity is useful in investment planning over the long run, during which the facility can be significantly altered. Theoretically, the economic efficiency of a large number of alternatives for investment decision can be examined by constructing the long-run relationship, and the alternative with maximum economic efficiency as observed from this long-run relationship can be selected. Once a particular facility is chosen, the profits in day-to-day operations can also be maximized on the basis of the short-run relationship. In practice, this analytical process is very complex, even if certain assumptions are introduced to simplify the analysis.

In this chapter, we discuss briefly the relationships of cost, benefit, and output volume, and the strategies for maximizing profit or net benefit in the short run. We examine the basic assumptions that simplify the analysis. Specifically, we consider linear break-even analysis and nonlinear optimization models as decision tools in the management of production.

13.2 PRODUCTION COSTS AND OUTPUT VOLUME

The costs incurred in production are related both to the cost of the facilities and the volume of output produced. Within the capacity of the available facilities, inputs of labor and material required to manufacture a given number of units of product vary with the volume of output. The relationship between the inputs and outputs is called the *production function* and depends on the technology of production that is used. Naturally, a firm will want to choose the least-cost technology if suitable. Some determinants of the cost of various technologies, and thus the technology chosen, include the general state of technological development, the relative costs of various factors of production, the volume of output, and the interrelationship of the selected production process with the production of other products of the firm.

Once a facility for production has been installed, there are certain types of expenses called *fixed costs,* which cannot be changed substantially over the short run, regardless of the output volume; other types of expenses which do vary directly with output volume are called *variable costs*. The total cost of production over a specified period of time is the sum of the fixed cost and the variable cost. The relationship between the total cost and the output volume is called the *cost function*.

Fixed costs are constant over the specified time period and must be taken as given over this relevant period of time. For instance, the cost of the plant is a fixed cost over its estimated useful life. Once the plant has been built, the costs associated with its use in the production of output will be almost the same within a certain level of production. The fixed costs of facilities with a useful life over several years may be amortized over the estimated useful life. The remuneration of salaried workers such as executives, manager, and engineers is generally considered to be a fixed cost over the short term because these salaries are definite amounts. However, over the long term these salaries can be controlled by employment policies.

Costs that are related directly or indirectly to the volume of output are called variable costs. The cost of raw materials is a variable cost: so many tons of raw material (and therefore an equivalent cost) *per unit* of product. The wages of hourly workers are generally considered variable costs, because a certain number of hours and hence a certain amount of pay are associated with each unit of input, and workers can be shifted to or from other products, given overtime, or laid off as output levels dictate. Some costs, such as telephones and electricity, have both fixed and variable components: a flat charge for the installation of the facilities and an hourly usage charge.

The characteristics of a typical cost function is illustrated by Fig. 13.1 in which the total cost of production is plotted against the output volume. Once a facility is committed to production, a fixed cost is incurred. As the volume of output increases, the variable cost also increases but usually not at a uniform rate. Beyond the threshold level of production, the total cost generally increases with the output volume at a decreasing rate as the work force adjusts itself to the production and improves the efficiency. However, when the output volume is increased beyond a certain point, it begins to cause strains on resources. Then, the total cost of production will increase at a higher rate until it explodes exponentially.

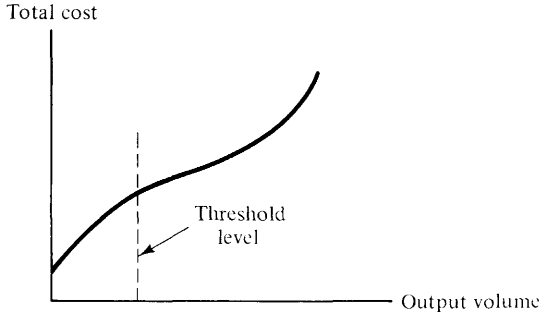


Figure 13.1 Total cost and output volume.

Let x be the output volume and $C(x)$ be the corresponding total cost. Then, the typical cost function in Figure 13.1 can be represented by a cubic equation which depicts the reversed curvature of the curve as follows:

$$C(x) = F + Dx - Ex^2 + Gx^3 \tag{13.1}$$

where D, E, F, and G are constants. For example, the cost function representing the total cost of production of mobile homes (in \$ million) with respect to the output volume (in thousands of housing units) may be given by

$$C(x) = 100 + 30x - 6x^2 + x^3$$

Thus, the total cost $C(x)$ is found to be \$125 million for $x = 1$ or 1,000 housing units.

13.3 AVERAGE COST AND MARGINAL COST

For the typical cost function for production in Fig. 13.1, it can be surmised that up to a certain point it is cheaper to produce a product in larger quantity. The phenomenon in a production process is referred to as the *economies of scale*. This concept can easily be understood by examining the cost per unit of product or the unit cost.

The *average cost* (AC) or *average total cost* (ATC) of a product is defined as the total cost divided by the number of units of production during a specified time period:

$$AC = \frac{C(x)}{x} \tag{13.2}$$

The *marginal cost* (MC) is the cost of producing one more unit of the product at the specified level of production. Numerically, it is equal to the derivative of $C(x)$ with respect to x evaluated at the specified level. Thus, for the production level of x,

$$MC = \frac{dC(x)}{dx} \tag{13.3}$$

The marginal cost depicts the rate of change of the total cost as the production volume increases.

The unit costs of production as represented by the average cost and the marginal cost for various output volumes are illustrated by Fig. 13.2 in which part (a) shows the variation of the total cost with output volume, while part (b) shows the variation of the unit cost with output volume. In part (a), the point $C(x_0)$ on the curve corresponding to $x = x_0$ is a point at which the tangent has the smallest slope relative to the tangents at adjacent points on the curve, while the line joining the point $C(x^*)$ corresponding to $x = x^*$ and the origin has a slope of $C(x^*)/x^*$ and is the tangent to the curve at $x = x^*$. In part (b), the marginal cost MC is seen to be minimum at $x = x_0$, and the average cost AC is minimum at $x = x^*$ where the AC curve intersects the MC curve. While the marginal cost increases after $x = x_0$ is reached, the most important point is $x = x^*$ beyond which the average cost will rise and the economies of scale can no longer be expected.

The minimum average cost can be obtained by finding the derivative of AC and setting it equal to zero to solve for the value of x^*. Thus, from Eq. (13.2)

$$\frac{d(AC)}{dx} = \frac{1}{x}\left[\frac{dC(x)}{dx}\right] - \frac{C(x)}{x^2} = \frac{1}{x}(MC - AC) \tag{13.4}$$

The result of setting $d(AC)/dx = 0$ confirms the fact that at $x = x^*$, MC = AC. Thus,

$$\left.\frac{dC(x)}{dx}\right|_{x=x^*} = \left.\frac{C(x)}{x}\right|_{x=x^*} \tag{13.5}$$

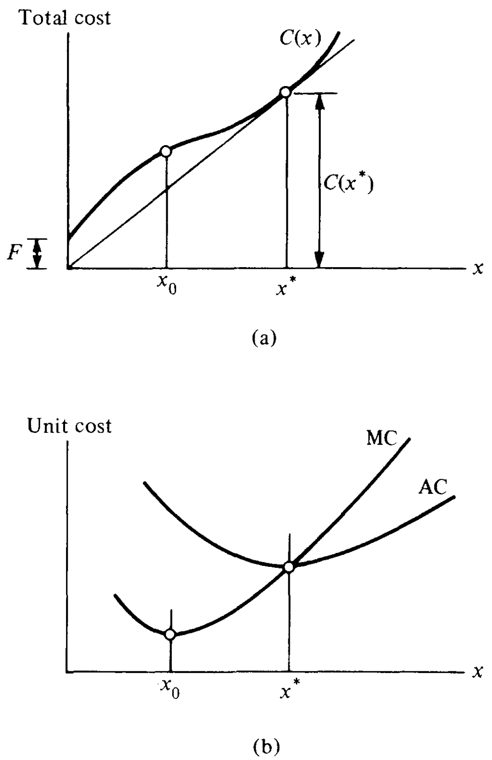


Figure 13.2 Average cost and marginal cost.

Example 13.1

The cost function representing the total cost of production of mobile homes (in \$ million) with respect to output volume (in thousands of homes) is given by

$$C(x) = 100 + 30x - 6x^2 + x^3$$

Find the total cost, average cost and marginal cost for $x = 1, 2, 3, 4, 5$, and 6. What is the minimum average cost and the corresponding output volume?

The average cost for this production process can be obtained by Eq. (13.2) as follows:

$$\text{AC} = \frac{100}{x} + 30 - 6x + x^2$$

Note that AC has a unit of \$ thousand per home resulting from \$ million per 1,000 homes. The marginal cost is given by Eq. (13.3) as follows:

$$\text{MC} = 30 - 12x + 3x^2$$

The unit of MC is also \$ thousand per home.

The total cost, average cost, and marginal cost for $x = 1$ through 6 are tabulated in Table 13.1.

TABLE 13.1 AVERAGE COST AND MARGINAL COST VERSUS OUTPUT VOLUME

x (1,000)	$C(x)$ ($ million)	AC ($ thousand)	MC ($ thousand)
1	125	125	21
2	144	72	18
3	163	54.3	21
4	188	47	30
5	225	45	45
6	280	46.7	66

From Table 13.1, it can be seen that AC = 45 is minimum at $x = 5$. This can be verified by Eq. (13.5). By letting MC = AC,

$$30 - 12x + 3x^2 = \frac{100}{x} + 30 - 6x + x^2$$

from which we get

$$2x^2 - 6x - \frac{100}{x} = 0$$

By factorization after clearing the fraction,

$$\frac{2}{x}(x - 5)(x^2 + 2x + 10) = 0$$

Hence, $x = 5$ is the output level leading to minimum average cost. By substituting $x = 5$ into the equation for AC, we get AC = 45, as indicated in Table 13.1.

13.4 BENEFIT ESTIMATION IN PRODUCTION

The total benefit in a production process is measured by the value of its product. In the private sector, it refers to the total revenue to the firm from the sales of the product. The relationship between the total benefit and the output volume is referred to as the *revenue function*. In the public sector, total benefit includes user benefits not necessarily accrued to the producer in addition to revenues, if any are collected. Hence, the term *benefit function* is used to denote the relationship between the total benefit and the output volume in this broader context.

The characteristics of a typical benefit function are illustrated by Fig. 13.3. Generally, there is a threshold output volume below which the benefit to the producer is regarded as insignificant. Above the threshold level, the total benefit increases substantially but at a decreasing rate because higher output volume will drive down the price of the product, thus increasing the sales. From the consumer perspective, the value (which is sometimes referred to as *utility*) of having more units

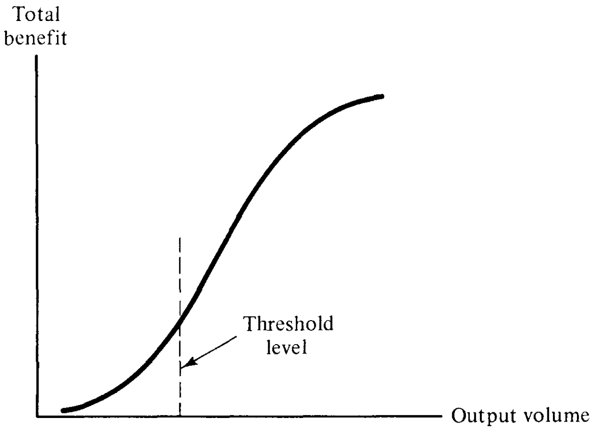


Figure 13.3 Total benefit and output volume.

of a product does not increase proportionately with quantity, but grows at a slower pace until it reaches saturation. Consequently, the decreasing rate of the total benefit with respect to the output volume beyond the threshold level is referred to as the *diminishing marginal utility*. The rate of change of the benefit function with respect to the volume can be obtained from the first derivative of the benefit function, and the curve representing the diminishing marginal utility is depicted in Fig. 13.4.

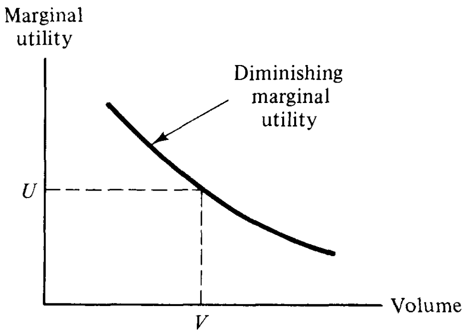


Figure 13.4 Diminishing marginal utility of a product.

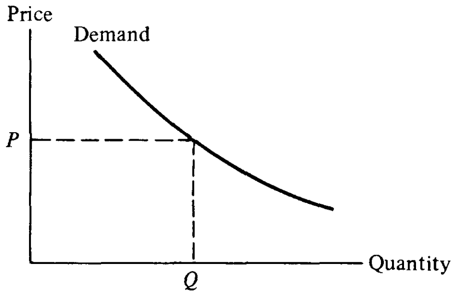


Figure 13.5 Demand curve for a product.

There is a close connection between the diminishing marginal utility curve in Fig. 13.4 and the demand curve in Fig. 13.5, which represents the quantity of a product in demand corresponding to the price level of the product. The price that a consumer is willing to pay for each additional unit of a product is related to the marginal utility of that product to the consumer.

Example 13.2

The benefit function $B(x)$ of a production process with output x is represented by the equation

$$B(x) = 10{,}000\sqrt{x} \text{ for } 100 < x < 2{,}500$$

Show that this function has a diminishing marginal utility.

The marginal utility for the benefit function $B(x)$ is given by its first derivative. Thus,

$$\frac{dB}{dx} = \frac{5{,}000}{\sqrt{x}}$$

This derivative is decreasing from $x = 100$ to 2,500, as can be seen from the values of dB/dx indicated below, which confirms that the benefit function has a diminishing marginal utility.

x	$B(x)$	dB/dx
100	100,000	500
400	200,000	250
900	300,000	167
1,600	400,000	125
2,500	500,000	100

13.5 PRICING STRATEGIES AND PROFIT MAXIMIZATION

A crucial decision in production is the selection of a pricing strategy that will lead to maximizing the profit of a private firm, or maximizing the net benefit to society for a public project. The examination of various pricing strategies and the suitability of their applications to different situations or industries will contribute materially to the management of production. For example, the items of production are often priced at marginal cost to maximize economic efficiency. Such a practice is referred to as *marginal cost pricing*.

The main reason that we must be careful to distinguish between fixed and variable costs is because variable costs are reflected in day-to-day operations, while fixed costs are committed in capital investment decisions made at the *beginning* of the relevant time period. The decision to produce or not to produce, or to select an alternative over the method currently used, is made only on the basis of benefits versus variable costs, but the decision of whether or not to *invest* in a given production facility is made on the basis of benefits versus both fixed and variable costs.

This concept can be explained further as follows. Once a plant is built, the fixed costs have already been incurred and can no longer be controlled. If revenues on the sale of a product are less than the variable costs, the company is incurring additional costs (losses) on each unit produced and should stop producing and shut down the plant. Examples of this can be seen in the hard-hit steel and auto industries. If the revenues are greater than variable costs but less than the sum of variable and fixed costs, the company should continue to produce over the life of the plant to recover *part* of the fixed costs, unless it is more profitable to dispose of the plant. If the revenues are greater than the sum of variable and fixed costs, then all of the costs have been covered, and the company is making a profit.

A private firm that produces a product for sale expects to make a profit, which is the difference between the total revenue from sales and the total cost. The size of the profit can be highly variable, depending on the price and the volume of sales. The firm may try to increase the total profit while maintaining the output volume and therefore increasing the unit price, or it may try to increase the output volume (within the capacity of its facilities) while maintaining the unit price, thus increasing the total profit because the fixed cost remains constant while the output volume is increased. Obviously, raising the unit price is the simplest of all options, but the firm cannot do so without considering the competitive market. The relationship between the unit price of a product and the output volume reflecting certain pricing strategies is called the *price function* based on supply.

The total benefit to the firm as reflected by the total revenue from sales depends on the demand for the product, which is determined by its real or perceived value to the customers. The volume of sales in a competitive market also depends on the unit price of the product. At the unit price charged by the firm, the volume of sales realized represents the "equilibrium" volume between the supply and the demand. At the point of equilibrium, the volume of production equals the volume of sales. Thus, the total benefit to the firm varies with the volume of sales.

13.6 LINEAR BREAK-EVEN ANALYSIS

If both the cost function $C(x)$ and the revenue function $B(x)$ are assumed to be linear with respect to the output volume x of a single product, the analysis of the profit under different levels of production and sales can be greatly simplified. The cost-volume-profit analysis based on linear cost and revenue functions is referred to as *linear break-even analysis*.

The linear model for break-even analysis represents an approximation for some industrial operations. Since it implies that profit increases linearly with increasing output volume beyond the break-even point, it is important to specify a planned output volume at which the production stops. The planned output volume is determined by the market demand based on the price-volume-profit relationship. Generally, if the price is lowered, the sales volume will increase and the profit may increase as a result of increased sales. On the other hand, if the price is increased, the sales volume will decrease but the profit may increase as a result of the higher price. However, the profit may also drop because of either of these actions. The planned output volume is dependent on market demand and price elasticity.

Let F denote the fixed cost over a time period that is constant for the given capacity of the facility, and $V(x)$ denote the variable cost that is proportional to x, i.e., $V(x) = Wx$ where W is the average variable cost per unit of production. Then, the total cost is the sum of F and $V(x)$, i.e.,

$$C(x) = F + Wx \tag{13.6}$$

The total revenue from the volume x sold is represented by the revenue function

$$B(x) = Px \tag{13.7}$$

where P is the unit price for the product. The profit or net income from the sales of volume x is the difference between $B(x)$ and $C(x)$, i.e.,

$$N(x) = (P - W)x - F \tag{13.8}$$

Let x' be the output volume planned for the time period. Then, at $x = x'$, the fixed cost F, the variable cost $V(x')$, the total cost $C(x')$, the total revenue $B(x')$, and the profit $N(x')$ are as shown in Fig. 13.6.

The average cost (AC) and the marginal cost (MC) of the linear cost function in Eq. (13.6) are given respectively by

$$\text{AC} = \frac{F}{x} + W$$

and

$$\text{MC} = W$$

Noting that the average variable cost (AVC) is given by $V(x)/x$, it follows that

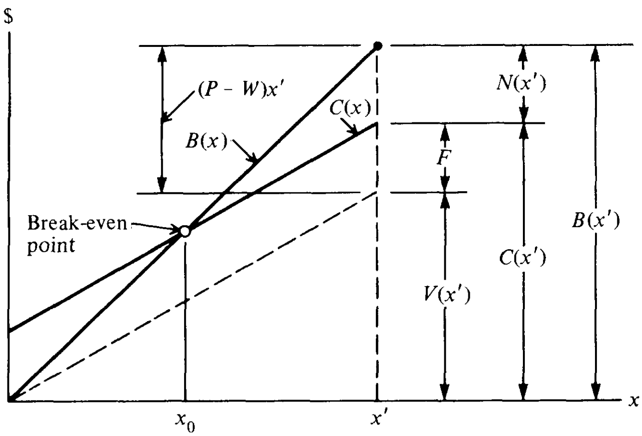


Figure 13.6 Linear break-even analysis.

AVC = W. In other words, for a linear model, the marginal cost is the same as the average variable cost for all levels of output volume.

The term $(P - W)x$ in Eq. (13.8) is called the *contribution margin* and is denoted by

$$z = (P - W)x \tag{13.9}$$

The term $(P - W)$ represents the *contribution margin per unit*. The *contribution margin ratio* as the percentage of the total revenue is defined by

$$r = \frac{(P - W)x}{Px} = 1 - \frac{W}{P} \tag{13.10}$$

The intersection of $B(x)$ and $C(x)$ on the graph in Fig. 13.6, indicating that $B(x) = C(x)$ at $x = x_0$, is called the *break-even point*. The volume $x = x_0$ is called the *break-even volume,* since at $x = x_0$,

$$N(x_0) = B(x_0) - C(x_0) = 0$$

From Eq. (13.8), we find that for $N(x_0) = 0$,

$$x_0 = \frac{F}{P - W} \tag{13.11}$$

and

$$F = (P - W)x_0 \tag{13.12}$$

Thus, the contribution margin per unit beyond $x = x_0$ represents an increment of profit of $(P - W)$.

Example 13.3

A manufacturer of power garden tools is planning to introduce a new power shear that will sell for \$20. The fixed cost for 1 year of operation is \$96,000 and the variable cost per unit is \$12. If the planned volume of sales is 15,000 units for the period, determine the break-even volume and the contribution margin and profit at both the planned volume and the break-even volume.

Since P = \$20, W = \$12, and F = \$96,000, we find from Eq.(13.11) that the break-even volume

$$x_0 = \frac{96{,}000}{20 - 12} = 12{,}000 \text{ units}$$

The remaining quantities may be obtained from Eqs. (13.6) through (13.9) and tabulated as follows:

	Planned volume	Break-even volume
Total revenue	\$ 300,000	\$ 240,000
Variable cost	180,000	144,000
Contribution margin	120,000	96,000
Fixed cost	96,000	96,000
Profit	24,000	0

Example 13.4

An import company buys foreign-made toy train sets at \$25 per unit. The fixed cost of the importing operation is \$40,000 per year. The train sets are sold on commission by sales representatives who receive 30% of the selling price for each set sold. At what price should a set be sold to allow the importer to break even on a total shipment of 4,000 sets per year? If the commission of the sales representatives is reduced to 15%, it is expected that \$8,000 must be added to the fixed cost for advertising in order to sell all 4,000 units at the same price. Will this latter course of action produce a profit or loss?

On the basis of a 30% commission, the total revenue per year will be 70% of the total selling price. Let P be the selling price of each set. At the break-even volume $x_0 = 4{,}000$, we have

$$B(x_0) = (0.7P)(4{,}000) = 2{,}800P$$

$$C(x_0) = 40{,}000 + (25)(4{,}000) = 140{,}000$$

At the break-even point, $B(x_0) = C(x_0)$. Hence,

$$2{,}800P = 140{,}000$$

from which we obtain $P = \$50$.

If the commission is reduced to 15%, then the total revenue per year will be 85% of the total price. On the other hand, the fixed cost is increased to $40{,}000 + 8{,}000 = \$48{,}000$. Hence, at $x = 4{,}000$,

$$B(x) = (0.85)(50)(4{,}000) = 170{,}000$$

$$C(x) = 48{,}000 + (25)(4{,}000) = 148{,}000$$

Hence,

$$N(x) = 170{,}000 - 148{,}000 = \$22{,}000$$

This action is obviously profitable.

13.7 NONLINEAR OPTIMIZATION MODELS

In many situations, nonlinear cost and revenue functions are more appropriate for representing the production process. While one or both of these functions may be nonlinear, the equations used to depict these functions are often simplified to facilitate an analytic solution. Quadratic approximations are by far the most common form of nonlinear models for illustration because they are easy to solve algebraically. Such models contain the notion of profit maximization since the profit does not vary proportionally with the increase of output volume as in the linear model. Consequently, the break-even volumes are not significant, and more than one such volume may occur in the model where the cost and benefit functions intersect.

Let us illustrate the general case of the nonlinear cost and revenue functions as shown in Fig. 13.7(a). The profit or net income $N(x)$ is given by

$$N(x) = B(x) - C(x) \tag{13.13}$$

The total revenue $B(x)$ increases as the output volume x increases up to a point, and may or may not decline as x continues to increase. Without losing generality, let us simplify the discussion by considering only the situation that $B(x)$ increases with x at

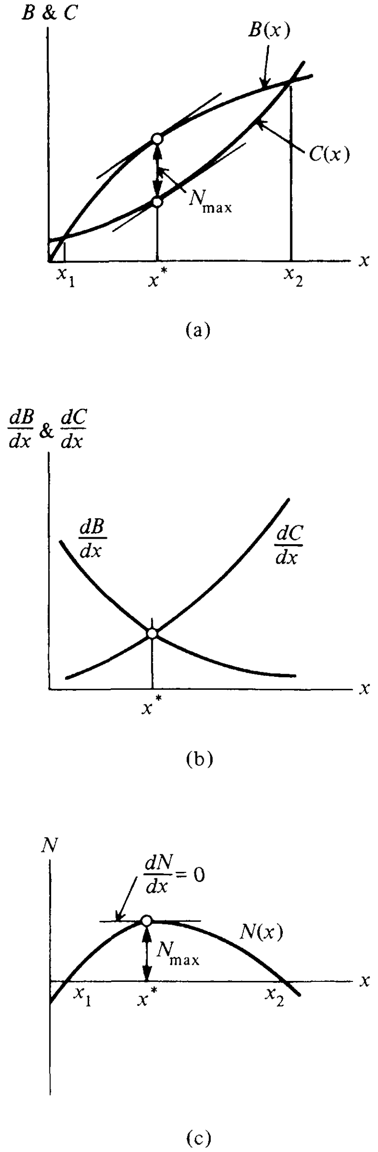


Figure 13.7 Nonlinear optimization model.

progressively lower rates. Then, the slope of $B(x)$ is positive but becomes progressively smaller. On the other hand, the total cost $C(x)$ increases with the output volume x at progressively higher rates, and its slope becomes progressively larger.

The variations in the slopes of $B(x)$ and $C(x)$ as represented by dB/dx and dC/dx, respectively, are shown schematically in Fig. 13.7(b). The derivative dB/dx is called the *marginal revenue* and dC/dx is called the *marginal cost*. The marginal revenue (or cost) represents the revenue (or cost) of producing one more unit of the product *at a specified output volume*. Characteristically, the marginal revenue (or cost) changes at different levels of production since the slope of the revenue (or cost) function changes with increase or decrease of x. The intersection of dB/dx and dC/dx indicates that

$$\left.\frac{dB}{dx}\right|_{x=x^*} = \left.\frac{dC}{dx}\right|_{x=x^*} \tag{13.14}$$

That is, the marginal revenue equals the marginal cost at the output volume x^*. Below x^*, the marginal revenue is greater than the marginal cost, and above x^*, the marginal revenue is less than the marginal cost.

In view of Eq. (13.14), we see that the slopes of $B(x)$ and $C(x)$ at $x = x^*$ in Fig. 13.7(a) are parallel to each other. Then, the value of $N(x)$ is maximized at $x = x^*$, i.e., $N_{\max} = N(x^*)$, since the difference between $B(x)$ and $C(x)$ is at a maximum when the slopes of two tangents are parallel to each other. If we plot the values of $N(x) = B(x) - C(x)$ in Fig. 13.7(a) on a horizontal base as shown in Fig. 13.7(c), we note that the maximum value of $N(x)$ occurs at $x = x^*$ when the slope of the tangent is 0, i.e.,

$$\left.\frac{dN}{dx}\right|_{x=x^*} = 0 \tag{13.15}$$

Note also that the sufficient condition for $N_{\max} = N(x^*)$ is that $d^2N/dx^2 < 0$ at $x = x^*$.

The relation in Eq. (13.15) can be easily verified by differentiating Eq. (13.13) with respect to x. Then,

$$\left.\frac{dN}{dx}\right|_{x=x^*} = \left.\frac{dB}{dx}\right|_{x=x^*} - \left.\frac{dC}{dx}\right|_{x=x^*} = 0 \tag{13.16}$$

Since Eq. (13.16) leads to the same result as Eq. (13.14), we can conclude that the output volume $x = x^*$ which produces a maximum profit $N_{\max} = N(x^*)$ is the same volume at which the marginal revenue equals the marginal cost. Note also that in both parts (a) and (c) of Fig. 13.7, $x = x_1$ and $x = x_2$ are break-even points where the profit $N(x)$ is zero.

Consequently, when the revenue and cost functions for the operation of a facility are nonlinear, we can determine the volume $x = x^*$ that will produce the maximum profit by using either Eq. (13.14) or Eq. (13.15). Then, the maximum profit $N_{\max} = N(x^*)$ can be obtained by substituting the value $x = x^*$ into Eq. (13.13).

For nonlinear optimization models, the average total cost and the marginal cost may be obtained directly from Eqs. (13.2) and (13.3), respectively, once the cost function $C(x)$ is given. If the nonlinear model is a quadratic approximation, the cost function $C(x)$ will be represented by a convex function with no reversal of curvature.

Example 13.5

A bicycle manufacturer is planning the production for next year and has found the cost and revenue functions, respectively, for the output volume x during this period:

$$C(x) = Dx^2 + E$$

$$B(x) = G - \frac{F}{x}$$

where D, E, F, and G are numerical constants. Determine the output volume that maximizes the profit of the operation and find the maximum profit in terms of D, E, F, and G. Specifically, if $D = 10^{-3}$, $E = 10^6$, $F = (16)(10^9)$, and $G = (3)(10^6)$ such that $B(x)$ and $C(x)$ are in dollars, determine the maximum profit, and the marginal revenue and marginal profit at the output volume which produces the maximum profit.

The profit function $N(x)$ for the planned operation is obtained by noting Eq. (13.13). Thus,

$$N(x) = G - \frac{F}{x} - Dx^2 - E$$

Taking the derivative of $N(x)$ and setting it equal to 0 yields

$$\frac{dN}{dx} = \frac{F}{x^2} - 2Dx = 0$$

from which we obtain

$$x^3 = \frac{F}{2D} \quad \text{and} \quad x = \sqrt[3]{\frac{F}{2D}}$$

We substitute this value of x into $N(x)$ to obtain the maximum profit. Thus,

$$N_{\max} = G - F\left(\sqrt[3]{\frac{2D}{F}}\right) - D\left(\sqrt[3]{\frac{F^2}{4D^2}}\right) - E$$

Simplifying, we get

$$N_{\max} = G - E - (1.5)(\sqrt[3]{2DF^2})$$

For the numerical values of D, E, F, and G given for the problem, we obtain the optimal production volume and the maximum profit as follows:

$$x = \sqrt[3]{\frac{(16)(10^9)}{(2)(10^{-3})}} = 20{,}000 \text{ units}$$

$$N_{max} = (3)(10^6) - 10^6 - (1.5)[\sqrt[3]{(2)(10^{-3})(16)^2(10^9)^2}]$$
$$= \$800{,}000$$

The marginal cost and marginal revenue may be obtained from

$$\frac{dC}{dx} = 2Dx \qquad \frac{dB}{dx} = \frac{F}{x^2}$$

at $x = 20{,}000$,

$$\frac{dC}{dx} = (2)(10^{-3})(20{,}000) = \$40/\text{unit}$$

$$\frac{dB}{dx} = \frac{(16)(10^9)}{(20{,}000)^2} = \$40/\text{unit}$$

When the cost function is nonlinear, both the average total cost and the marginal cost vary with the volume x. Thus,

$$\text{AC} = \frac{Dx^2 + E}{x} = \frac{x}{10^3} + \frac{10^6}{x}$$

and

$$\text{MC} = 2Dx = \frac{2x}{10^3}$$

The values of AC and MC at several values of x are shown below:

x	10,000	20,000	30,000	40,000	50,000
AC	\$110	\$70	\$63	\$65	\$ 70
MC	\$ 20	\$40	\$60	\$80	\$100

Note that for this example the value of AC first decreases and then increases with x while the value of MC increases with x. The variation of the average cost results from decreasing unit fixed cost and rising unit variable cost as the volume x increases. In fact, the output volume x that produces a minimum value of AC may be determined by setting the derivative of AC equal to zero, or

$$\frac{d}{dx}(\text{AC}) = \frac{1}{10^3} - \frac{10^6}{x^2} = 0$$

from which $x^2 = 10^9$ or $x = 31{,}622$. Hence, the minimum value of AC is obtained by back substitution, i.e.,

$$(\text{AC})_{min} = \frac{31{,}622}{10^3} + \frac{10^6}{31{,}622} = 31.62 + 31.62 = \$63.24$$

It should be noted also that at $x = 31{,}622$,

$$MC = \frac{(2)(31{,}622)}{10^3} = \$63.24$$

13.8 MINIMIZATION OF TOTAL COST

Since benefits are more difficult to measure than costs in many situations, it is often tempting to minimize the cost over the relevant range of the variable, provided that the benefit is constant in the same range. If both cost and benefit functions are nonlinear, this condition cannot hold. However, we can often modify the problem so that we can determine the desired level of output by minimizing a modified form of the total cost without explicitly determining the total benefit.

Let $C(x)$ and $B(x)$ be the nonlinear cost and benefit functions of the variable x as previously defined. Let $\hat{B}$ be the maximum total benefit (a constant value) that can ever be achieved in the relevant range of the variable x, and let $L(x)$ be the loss function of x which represents the opportunity (or benefit) lost in achieving the ideal maximum total benefit. That is,

$$B(x) = \hat{B} - L(x) \tag{13.17}$$

where $\hat{B}$ is constant but $B(x)$ and $L(x)$ are nonlinear functions. Also let $T(x)$ be defined as the *grand total cost* such that

$$T(x) = C(x) + L(x) \tag{13.18}$$

Then, the net benefit can be expressed as follows:

$$N(x) = B(x) - C(x) = \hat{B} - L(x) - C(x)$$

Hence,

$$N(x) = \hat{B} - [L(x) + C(x)] = \hat{B} - T(x) \tag{13.19}$$

It can be seen from Eq. (13.19) that the level of x that maximizes $N(x)$ also minimizes $T(x)$. That is,

$$\begin{aligned} \max N(x) &= \max\,[\hat{B} - T(x)] \\ &= \hat{B} + \max\,[-T(x)] \end{aligned}$$

or

$$\max N(x) = \hat{B} - \min T(x) \tag{13.20}$$

Thus, we can modify the original problem of maximizing net benefit $N(x)$ to form a new problem of minimizing the grand total cost $T(x)$. If $\hat{B}$ is known, then the maximum value of $N(x)$ can be obtained from Eq. (13.20) after the minimum value of

$T(x)$ is found. However, in many situations, we are only interested in the level of the variable x which maximizes $Nx)$ and not the maximum value of $N(x)$. Then, we can avoid the difficult task of determining $\hat{B}$ and simply try to find the optimal level of the variable x by minimizing $T(x)$. Thus,

$$\min T(x) = \min [C(x) + L(x)] \tag{13.21}$$

The grand total cost functions $T(x)$, the cost function $C(x)$, and the loss function $L(x)$ are shown schematically in Fig. 13.8(a). Note that by differentiating Eq. (13.17), we have

$$\frac{dB}{dx} = -\frac{dL}{dx}$$

Thus, at $x = x^*$ where $T(x^*)$ is a minimum, we have

$$-\frac{dL}{dx} = \frac{dC}{dx} \tag{13.22}$$

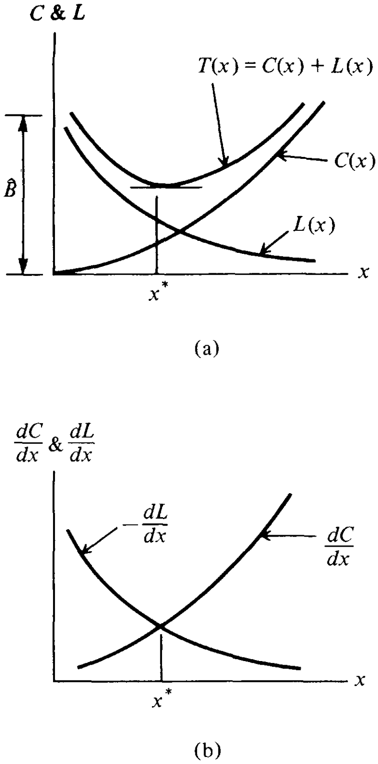


Figure 13.8 Minimization of grand total cost.

as shown in Fig. 13.8(b). Note that dL/dx is negative because the slope of $L(x)$ is negative in the relevant range of x; hence, $(-dL/dx)$ is a positive quantity.

Thus, when a problem is properly formulated to include the "loss" in total benefit which can theoretically be achieved as well as the actual expenses in the total cost, the minimization of the total cost will result in the optimal level of the variable which will also maximize the net benefit or profit.

Example 13.6

The level of detail in cost estimating for economic evaluation may be denoted by the variable x. The cost and benefit functions of cost estimating are, respectively,

$$C(x) = Dx^2 + E$$

$$B(x) = G - \frac{F}{x}$$

where D, E, F, and G are constants. The constant G represents the maximum total benefit that can ever be achieved ideally but cannot easily be determined. Thus, $L(x) = -F/x$ Noting that

$$N(x) = G - \frac{F}{x} - Dx^2 - E$$

find the level of x that minimizes the grand total cost

$$T(x) = E + Dx^2 + \frac{F}{x}$$

We can find the optimal level of x by setting the derivative of $T(x)$ equal to zero. Thus,

$$\frac{dT}{dx} = -\frac{F}{x^2} + 2Dx = 0$$

from which we obtain

$$x^3 = \frac{F}{2D} \quad \text{and} \quad x = \sqrt[3]{\frac{F}{2D}}$$

The same result may be obtained by maximizing $N(x)$.

If we examine the grand total cost $T(x)$ closely, we note that the term E represents the fixed cost, which does not change with x; the term Dx^2 represents the variable cost, which increases with x; and the term F/x represents the loss resulting from the inaccuracy of estimating, which decreases with x. The sum of the first two terms constitutes the amount paid for preparing the cost estimate. The last term is not an amount paid out but rather the loss in benefits that could have been received if the cost estimate were carried to perfection instead of stopping at the level x.

13.9 ALLOCATION OF JOINT COSTS

In some situations, the costs in production are accrued to a joint account and allocated later to different operations according to an equitable basis. The allocation of overhead costs to different operations is a typical example. The allocation of highway operation and maintenance costs to different classes of vehicles whereby the user fees for such vehicles can be charged accordingly is another example.[1]

Ideally, the allocation of joint costs should be causally related to the category of basic costs in an allocation process. In practice, a causal relationship between the allocation factor and the cost item cannot be clearly identified, and the allocation procedure may be quite arbitrary. For example, labor expenses for various operations are often relied upon as a means of allocating overhead burdens in production. As labor expenses become a small portion of expenditures in production due to automation, this basis for allocating overhead burdens introduces distortion and unintended cross subsidies among different operations.[2]

Example 13.7

The cost accounting records of a university indicate that its Office Service Department is allocated an overhead cost of $50,000 per year on the basis of the floor space it occupies. The Office Service Department charges the use of its services for printing on the basis of the direct labor and material costs plus the indirect costs assigned to the job in proportion to its size in order to recover its overhead cost. The Registrar's Office plans to print 5,000 copies of a catalog, and, contrary to the past practice of routinely sending the work to the Office Service Department, it solicits a price quotation first from both a commercial printer and that department. The quotation from the outside printer is $4,500 and that from the Office Service Department is $4,600, with the following breakdowns:

Direct labor cost	$1,850
Direct material cost	750
Overhead cost	2,000
Total	$4,600

The business manager of the university decides that it is in the interest of the university to have the catalog printed in the Office Service Department. Why?

The Office Service Department has been counting on the printing of the catalog for the Registrar's Office as usual in its budget for the year. The overhead cost of $2,000 for printing the catalog is quite substantial (4% of the annual total of $5,000). The department does not expect other sources of revenues from which it can recover this amount of overhead, and it cannot adjust

[1] See Ref. 13.2 for a case study of turnpike toll design.

[2] See Ref. 13.3, especially the example of an actual case on p. 185.

its operation to reduce its share of overhead allocated by the university this year. Consequently, it is in the interest of the university to have the catalog printed internally since the additional cost of $100 charged to the Registrar's Office will be compensated by the increase in overhead recovery of $2,000 by the Office Service Department.

13.10 SUMMARY AND STUDY GUIDE

In this chapter, we discussed the relationships of cost, benefit, and output volume of production to determine the best way to operate the facility in the short run when the capacity and technology of the facility cannot be significantly altered. The basic concepts of production function, cost function, benefit function (or revenue function), and price function were introduced in an intuitive manner.

We introduced both the linear model for break-even analysis and nonlinear models for profit maximization. Examples were given to illustrate the concepts of average cost and marginal cost. We also applied the concept of profit maximization to determine the desired level of output by minimizing the grand total cost without explicitly determining the net benefit.

Finally, we discussed the allocation of joint costs and the causal relationships between the allocation factor and the cost item. The allocation of overhead burdens is cited as an example that could become such a problem.

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PROBLEMS

P13.1 The cost function for a production line of recreational vehicles (in $ million) with respect to output volume (in thousands of vehicles) is given by

$$C(x) = 108 + 27x - 9x^2 + x^3$$

Find the total cost, average cost, and marginal cost for $x = 1, 2, \ldots, 8$. Plot the total cost function and the unit cost function depicting the average cost and marginal cost.

P13.2 Suppose that the cost function for a production line of recreational vehicles (in \$ million) with respect to output volume (in thousands of vehicles) can be represented by a linear approximation as follows:

$$C(x) = 108 + 9x$$

Find the total cost, average cost, and marginal cost for $x = 1, 2, \ldots, 8$.

P13.3 Suppose that the cost function (in \$ million) for a production line of recreational vehicles (in thousands of vehicles) can be represented by a quadratic approximation as follows:

$$C(x) = 216 - 45x + 6x^2$$

Find the total cost, average cost, and marginal cost for $x = 1, 2, \ldots, 8$.

P13.4 Suppose that the cost function for the production of mobile homes (in \$ million) with respect to output volume (in thousand of homes) is given by a linear approximation:

$$C(x) = 90 + 27x$$

Find the total cost, average cost, and marginal cost for $x = 1, 2, \ldots, 6$. Also plot the total cost function and the unit cost function depicting the average cost and the marginal cost.

P13.5 Suppose that the cost function for the production of mobile homes (in \$ million) with respect to output volume (in thousands of homes) is given by a quadratic approximation:

$$C(x) = 150 - 15x + 6x^2$$

Find the total cost, average cost, and marginal cost for $x = 1, 2, \ldots, 6$. Also plot the total cost function and the unit cost function depicting the average cost and the marginal cost.

P13.6 In a construction project involving the excavation of 50,000 m^3 of earth, a mechanical excavator that can excavate 500 m^3 per day was rented at \$100 per day, plus a one-way shipping cost of \$300 to the construction site. After 40 days of operation, the contractor discovers another similar excavator for rent which can also excavate 500 m^3 per day, but will cost only \$90 per day, plus a one-way shipping cost of \$300 to the site. Assuming each excavator will be left on site until it is no longer needed and the return shipping cost must be borne by the contractor, is it cheaper to continue to rent the first excavator or to change to the second excavator?

P13.7 An intercity bus service between two communities requires a fixed cost of \$36,000 per month and can handle a maximum of 2,000 passengers per month. The contribution margin per passenger is 60% of the \$75 ticket price for a one-way trip. What is the number of passengers per month that the service must attract to break even?

P13.8 A subcontractor specializing in foundation excavation has estimated the cost and revenue functions with respect to the volume of excavation work (in units of 1,000 yd^3) for the next year as follows:

$$C(x) = Dx + E$$

$$B(x) = G - \frac{F}{x}$$

where $D = 500$, $E = 20$, $F = 200{,}000$, and $G = 100{,}000$ such that both $C(x)$ and $B(x)$ are expressed in dollars. Determine the output volume x that will maximize the profit.

P13.9 For Problem P13.8 determine the marginal cost function and the marginal revenue functions, and find their values at the output volume for maximum profit.

P13.10 The building energy performance (BEP) can be expressed in an index x (which is inversely proportional to annual energy consumption in millions of BTU). Both the amortized uniform annual cost C for energy conservation measures and the annual cost for net energy consumption E are functions of the index x. Suppose that the annual energy conservation cost C (in dollars) for a building is expressed as

$$C = 50(x^2 + 1)$$

where x has a range of values from 1 to 10, and the annual net energy consumption cost E (in dollars) for that building is expressed as

$$E = -200 + \frac{21{,}600}{x}$$

The total equivalent uniform annual cost T to the owner of the building is the sum of E and C. Determine the index x that will minimize this total cost. Also plot the functions E, C, and T at $x = 1, 2, \ldots, 10$.

P13.11 For Problem P13.10, determine the functions for the marginal energy conservation cost and the marginal net energy consumption cost. If $B = \hat{B} - E$ where B is the benefit of energy conservation measures and $\hat{B}$ is a constant representing the theoretical maximum benefit that can be achieved when there is no net energy consumption, plot the values of the marginal benefit of energy conservation and those of the marginal cost of net energy consumption at $x = 1, 2, \ldots, 10$.

P13.12 Charles Manufacturing Company distributes overhead costs according to the number of employees in each department. Within a department the overhead is allocated according to direct labor-hours. According to the incomplete records, the monthly data for two departments A and B are as follows:

	Dept. A	Dept. B	Total A & B
Direct labor-hours	750	1200	1950
No. of employees	5	7	12
Overhead costs			\$18,000

The company has just received a job order that requires a total of \$800 for direct material, and 60 hours and 100 hours of direct labor from departments A and B, respectively. If the costs for direct labor in departments A and B are \$10 per hour and \$8 per hour, respectively, determine the total cost for the job order including direct material, direct labor, and overhead costs.

CHAPTER

14 Financial Statements

14.1 THE IMPORTANCE OF FINANCIAL REPORTS OF A FIRM

The decision to undertake any capital investment project cannot be made on the basis of the profitability of the project alone but must be considered in the context of its impact on the financial strength and position of the firm that is proposing to undertake the project. A firm that is currently in a strong financial position will be more willing and able to undertake projects than the same firm would be when it is weaker. The purpose of this chapter is to provide an overview of the major determinants of the financial position of a firm as reflected periodically in its financial statements.

The financial statements are contained in the annual report of a corporation, which includes the *balance sheet*, the *income statement*, the *statement of changes in financial position*, and the *auditors' report*. These statements are usually preceded in the annual report by a brief description of the corporation and its operations. The balance sheet summarizes the financial position of the corporation and lists the values of its assets and financial obligations or liabilities. The income statement itemizes revenues and expenses for the year and provides an overview of the operations for the year. The statement of changes in financial position lists the sources and application of funds. The auditors' report is an independent appraisal of the financial statements of the corporation by a team of professional accountants.

Financial statements are prepared on an annual basis because the year is a standard unit of time and because the year is a good period of time for evaluating the activities of a firm. It is long enough to provide a meaningful measure of the operations, but short enough so that statements do not become outdated. It includes all four seasons and thus eliminates the effect of trends that are purely seasonal. It normally includes several cycles of production and sale of goods (except for a handful of industries such as the tobacco industry where the growing period is longer than a year). Publicly held corporations are also required to prepare financial statements on a quarterly basis, and all corporations must prepare financial statements when they wish to raise capital by either borrowing money or selling stock.

14.2 BALANCE SHEETS

The balance sheet is a statement of the financial position of a firm. It balances the assets and the claims against those assets of a firm. The assets and claims are, by definition, equal. Claims against those assets include both liability and stockholders' equity. According to convention, the balance sheet has a standard form which lists

TABLE 14.1 BALANCE SHEET G. FITZGERALD CORPORATION

December 31	1990	1989
ASSETS		
Current assets		
Cash	$ 100,000	$ 90,000
Marketable securities	50,000	50,000
Accounts receivable	150,000	115,000
Inventories	290,000	250,000
Prepaid expenses	10,000	20,000
Total current assets	$ 600,000	$525,000
Fixed assets		
Land and buildings	$ 350,000	$350,000
Machinery and equipment	200,000	180,000
Total cost of fixed assets	550,000	530,000
Less accumulated depreciation	150,000	110,000
Total fixed assets net of depreciation	400,000	420,000
TOTAL ASSETS	$1,000,000	$945,000

assets on the left side and *liabilities and stockholders' equity* on the right side. Other formats are sometimes used but they are less common. Usually, a balance sheet presenting the information for the year just ended and the previous year is given in the annual report for the purpose of comparison. An example is shown in Table 14.1.

Assets are listed on the balance sheet in descending order of liquidity. At the top of the list, under *current assets,* is *cash,* which represents the ultimate degree of liquidity because it can be used to acquire assets or pay off debts. The term *cash* refers to monies held on company premises and in banks. The next item after cash is usually *marketable securities,* which include such items as stocks, bonds, and Treasury bills, which are held to earn interest but may be sold at any time to raise the funds needed for operations. Next is *accounts receivable,* which are amounts owed by customers on credit sales, but not yet paid. Since cash is soon collected on accounts receivable, they are considered a near-cash item.

TABLE 14.1 (CONTINUED)

December 31	1990	1989
LIABILITIES AND STOCKHOLDERS' EQUITY		
Current liabilities		
Accounts payable	$ 125,000	$110,000
Salaries payable	20,000	20,000
Taxes payable	20,000	15,000
Dividends payable	10,000	7,500
Unsecured short-term notes payable	10,000	5,000
Mortgage notes payable, current maturities	20,000	20,000
Interest payable	15,000	12,500
Total current liabilities	$ 220,000	$190,000
Long-term liabilities		
Long-term mortgage notes payable, net of current maturities	$ 180,000	$220,000
Stockholders' equity		
Preferred stock, 10%, $1 par value 50,000 shares issued and outstanding	$ 50,000	$ 50,000
Common stock, $2 par value, 200,000 shares authorized; 100,000 shares issued and outstanding in 1990 and 1989	200,000	200,000
Additional paid-in capital	25,000	25,000
Retained earnings	325,000	260,000
Total stockholders' equity	$ 600,000	$535,000
TOTAL LIABILITIES AND EQUITY	$1,000,000	$945,000

Inventory refers to goods produced or purchased and held for sale or used in the production of services. It does not represent a definite commitment that will generate cash and income for the firm; nevertheless, it is held with the expectation that it will be sold within the year. Revenue is accrued when an order is placed by a customer for inventory, and the order is charged to accounts receivable. When the cash is collected from the accounts receivable, it is either used to purchase more inventory or set aside for other investment purposes. Thus, the operating cycle begins with the production or purchase of inventory, continues with the sale of inventory and the accumulation of accounts receivable, and is completed with the liquidation of these accounts for cash.

Other items commonly listed as current assets on the balance sheet may include *prepaid expenses,* such as rent and insurance. Prepaid expenses are payments made in advance to satisfy certain necessary expenses. Such payments are sometimes made several years in advance, but if the portion beyond the current year may be reclaimed, that portion is included as a current asset.

Noncurrent assets consist largely of *fixed assets* such as land, buildings, machinery, and equipment. Sometimes these items are included under the one heading, but usually one or more of these items are listed separately. Other noncurrent assets include the value of such *intangible items* as *patents, trademarks,* and *goodwill.* Noncurrent assets are used by the firm in the production of goods and services and are expected to generate revenues on a long-term basis.

The *market value* of most assets fluctuates greatly over a period of time. The values of all assets except cash on the balance sheet are only *estimates* of the true value of the assets. Assets are subject to declines in value due to risk, use, technological obsolescence, and other factors. When it occurs, the decline in value for an asset is estimated and subtracted from the original value to arrive at a revised lower estimate.

On the same balance sheet, the asset may be shown at its original value, minus an allowance for the decline in value in an offsetting account called a *contra account*. Thus, *accounts receivable* will be offset by an account called *allowance for doubtful accounts* (estimated uncollectable receivables). The *original value of machinery and equipment* will be offset by an account for the *depreciation accumulated to date*. The offsetting account for intangible assets such as patents is called *allowance for amortization.*

The examination of the contra accounts, if they exist, can give an indication of the firm's financial position. Allowance for doubtful accounts should be only a small percentage of the accounts receivable, and if this figure exceeds 5%, there is cause for concern that the firm is making a substantial proportion of credit sales to customers with questionable credit ratings. Accumulated depreciation can range anywhere from 0% on equipment just put into service to 100% on equipment that has been in service longer than its estimated useful life. If the accumulated depreciation of the equipment is a high proportion of the original value, it may be a sign that the firm is using obsolete equipment. A high percentage of amortization of the original value of patents may also indicate that the patent is not as valuable as it once was, due to its possible obsolescence or the development of similar processes by competitors.

Inventory poses a special problem in valuation. It can have any one of several stated values: original cost to purchase or produce, present cost to purchase or produce, or resale value at market retail prices. The most conservative practice is to record the inventory at the original cost of acquisition, except that when the current retail market value is less than the original cost, the inventory is revalued at the market price. The loss in the revaluation is reflected on the income statement, not the balance sheet, and is referred to as valuing inventory at the *lower of cost or market*.

Liabilities are the debts incurred to acquire assets and therefore are obligations for definite amounts of money. Like assets, liabilities are classified as current or noncurrent. Liabilities are listed in descending order of priority or urgency on the balance sheet.

At the top of the *current liabilities* are obligations that are due within a few days or weeks. *Accounts payable* are the debts that the firm owes to suppliers of goods and services, and are usually due within 30 days. *Wages and salaries payable* are amounts due to the employees of the firm that have been earned during the pay period of 2 weeks or a month. *Taxes payable* are taxes that are due to federal, state, and local governments, with the taxes being due at the end of each quarter. *Dividends payable* are dividends that have been declared, but not yet paid. *Short-term debts* to banks and other lending institutions are, by definition, current liabilities. The *current portion of long-term debt* refers to the principal portion of the long-term debt that is due in the upcoming year. Closely related to debt liability is *interest payable* which is accrued on a current basis.

Long-term liabilities include notes payable and bonds payable. *Notes payable* are written contracts for the incurrence and repayments of indebtedness, signed by both borrower and lender. This contrasts with accounts payable which is an informal debt arrangement evidenced only by a seller's invoice. Notes may either be unsecured or secured by collateral such as inventory or accounts receivable. *Mortgage notes* are a special kind of notes payable that is secured by real property, i.e., the fixed assets of the firm. A *bond* is a long-term debt instrument that commits the firm to make periodic interest payments and to repay the principal sum stated on the face of the instrument at the maturity date. Schedules of the terms of long-term debt, disclosing the interest rate and the maturity dates, are included in the financial statement. Such information indicates whether the company is in a comfortable financial situation, with low interest rates and dispersed maturity dates of liabilities, or is in an upcoming financial bind, with high interest rates and a large amount of shortly maturing liability. Under the long-term liabilities section of the balance sheet, the value of notes payable is stated "less current portion," i.e., less the part that is listed under current liabilities.

The *stockholders' equity* is the portion of capital that belongs to the investors. There are several classes of stock which entitle their holders to varying degrees of rights regarding income and participation in the affairs of management. There are three major types of stock: preferred stock, common stock, and treasury stock. Preferred stock is a privileged class of stock that gives its holders the right to annual dividends at a stated rate and to receive such payments before other classes of shareholders if the management declares dividends. Preferred stock does not give its holders the right to receive dividends above this rate unless it is *participating*, and it is not normally accompanied by the right to vote for the members of the board of directors. Common stock is the general class of stock that entitles its holders to receive dividends and to vote for directors and on certain issues at the rate of one vote per share of stock. Treasury stock is stock repurchased by the firm and removed from the market.

Under stockholders' equity, the number of shares of each class of stocks authorized and/or issued as well as the *par value* for each share are listed separately. The par values for each share of preferred stock and common stock are values designated arbitrarily by the management, including the option of not designating any par value. The total value of the stocks issued in excess of par value is recorded in an account called *additional paid-in capital*. Retained earnings is the sum of all net income, less dividends of the corporation, that has been retained or plowed back into the business since its beginning.

The stockholders' equity section is important both in analyzing the structure of the firm and in providing investment opportunities. This section poses some of the most critical and complex problems in finance, but we are only concerned with stockholders' equity in terms of the structure of ownership and therefore will not discuss these problems here in detail.

Example 14.1

Verify the contra account for accumulated depreciation (under fixed assets) of the G. Fitzgerald Corporation from the following data:

	Land & Buildings	Machinery & Equipment
Costs of items purchased before 1990	$350,000	$180,000
Costs of addition in 1990	0	20,000
Accumulated depreciation before 1990	75,000	35,000
Depreciation allowance in 1990	25,000	15,000

For the given data, we can compute the contra accounts for accumulated depreciation as follows:

	1990	1989
(1) Land and buildings		
Costs of acquisition	$350,000	$350,000
Less accumulated depreciation	100,000	75,000
Amount net of depreciation	250,000	275,000

	1990	1989
(2) Machinery and equipment		
Costs of acquisition	200,000	180,000
Less accumulated depreciation	50,000	35,000
Amount net of depreciation	150,000	145,000
(3) Total fixed assets, (1) + (2)		
Costs of acquisition	550,000	530,000
Less accumulated depreciation	150,000	110,000
Total net of depreciation	$400,000	$420,000

Example 14.2

The G. Fitzgerald Corporation issued 100,000 shares of common stock with a par value of $2 per share. The total market value of these shares is $225,000. How would this total value be recorded on the balance sheet?

Of this total amount, $200,000 would be recorded under common stock (100,000 shares outstanding at $2 par value per share), and $25,000 would be credited to additional paid-in capital under the stockholders' equity on the balance sheet.

14.3 INCOME STATEMENTS

The income statement lists the revenues and expenses during the year, together with the net income and retained earnings at the end of the year. It is sometimes referred to as the statement of income and reinvested earnings. The income statement usually includes the information for the year just ended and the previous year for the purpose of comparison. An example is shown in Table 14.2.

At the top of the income statement are *revenues from sales and other operations*. Production and other property expenses generally include the *cost of goods sold* and *cost of selling*. The cost of goods sold refers to the costs incurred in the production, including materials, wages of workers, and certain overhead items. The cost of selling covers the cost of advertising and the remuneration of sales persons. The *general and administrative expenses* include the salaries of managers and the administrative personnel who are not directly engaged in production, and other items of expenditure not directly related to production. Although the depreciation allowance is a deduction and not a "charge" because it does not involve an outlay of cash, it is usually considered an operating expense. Hence, the amount of depreciation allowance is a part of the costs and expenses which are deducted from revenues of the current year to arrive at net income. Other deductions include *interest* paid, but income taxes are usually listed separately.

Extraordinary items are financial gains and losses due to unusual and nonrecurring events such as mergers or natural disasters and are usually shown separately to emphasize the unusual nature of the transaction. After all income before taxes is computed, the applicable tax rate is used to compute the tax expense, which is de-

TABLE 14.2 INCOME STATEMENT G. FITZGERALD CORPORATION

For years ended December 31	1990	1989
REVENUES		
Sales and other operating revenues	$1,100,000	$940,000
COSTS AND EXPENSES		
Production and operating expenses	600,000	545,000
General and administrative expenses	235,000	215,000
Depreciation	40,000	30,000
Total	875,000	790,000
REVENUES LESS COSTS AND EXPENSES	225,000	150,000
OTHER DEDUCTIONS		
Interest expense, net of interest income	15,000	10,000
INCOME BEFORE TAXES	$ 210,000	$140,000
TAXES ON INCOME	105,000	75,000
NET INCOME	$ 105,000	$ 65,000
CASH DIVIDENDS		
Preferred stock (per share, $0.10)	$ 5,000	$ 5,000
Common stock (per share, 1990, $0.35; 1989, $0.25)	35,000	25,000
	40,000	30,000
RETAINED EARNINGS		
Beginning of year	260,000	225,000
Current year	65,000	35,000
End of year	$ 325,000	$260,000
EARNINGS PER SHARE OF COMMON STOCK	$1.00	$0.60

ducted from before-tax income to arrive at after-tax income. The after-tax income is referred to as *net income* or *net earnings*.

Cash dividends declared for various classes of stock are listed next. They are deducted from the net income to obtain the retained earnings for the current year, which in turn are added to the retained earnings at the beginning of the year to arrive at the retained earnings at the end of the year.

At the bottom of the income statement is the required item of *earnings per share*. It is computed by first finding the difference of net income after taxes and preferred dividends, which is then divided by the number of shares of common stock or *common stock equivalent*. This is a figure of great interest to investors with holdings of stock because it is one measure of the value of their stock. The computation of equivalent shares of common stock is a very complicated one for some firms because of the number and types of outstanding convertible securities. It is sufficient to point out that there are two types of figures for earnings per share that are usually presented: primary earnings per share, which assumes that only certain types of securities would be converted into common stock, and a lower figure for fully diluted earnings per share, which assumes that all convertible securities will be converted. The fully diluted figure is lower because of the larger number in the denominator representing more equivalent shares of common stock.

There is a close connection between the items listed on the balance sheet and the transactions summarized in the income statement. This is due to the fact that the balance sheet consists of the assets used to produce income, and the income statement reflects the addition to the firm's capital, and eventually its assets, from net earnings generated and retained each year. For this reason, the two statements are said to *articulate* with each other.

Income statements in modified forms are also presented for the previous 5 or 10 years in the annual report of a firm. They often represent a financial and statistical summary emphasizing special features such as the revenue, income, and asset composition of the firm. Earnings per share data are also presented on a year-to-year basis. The historical presentation of income statements shows the trend of revenues and income over a period of time. It allows prospective investors to see the growth (or decline) of a firm, and assists them in forming an opinion about the firm's future prospects.

Example 14.3

Find the earnings per share of the common stock of the G. Fitzgerald Corporation in 1990 from the data in Tables 14.1 and 14.2.

Since the net income after taxes is $105,000 and the cash dividends paid to preferred stockholders are $5,000 in 1990 (in Table 14.2), the difference is $100,000. The number of outstanding shares of common stock is 100,000 in 1990 (in Table 14.1). Hence, the earnings per share would be $1.00.

14.4 STATEMENTS OF CHANGES IN FINANCIAL POSITION

The sources of revenues and expenses are often indicators of the strengths and weaknesses of the operations of a firm. In addition, the inflow and outflow of income usually have a differential effect on the various balance sheet accounts of the firm. These effects are summarized in the *statement of changes in financial position*. This statement details how the firm acquired funds and how the funds were used. In the words of the Accounting Principles Board, such a statement "summarizes the financing and investing activities of the firm."[1] Because it makes the analysis of the flow of funds so much more convenient, as well as providing greater insight into the results of the firm's operations, the inclusion of the statement

[1] American Institute of Certified Public Accountants, Accounting Principles Board (APB) Opinion No. 19, "Reporting Changes in Financial Position." New York: AICPA, 1971.

of changes in financial position in a firm's financial report is required. An example of such a statement is shown in Table 14.3.

The term *funds* has been variously defined as cash, monetary assets such as cash plus marketable securities, and working capital. *Working capital* refers to the difference between current assets and current liabilities. It is the most generally accepted definition for the term funds in the statements of large corporations, because it is a measure of liquidity. However, it has the disadvantage of including certain relatively nonliquid items such as inventory. For *small firms* that do business on a

TABLE 14.3 STATEMENT OF CHANGES IN FINANCIAL POSITION
G. FITZGERALD CORPORATION

For years ending December 31	1990	1989
SOURCES OF FUNDS		
Operations		
Net Income	$105,000	$ 65,000
Depreciation	40,000	30,000
Total from operations	145,000	95,000
Additions to long-term debt	0	35,000
Total sources of funds	145,000	130,000
USES OF FUNDS		
Cash dividends	40,000	30,000
Additions to land, buildings, and equipment	20,000	15,000
Reduction of long-term debt		
Total application of funds	100,000	80,000
INCREASE (DECREASE) IN WORKING CAPITAL	$ 45,000	$ 50,000
INCREASE (DECREASE) IN COMPONENTS OF WORKING CAPITAL		
Cash	$ 10,000	$ 20,000
Marketable securities	0	(5,000)
Accounts receivable	35,000	30,000
Inventories	40,000	15,000
Prepaid expenses	(10,000)	0
Accounts payable	(15,000)	(10,000)
Salaries payable	0	5,000
Taxes payable	(5,000)	(5,000)
Dividends payable	(2,500)	(5,000)
Unsecured short-term notes payable	(5,000)	2,500
Mortgage notes payable, current maturities	0	0
Interest payable	(2,500)	2,500
Increase (decrease) in working capital	45,000	50,000
Working capital, beginning of year	335,000	285,000
Working capital, end of year	$380,000	$335,000

cash rather than an accrual basis, *cash* may be a better definition of funds than working capital.

The statement of changes in financial position is divided into two parts. The upper portion of the statement lists the sources and uses of funds. If the conventional definition (working capital) is used to denote funds, then the change of working capital refers to funds added to or subtracted from working capital which arise from transactions involving nonworking capital items.

Specifically, *sources of funds* include net income, depreciation or amortization, decrease in long-term assets through sales, increase in long-term liabilities by incurring new debt, and increase in equity through issuance of new stock. *Uses of funds* refer to cash dividends, increase in long-term assets through acquisition, decrease in long-term liabilities through redemption of debt, and decrease in equity through repurchase of stock. The difference between the total of funds from all sources and the total uses of funds is the change (increase or decrease) in working capital.

In the second part of the statement, the changes in the individual working capital accounts, such as cash, inventory, or accounts payable, are itemized. The changes in the various accounts are then added (with increases and decreases in liabilities inversely related to increases and decreases in assets). The sum of all working capital accounts equals the changes in working capital and should match the figure arrived at by taking the difference between sources and uses of funds.

The statement of changes in financial position in Table 14.3 is compiled from information on the balance sheets, the income statement and the pertinent contra accounts concerning the acquisition and sales of long-term assets, the issuance and redemption of long-term debt, and the issuance and repurchase of stock. In the upper portion of Table 14.3, the category "operations" is an accounting term which covers both net income and depreciation. These items together with cash dividends can be found from the income statement in Table 14.2. Without the contra accounts, we cannot know the exact transactions regarding assets, debt or equity during the period. From the balance sheet in Table 14.1, we can only derive their net increases or decreases but they are not sufficient to complete the first part of Table 14.3. The second part of Table 14.3 can be obtained from the balance sheet in Table 14.1.

In analyzing the sources and uses of funds, as shown in the first part of Table 14.3, we adopt the following notation:

S_x = a category of m sources of funds ($x = 1, 2, \ldots, m$)
U_y = a category of n uses of funds ($y = 1, 2, \ldots, n$)
S = total of funds from all sources
U = total uses of funds
ΔW = change in working capital, with a positive value indicating an increase and a negative value a decrease.

Then,

$$S = \sum_{x=1}^{m} S_x \tag{14.1}$$

$$U = \sum_{y=1}^{n} U_y \tag{14.2}$$

$$\Delta W = S - U \tag{14.3}$$

The change in components of working capital, as shown in the second part of Table 14.3, can be analyzed by considering the components of current assets and liabilities. We adopt the following notation:

$\Delta W'$ = change in working capital, with a positive value indicating an increase and a negative value indicating a decrease
$\Delta A'_x$ = increase (positive) or decrease (negative) in each current asset account x, such as cash, accounts receivable, and inventory
$\Delta L'_y$ = decrease (positive) or increase (negative) in each current liability account y, such as accounts payable, salaries, and taxes payable
$\Delta A'$ = total change in current assets (positive for an increase and negative for a decrease)
$\Delta L'$ = total change in current liabilities (positive for a decrease and negative for an increase)

Thus, the change in current assets represents the value of all the current asset accounts ($x = 1, 2, \ldots, m$):

$$\Delta A' = \sum_{x=1}^{m} \Delta A'_x \tag{14.4}$$

and the change in current liabilities represents the value of all the current liability accounts ($y = 1, 2, \ldots, n$):

$$\Delta L' = \sum_{y=1}^{n} \Delta L'_y \tag{14.5}$$

Hence, the change in working capital equals the sum of the change in current assets and the change in current liabilities:

$$\Delta W' = \Delta A' + \Delta L' \tag{14.6}$$

Since the first and second parts of the statement of changes in financial position must show the same increase or decrease in working capital, it is obvious that

$$\Delta W = \Delta W' \tag{14.7}$$

Thus, the statement provides an insight into the operations of the firm by including the results of operations under sources of funds. The first and most obvious source of funds is net income. Then, items such as depreciation and amortization are added back to net income to arrive at working capital generated from operations. This is because depreciation and amortization are *accounting* deductions that reduce gross income but do not involve an outflow of working capital. Hence, they must be added back to income to arrive at working capital generated from operations. Then, other sources of funds caused by changes in balance sheet items are added to funds provided by operations to arrive at total sources of funds. Uses of funds represent investments made by the firm in the current year to benefit future years and are therefore subtracted from sources of funds to arrive at the changes in working capital.

The strength of the financial position of a firm cannot be judged by the increase or decrease of working capital alone. A decrease in working capital may reflect the use of available resources to fund a major expansion of capital investment. On the other hand, improved management of inventory, receivables, and payables may contribute to the increase of working capital. Consequently, an evaluation of the changes in the financial position of the firm must include a detailed analysis of the inflow and outflow of funds and an itemization of changes in components of working capital in comparison to the previous year. It is imperative that the statement of changes in financial position be viewed within the larger context of the firm's overall operations. However, a steady decrease in working capital year after year is an indication that the firm may be in financial trouble, and a steady increase in working capital accompanying the growth of the firm as a whole is a good sign.

Example 14.4

Find the working capital of the G. Fitzgerald Corporation for 1990 by obtaining sources and uses of funds on the basis of the information in Tables 14.1 and 14.2. Without additional information from the contra accounts, assume any net increase or decrease in asset, debt, or equity in Table 14.1 as an actual transaction in that category against a base of zero. Also find the increases and decreases in the components of working capital for 1990.

From the information in Tables 14.1 and 14.2, various sources of funds can be obtained as follows:

1. Net income: $S_1 = 105{,}000$
2. Depreciation: $S_2 = 40{,}000$
3. Decrease in assets: $S_3 = 0$ (zero base for increase)
4. Increase in debt: $S_4 = 0$ (zero base for decrease)
5. Increase in equity: $S_5 = 0$ (zero base for decrease)

Various uses of funds are:

1. Cash dividends: $U_1 = 40{,}000$
2. Increase in assets: $U_2 = 550{,}000 - 53{,}000 = 20{,}000$
3. Decrease in debt: $U_3 = 220{,}000 - 180{,}000 = 40{,}000$
4. Decrease in equity $U_4 = 0$

Hence, the total of all sources and that of all uses are obtained from Eqs. (14.1) and (14.2), respectively:

$$S = 105{,}000 + 40{,}000 = 145{,}000$$

$$U = 40{,}000 + 20{,}000 + 40{,}000 = 100{,}000$$

Then, from Eq. (14.3)

$$\Delta W = 145{,}000 - 100{,}000 = 45{,}000$$

From the list of current assets in Table 14.1, we can compute the increase or decrease in each component of working capital in 1990 in comparison with the corresponding component in 1989:

1. Cash: $\Delta A'_1 = 100{,}000 - 90{,}000 = 10{,}000$
2. Marketable securities: $\Delta A'_2 = 50{,}000 - 50{,}000 = 0$
3. Accounts receivable: $\Delta A'_3 = 150{,}000 - 115{,}000 = 35{,}000$
4. Inventories: $\Delta A'_4 = 290{,}000 - 250{,}000 = 40{,}000$
5. Prepaid expenses: $\Delta A'_5 = 10{,}000 - 20{,}000 = -10{,}000$

From the list of current liabilities in Table 14.1, we can do likewise:

1. Accounts payable: $\Delta L'_1 = -(125{,}000 - 110{,}000) = -15{,}000$
2. Salaries payable: $\Delta L'_2 = -(20{,}000 - 20{,}000) = 0$
3. Taxes payable: $\Delta L'_3 = -(20{,}000 - 15{,}000) = -5{,}000$
4. Dividends payable: $\Delta L'_4 = -(10{,}000 - 7{,}500) = -2{,}500$
5. Notes payable: $\Delta L'_5 = -(10{,}000 - 5{,}000) = -5{,}000$
6. Mortgage payable: $\Delta L'_6 = -(20{,}000 - 20{,}000) = 0$
7. Interest payable: $\Delta L'_7 = -(15{,}000 - 12{,}500) = -2{,}500$

Consequently, the total changes in current assets and current liabilities are obtained from Eqs. (14.4) and (14.5), respectively, as follows:

$$\Delta A' = 10{,}000 + 35{,}000 + 40{,}000 - 10{,}000 = 75{,}000$$

$$\Delta L' = -15{,}000 - 5{,}000 - 2{,}500 - 5{,}000 - 2{,}500 = -30{,}000$$

Then, from Eq. (14.6),

$$\Delta W' = 75{,}000 - 30{,}000 = 45{,}000$$

As a final check, we can compare the total working capital in the current year (1990) with that in the previous year (1989), as shown in Table 14.3. Hence,

$$\Delta W = 380{,}000 - 335{,}000 = 45{,}000$$

14.5 ANALYSIS OF THE FINANCIAL POSITION OF A FIRM

The analysis of financial statements has so far been limited to a line-by-line examination of all the categories under a firm's balance sheet and income statement, and the possible implications of certain items. However, the figures shown on the financial statements do not stand alone; they are meaningful only in the context of their relationships to the other figures in the financial statements. A handful of key relationships can tell a great deal about the firm's financial position. A much more effective analysis can be accomplished by examining these relationships.

The financial areas of a firm that concern a prospective investor the most are liquidity, solvency, and profitability. *Liquidity* is the ability of a firm to raise enough cash to pay its liabilities as they become due. *Solvency* refers to the long-term ability of a firm to meet its obligations, based on the structure of its debt in relationship to its assets. *Profitability* is the ability of the firm to generate profits. Managers, investors, and lenders watch these areas closely to make sure that the firm is able both to stay afloat and to provide a return on investment.

Liquidity can be determined from the relationship between current assets and current liabilities. One measure of liquidity is the difference between the two, which is the working capital. However, this measure is often misleading because a firm with $200,000 in current assets and $100,000 in current liabilities and another firm with $1,100,000 in current assets and $1,000,000 in current liabilities have the same amount of working capital, i.e., $100,000, but it is intuitively obvious that the first firm is more liquid percentage-wise.

A better measure of liquidity than net working capital is the *current ratio,* which is defined as the ratio between current assets and current liabilities. In the above example, the current ratio for the first firm is 2, while the current ratio for the second firm is only 1.1. Generally, the current ratio should range between 2 and 3. If the current ratio is much below 2, the firm may have some trouble generating enough cash in the short term to meet its short-term obligations, because some of its accounts receivable and inventories included under current assets may not be converted into cash quickly. If the current ratio is much beyond 3, the firm is maintaining overly large balances of liquid assets relative to its current needs. Inventories may be sitting idle and not earning a profit while storage costs mount, or the payments on accounts receivable may be slow, increasing both the lost interest and the risk of default. If it is substantial in amount, excess cash should be invested in more fixed assets to generate more income.

The liquidity of certain current asset items leads to another test of liquidity. Cash or near-cash assets, such as marketable securities and accounts receivable, are called *quick assets*. In general, current assets minus inventories represent quick assets which can be mobilized quickly enough to pay off current liabilities. The ratio of quick assets to current liabilities is called the quick ratio or *acid-test ratio*. For this reason, the quick ratio should not be less than 1 and in fact, should be slightly higher. A quick ratio of greater than 2, however, may indicate the presence of unused funds.

Closely related to the problem of liquidity is solvency, the ability to pay off long-term debt commitments. The two measures of solvency are the ability to meet interest (and principal) payments, and the proportion of debt financing of the firm. The income statement gives an indication of the ability to pay interest, while the balance sheet is the indicator of the proportion of debt.

The ratio of income *before* taxes and interest deduction to interest expense is generally taken as a sign of "ability to pay." Normally, a firm with an income-to-interest ratio of less than 2 is in financial trouble, because it has little money left over after making its current debt repayments. Not only is the ability to pay dividends on equity or purchase new assets impaired, but the firm will get into an even tighter situation if its income should decline further or if it needs to borrow even more money. A more comfortable income-to-interest ratio is 5. A much higher income-to-interest ratio may be an indication of the ability to expand by borrowing more money.

The ratio of debt to stockholders' equity is another measure of ability to absorb more borrowed capital. Lenders are very risk conscious and do not look favorably on a firm that has an unduly high proportion of debt and does not have a sizeable amount of equity capital. Since lenders have first priority over stockholders for repayment of debt in the event of a failure and can in fact appropriate invested capital (or assets purchased with invested capital) in repayment of the debt, invested capital is a financial "cushion" for lenders. American bankers and bond purchasers normally do not like to see a debt-to-equity ratio of much more than 1 when a firm "owes what it owns." A higher ratio will make it difficult to borrow more money, but borrowing money is usually easy with a debt equity of less than 1:1. However, banks in certain foreign countries, notably Japan, where loans are guaranteed by the government, are willing to finance higher debt-to-equity ratios.

The measures of profitability of an investment may be based on two criteria: capital gains through increases in the price of the stock or through dividend payments. The profitability of a firm will affect both the desirability of the stock (and hence its price) and the ability of the firm to pay out dividends. Naturally, it is in the interest of the firm to maximize earnings, but the question of whether to attract capital by paying out earnings in the form of dividends or by reinvesting the earnings in the company is often a tricky one.

The attractiveness of stock to an investor is based on both the profitability of the firm as a whole and the profitability to the stockholders. Sometimes a firm that is profitably operated will not be particularly attractive to a stockholder due to an unfavorable equity structure, such as a large amount of preferred stock, or a large number of shares already issued and outstanding; and sometimes a firm that is only moderately profitable may be attractive to stockholders because of the returns paid to stockholders. Therefore, measures of both the operating profitability of a firm and the return on stockholders' equity will be discussed.

The two most important measures of operating profitability are return on sales and return on assets, expressed in percentages. *Return on sales* is defined as the ratio of income after taxes to sales and other operating revenues. It varies from industry to industry and may range from as low as 2% in certain high-volume retail industries to 15% or more on low-volume, high-quality goods. The only way to judge a firm by return on sales is to compare it with other firms in the industry. A year-to-year comparison of the same firm's profitability will also show whether its financial position is improving or declining.

Return on assets, otherwise known as *return on investment,* is another widely used method of measuring profitability. It is defined as the ratio of income after taxes to total assets. Its advantage is its simplicity, because a certain rule of thumb may be used to gauge the profitability. For example, some business people consider a return on assets of 5% or less as poor, 10% as fair, and 15% or more as outstanding. Its disadvantage as a measure is that it does not take into account the asset structure of the firm. A firm that is currently building up its assets, especially fixed assets, may show a low return on assets temporarily but may have good long-term prospects. In a time of high inflation, however, a firm with a large proportion of fixed assets acquired at times of lower prices and/or highly depreciated assets may show high rates of return on assets based on high current earnings in inflated dollars generated by older or even obsolete assets.

Stockholders, however, are even more interested in what company earnings mean to them. The most important measure of return to common stockholders is the *earnings per share of common stock.* The earnings accrued to common stock refers to the net income (after tax) minus the dividends paid to the holders of preferred stock. Thus, a firm with a low return on assets may have a high earnings per share of common stock because it is largely financed with debt and there are relatively few shares of common stock. An alternative measure is the *dividend payout ratio* which expresses the percentage of dividends paid out on common stock to the earnings accrued to common stock.

The conventional measures for liquidity, solvency, and profitability discussed in this section can be summarized in the following equations:

1. For liquidity:

$$\text{Current ratio} = \frac{\text{current assets}}{\text{current liabilities}} \tag{14.8}$$

$$\text{Quick ratio} = \frac{\text{current assets} - \text{inventories}}{\text{current liabilities}} \tag{14.9}$$

2. For solvency:

$$\text{Income-to-interest ratio} = \frac{\text{income before tax and interest deduction}}{\text{interest expense}} \tag{14.10}$$

$$\text{Debt-to-equity ratio} = \frac{\text{current liabilities} + \text{long-term liabilities}}{\text{stockholders' equity}} \tag{14.11}$$

3. For profitability:

$$\text{Return on sales} = \frac{\text{income after taxes}}{\text{sales and other operating revenues}} \tag{14.12}$$

$$\text{Return on assets} = \frac{\text{income after taxes}}{\text{total assets}} \tag{14.13}$$

$$\text{Earning per common share} = \frac{\text{net income} - \text{preferred dividends}}{\text{number of common shares}} \tag{14.14}$$

$$\text{Dividend payout ratio} = \frac{\text{dividend paid out to common stock}}{\text{net earnings} - \text{preferred dividends}} \tag{14.15}$$

Example 14.5

On the basis of the information for 1990, determine the liquidity, solvency, and profitability of the G. Fitzgerald Corporation, using the conventional measures.

Using the information on the balance sheet and the income statement in Tables 14.1 and 14.2, respectively, we get

$$\text{Current ratio} = \frac{600{,}000}{220{,}000} = 2.73$$

$$\text{Quick ratio} = \frac{600{,}000 - 290{,}000}{220{,}000} = 1.41$$

$$\text{Income-to-interest ratio} = \frac{210{,}000 + 15{,}000}{15{,}000} = 15$$

$$\text{Debt-to-equity ratio} = \frac{220{,}000 + 180{,}000}{600{,}000} = 0.67$$

$$\text{Return on sales} = \frac{105{,}000}{1{,}100{,}000} = 9.5\%$$

$$\text{Return on assets} = \frac{105{,}000}{1{,}000{,}000} = 10.5\%$$

$$\text{Earnings per common share} = \frac{105{,}000 - 5{,}000}{100{,}000} = \$1.00$$

$$\text{Dividend payout ratio} = \frac{35{,}000}{105{,}000 - 5{,}000} = 35\%$$

It can be concluded that the G. Fitzgerald Corporation is in a sound financial position. It has probably retained too much cash on hand that should be invested; it also has a tremendous capacity for expansion through borrowing. The firm is fairly profitable from the viewpoints of both the management and investors.

14.6 AUDITORS' REPORT

Before making a decision on the financial position of a firm from its financial statements, readers must satisfy themselves that the statements are reliable. What is to prevent a firm from unintentionally misrecording, or worse, intentionally falsifying the record of its transactions, thus giving a misleading impression of its strength?

The financial statements of most firms, except those of a small number of closely held firms, include a report from a team of independent certified public accountants who are not connected with the firm. Thus, the veracity of the records is attested to by an outside source.

Before rendering their opinion, accountants *audit* the firm in order to obtain "reasonable assurance" that the firm's records are reasonably accurate. The audit examination usually includes at least the following:

1. Checking the accuracy of the computations in the accounting records and the appropriateness of the accounting methods used
2. Verifying by inspection the physical existence and the company's ownership of the items listed under property, plant, equipment, and inventory, and making independent estimates of the value of such items
3. Confirming the amounts and condition of outside transactions with the outside sources involved, such as customers and bankers

After the team of accountants has formed an estimate of the reliability of the firm's financial statements and satisfied itself that the estimate is supported by its findings, it issues one of several types of opinions:

1. The unqualified opinion
2. The "subject to" qualified opinion
3. The "except for" qualified opinion
4. A disclaimer of opinion
5. An adverse opinion

The type of opinion rendered indicates the state of the firm's financial statements.

Although a typical auditors' report contains a preamble about "generally accepted auditing standards" and "generally accepted accounting principles," the heart of the report lies in the opinion. The most common type of opinion is the unqualified or "clean" opinion, which states that "In our opinion, the financial statements referred to above *present fairly* the financial position of XYZ Company as of . . . ." The unqualified opinion asserts the accuracy of the financial statements as presented by management but does not pass judgment on the results of operations, which is up to the reader. Thus, a set of financial statements showing losses from operations may have a clean opinion if these results are accurately reported.

If the auditors have less than full confidence in the accuracy and fairness of the financial statements, a qualifying phrase such as "subject to," or "except for" after the words "In our opinion" may be added. A "subject to" qualification is less serious because it refers only to uncertainty rather than inaccuracy in the firm's financial statements. Such an uncertainty might arise from a possible loss contingency such as a pending or ongoing lawsuit or adverse government ruling, where the probable amount of damage cannot be reasonably estimated or accounted for. The reader is thus placed on notice of a major contingency that might affect the financial position of the firm and make it less attractive than it appears in the financial statements. The "except for" qualification following "In our opinion" means that the financial statements are misleading in one or more major respects because of management accounting and reporting policies. A report with the "except for" opinion will both list the specific deficiencies in the financial statements and alert the reader to be suspicious of management's overall intentions.

Many companies, especially small, closely held companies will have outside accountants prepare their financial statements, without auditing the company, and issue a report. Such statements are more reliable than those issued by the company alone, but considerably less reliable than audited statements. The auditors' report will contain a *disclaimer of opinion* which reads: "We *have not audited* the statements of XYZ Company and accordingly, do not express an opinion on them."

If the financial statements of a firm are totally misleading, the auditors' report will issue an adverse opinion which states: "In our opinion, the financial statements above *do not present fairly* the financial position of XYZ Company . . . ." Such an opinion generally results from a situation in which the auditor has been unable to convince the client to amend the financial statements to reconcile the auditor's independent estimates according to generally accepted accounting principles. In that case, the adverse opinion must be accompanied by a statement of the reasons for issuing such an opinion.

14.7 INFLATION ACCOUNTING

The effects of price level changes have caused difficulties in measuring and comparing the performance of business and industrial firms over the years. In periods of inflation, the net earnings in the financial statements may not be indicative of actual performance due to the increasing costs of replacing existing inventories and operating assets.

In an effort to help readers of financial reports to understand the effects of price level changes, the historical cost financial data are supplemented by the inflation-adjusted financial information. Two approaches to disclosing the effects of price level changes have been suggested by the Financial Accounting Standards Board (FASB).[2] The *constant dollar method* measures the effects of general inflation by restating certain historical cost financial data to the average dollar value for the year just ending by use of a suitable price index. The *current cost method* measures the effects of changes in specific prices by taking into consideration the differential price changes.

According to the theory prescribed by FASB No. 33, the effects of inflation are greatest on property, plant and equipment, and inventories. This result occurs because fixed assets and inventory tend to rise or fall in nominal value with the changes in price levels while maintaining relatively constant real values. In times of rising prices, the historical values recorded for such assets on the balance sheet tend to understate the monetary values. On the other hand, liquid assets such as cash, marketable securities, and receivables have constant nominal values on the balance but decline in real values due to the effects of inflation.

Although inflation affects balance sheet values, the effect of price level changes is greatest on the income statement because past costs are matched against current revenues. As a result, a number of companies have prepared inflation-adjusted income statements for the purpose of comparison with nonadjusted income statements. This practice reflects general concern that expenses in real terms are understated, and thus income is overstated on the traditional income statement.

The rising costs of maintaining productive capacity in inflationary periods lead to a reduction in real net income. However, even though pretax income on a constant dollar basis is lower compared to the historical dollar amount, the provision for income taxes is unchanged, thus increasing the firm's effective tax rate.

To offset or aggravate the results of real income decline, the firm may realize a holding gain or loss on balance sheet items. On the theoretical assumption that fixed assets and inventory values keep up with inflation, only the firm's monetary assets such as cash, securities, and receivables are exposed to decline in purchasing power due to inflation. This exposure is offset by monetary liabilities such as current liabilities and long-term debt that will be repaid in cheaper dollars. If these monetary liabilities exceed monetary assets, there will be a net holding gain; otherwise there will be a net holding loss. Thus, inflation affects a firm's income directly by reducing its value and a firm's financial structure indirectly by encouraging smaller holdings of assets and large holdings of debt.

[2] American Institute of Certified Public Accountants, Financial Accounting Standards Board (FASB) Statement No. 33, "Financial Reporting and Changing Prices," New York: AICPA, 1979.

14.8 SUMMARY AND STUDY GUIDE

The chapter presented an overview of the major determinants of the financial position of a firm as reflected periodically in its financial statements. The financial statements are contained in the annual report of a corporation, which includes the balance sheet, the income statement, the statement of changes in financial position, and the auditors' report. Each of these statements was explained in detail in order to offer some insights into the financial position of a firm as well as to provide a general background for the discussions in Chapters 15 and 16.

The possible implications of the financial statements of a firm that concern a prospective investor the most are liquidity, solvency, and profitability. An effective analysis of the financial position of a firm can be obtained by examining a set of measures related to such areas. We have therefore considered the definitions of the following quantitative measures: current ratio, quick ratio, income-to-interest ratio, return on sales, return on assets, earnings per common share, and dividend payout ratio.

The financial statements of most firms include an auditors' report indicating that the veracity of the records is attested to by an outside independent source. The significance of different types of opinions that may be expressed in the auditors' report has been explained.

Finally, we have considered briefly the effects of price level changes on the measurement and comparison of business and industrial firms over the years. The methods of disclosing such effects have been suggested by the Financial Accounting Standards Board.

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14.1 Bernstein, L. A., *Analysis of Financial Statements,* 3d ed. Homewood, IL: Richard D. Irwin, 1990.

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14.4 Johnson, H. T., and R. S. Kaplan, *Relevance Lost: The Rise and Fall of Management Accounting.* Boston: Harvard Business School Press, 1987.

PROBLEMS

P14.1 From the balance sheet and the income statement of Company A (pp. 392–93), calculate the appropriate ratios in evaluating the company's liquidity, solvency, and profitability for year 1987. Comment on these ratios and indicate what additional information you wish to have in order to render a better judgment.

P14.2 From the financial statements of Company A, compute the source of funds, uses of funds, and the change of working capital for the year ending December 31, 1987. Without additional information from the contra accounts, assume any net increase or decrease in asset, debt, or equity as an actual transaction in that category. Compute also the increase or decrease in each component of working capital for the same year.

P14.3 If the auditors' report for Company A was qualified by a "subject to" opinion regarding a contingency liability of $100,000 due to a pending lawsuit, how would this affect your evaluation of the company's prospects?

P14.4 From the balance sheet and the income statement of Company B (pp. 394–95), calculate the appropriate ratios in evaluating the company's liquidity, solvency, and profitability for year 1990. Comment on these ratios and indicate what additional information you wish to have to render a better judgment.

P14.5 From the financial statements of Company B, compute the sources of funds, uses of funds, and the change of working capital for the year ending December 31, 1990. Without additional information from the contra accounts, assume any net increase or decrease in asset, debt, or equity as an actual transaction in that category. Compute also the increase or decrease in each component of working capital for the same year.

P14.6 If the auditors' report for Company B was qualified by an "except for" opinion indicating a possible additional liability of $30,000 for taxes, how would this affect your evaluation of the company's prospects?

P14.7 From the balance sheet and the income statement of Company C (pp. 396–97), calculate the appropriate ratios in evaluating the company's liquidity, solvency, and profitability for the year ending December 31, 1988.

P14.8 From the financial statements of Company C, compute the sources of funds, uses of funds, and the changes in working capital for the year ending December 31, 1988.

P14.9 From the financial statements of Company C, compute the increase or decrease in each component of working capital for the year ending December 31, 1988.

P14.10 From the balance sheet and the income statement of Company D (pp. 398–99), calculate the appropriate ratios in evaluating the company's liquidity, solvency, and profitability for the year ending December 31, 1989.

P14.11 From the financial statements of Company D, compute the sources of funds, uses of funds, and the change of working capital for the year ending December 31, 1989. Without additional information from the contra accounts, assume any net increase or decrease in asset, debt or equity as an actual transaction in that category.

P14.12 From the financial statements of Company D, compute the increase or decrease in each component of working capital for the year ending December 31, 1989.

BALANCE SHEET
COMPANY A

December 31	1987	1986
ASSETS		
Current assets		
Cash	$ 10,000	$ 15,000
Marketable securities	35,000	40,000
Accounts receivable*	80,000	70,000
Inventories	75,000	75,000
Total	$200,000	$200,000
Fixed assets		
Land	20,000	20,000
Plant and equipment	400,000	340,000
Total cost	420,000	360,000
Less accumulated depreciation	120,000	90,000
Total cost net of depreciation	300,000	$270,000
TOTAL ASSETS	$500,000	$470,000

	1987	1986
LIABILITIES AND EQUITY		
Current liabilities		
Accounts payable	$ 60,000	$ 55,000
Wages payable	5,000	0
Dividends payable	10,000	5,000
Notes payable	15,000	10,000
Income taxes payable	15,000	10,000
Long-term debt, current portion	5,000	10,000
Total	$110,000	$ 90,000
Long-term liabilities		
Long-term debt, less current portion	60,000	70,000
Bonds payable	100,000	100,000
Total	$160,000	$170,000
Stockholders' equity		
Capital stock, $10 par value	30,000	30,000
Additional paid-in capital	20,000	20,000
Retained earnings	180,000	160,000
Total	$230,000	$210,000
TOTAL LIABILITIES AND EQUITY	$500,000	$470,000

INCOME STATEMENT
COMPANY A

For year ending December 31	1987	1986
REVENUES		
Sales and operating revenues	$850,000	$780,000
COSTS AND EXPENSES		
Cost of goods sold	450,000	420,000
Selling expenses	200,000	175,000
General and administrative expenses	90,000	80,000
Depreciation	30,000	25,000
Total	770,000	700,000
REVENUES LESS COSTS AND EXPENSES	80,000	80,000
OTHER DEDUCTIONS		
Interest expense, net of interest income	20,000	10,000
INCOME BEFORE TAXES	60,000	70,000
PROVISION FOR INCOME TAXES	30,000	35,000
NET INCOME AFTER TAX	30,000	35,000
CASH DIVIDENDS	10,000	15,000
RETAINED EARNINGS		
Current year	20,000	20,000
Beginning of the year	160,000	140,000
End of the year	180,000	160,000
EARNINGS PER SHARE OF STOCK	$10.00	$11.67

BALANCE SHEET
COMPANY B

December 31,	1990	1989		1990	1989
ASSETS			LIABILITIES AND STOCKHOLDERS' EQUITY		
Current assets			Current liabilities		
Cash	$ 200,000	150,000			
Marketable securities	350,000	210,000	Accounts payable	$1,000,000	$950,000
Accounts receivable	940,000	800,000	Accrued taxes payable	50,000	45,000
Inventories	850,000	720,000	Notes payable, current	10,000	15,000
Total	2,340,000	1,880,000	Total	1,060,000	1,010,000
Fixed assets			Long-term liabilities		
Land	360,000	360,000	Mortgage notes, net of		
Buildings	500,000	500,000	current portion	100,000	130,000
Machinery	700,000	600,000			
Total cost	1,560,000	1,460,000	Stockholders' equity		
Less accumulated depreciation	300,000	260,000	Common stock, $1 par value	400,000	400,000
Total net of depreciation	1,260,000	1,200,000	Retained earnings	2,040,000	1,540,000
			Total	2,440,000	1,940,000
TOTAL ASSETS	$3,600,000	$3,080,000	TOTAL LIABILITIES AND EQUITY	$3,600,000	$3,080,000

INCOME STATEMENT
COMPANY B

For year ending December 31,	1990	1989
REVENUES		
Sales and operating revenues	$6,500,000	$6,000,000
COSTS AND EXPENSES		
Cost of goods sold	3,000,000	2,700,000
Selling expenses	1,500,000	1,200,000
General and administrative expenses	950,000	810,000
Depreciation	40,000	30,000
Total	5,490,000	4,740,000
REVENUES LESS COSTS AND EXPENSES	1,010,000	1,260,000
OTHER DEDUCTIONS		
Interest expense, net of interest income	10,000	10,000
INCOME BEFORE TAXES	1,000,000	1,250,000
PROVISION FOR INCOME TAXES	360,000	450,000
NET INCOME AFTER TAX	640,000	800,000
CASH DIVIDENDS	140,000	200,000
RETAINED EARNINGS		
Current year	500,000	600,000
Beginning of the year	1,540,000	940,000
End of the year	2,040,000	1,540,000
EARNINGS PER SHARE OF COMMON STOCK	$1.60	$2.00

BALANCE SHEET
COMPANY C

December 31	1988	1987	December 31	1988	1987
ASSETS			LIABILITIES AND STOCKHOLDERS' EQUITY		
Current assets			Current liabilities		
Cash	$ 15,000	$ 14,000	Accounts payable	$ 90,000	$ 76,000
Marketable securities	36,000	30,000	Wages payable	$ 20,000	$ 15,000
Accounts receivable	82,000	60,000	Taxes payable	25,000	20,000
Inventories	143,000	115,000	Notes payable, current portion	15,000	15,000
Total current assets	276,000	219,000	Total current liabilities	150,000	126,000
Fixed assets			Long-term liabilities		
Land	48,000	46,000			
Plant and equipment	360,000	360,000	Notes payable, net of current portion	150,000	160,000
Total cost of fixed assets	408,000	406,000	Bonds payable	75,000	80,000
Less accumulated depreciation	84,000	54,000	Total long-term liabilities	225,000	240,000
Total cost net of depreciation	324,000	352,000	Stockholders' equity		
			Common stock, $2 par value (40,000 shares)	80,000	80,000
TOTAL ASSETS	$600,000	$571,000	Paid-in surplus	0	0
			Retained earnings	145,000	125,000
			Total stockholders' equity	225,000	205,000
			TOTAL LIABILITIES AND EQUITY	$600,000	$571,000

INCOME STATEMENT
COMPANY C

For year ending December 31	1988	1987
REVENUES		
Sales and operation revenues	$1,000,000	$800,000
COSTS AND EXPENSES		
Cost of goods sold	700,000	500,000
Selling expenses	150,000	140,000
General and administrative expenses	60,000	50,000
Depreciation	30,000	40,000
Total	940,000	730,000
REVENUES LESS COSTS AND EXPENSES	60,000	70,000
OTHER DEDUCTIONS		
Interest expenses, less interest income	20,000	10,000
INCOME BEFORE TAXES	$40,000	$ 60,000
PROVISION FOR INCOME TAXES	20,000	30,000
NET INCOME AFTER TAX	$20,000	$30,000
RETAINED EARNINGS		
Current year	20,000	30,000
Beginning of the year	125,000	95,000
End of the year	$145,000	$125,000
EARNINGS PER SHARE COMMON STOCK	$0.50	$0.75

BALANCE SHEET
COMPANY D

December 31	1989	1988		1989	1988
ASSETS			LIABILITIES AND EQUITY		
Current assets			Current liabilities		
Cash	$ 10,000	$ 11,500	Accounts payable	$ 6,000	$ 5,000
Marketable securities	17,500	20,000	Wages payable	2,200	1,000
Accounts receivable	11,000	16,500	Dividends payable	500	400
Inventories	26,500	28,000	Notes payable	2,000	2,600
Total	$ 65,000	$ 76,000	Income taxes payable	600	1,000
			Long-term debt, current portion	1,100	1,000
Fixed assets			Total	$ 12,400	$ 11,000
Land	10,000	10,000			
Plant and equipment	40,000	30,000	Long-term liabilities		
Total cost	50,000	40,000	Long-term debt, less		
Less accumulated depreciation	6,000	5,000	current portion	15,000	20,000
Total cost net of depreciation	$ 44,000	$ 35,000			
			Total	$ 27,400	$ 31,000
TOTAL ASSETS	$109,000	$111,000			
			Stockholders' equity		
			Common stock ($6 par value)	30,000	30,000
			Additional paid-in capital	10,000	10,000
			Retained earnings	41,600	40,000
			Total	$ 81,600	$ 80,000
			TOTAL LIABILITIES AND EQUITY	$109,000	$111,000

INCOME STATEMENT
COMPANY D

For year ending December 31	1989	1988
REVENUES		
Sales and operating revenues	$96,000	$80,000
COSTS AND EXPENSES		
Cost of goods sold	55,000	44,200
Selling expenses	21,000	16,000
General and administrative expenses	12,000	10,000
Depreciation	1,000	800
Total	89,000	71,000
REVENUES LESS COSTS AND EXPENSES	7,000	9,000
OTHER DEDUCTIONS		
Interest expense, net of interest income	3,000	1,800
INCOME BEFORE TAXES	4,000	7,200
PROVISION FOR INCOME TAXES	1,400	2,600
NET INCOME AFTER TAX	2,600	4,600
CASH DIVIDENDS	1,000	1,000
RETAINED EARNINGS		
Current year	1,600	3,600
Beginning of the year	40,000	36,400
End of the year	41,600	40,000
EARNINGS PER SHARE OF STOCK	$0.52	$0.92

CHAPTER

15 Uncertainty and Risk

15.1 DECISIONS UNDER UNCERTAINTY

In our discussion of capital investment up to this point, we have assumed that an investment can be represented by a unique stream of cash flows. In reality, the cash flow profile of an investment will be affected by future events, the outcomes of which are uncertain and cannot be controlled by the decision maker. Such events can be international or national in scope, or of an industrywide or local character. For example, the international political situation affects the price of oil, and the monetary and fiscal policies of the federal government affect the level of business activities. On the other hand, the steel and automobile industries are confronted with problems that are characteristic of the particular industry. At the level of a firm, a change of its management or a natural disaster may produce profound changes in the viability of the firm. No one can predict exactly in advance the outcome of any of these events, let alone every event that may be relevant to an investment decision.

Let us consider specifically a simplified situation. Suppose that two prospective investments of 1-year duration will produce net benefits a year later under different conditions of the economy as shown in Table 15.1, and the chance of having each of these conditions is equally likely. Which investment is more profitable?

Intuitively, we may attempt to find the average of the net benefits for each investment. Having found that the averages of the two investments are equal ($1,000 for each case), however, we are no closer to an answer because we are unlikely to be indifferent in accepting either one of the two investments in view of the difference in the spreads of the net benefit for the two cases. In other words, the dispersion of net

TABLE 15.1 FORECASTS OF NET BENEFITS

Condition of economy	Net benefits of prospective investments	
	Household goods	Recreational supplies
Expanding	\$1,200	\$2,050
Stable	\$1,000	\$1,000
Contracting	\$ 800	−\$ 50

benefits is smaller for the investment in household goods than that in recreational supplies under varying conditions of the economy. If the economy is expanding, the investment in recreational supplies is more profitable, but if the economy is contracting, the investment in recreational supplies produces a loss. Therefore, the uncertainty associated with the net benefit for the first investment is visibly smaller than that associated with the second one. Even if the degree of uncertainty can be determined for each case, the choice depends to a large extent on our attitude toward risk. There is empirical evidence to indicate that most investors are averse to risk, and they demand higher expected returns from more uncertain investments to compensate for the higher levels of risk.

Under the conditions of uncertainty, we can no longer simply ask which investment produces the maximum net benefit because other factors must also be considered. The criterion of profit maximization, which has been used to evaluate prospective investments when the cash flows are assumed to be unique and certain, is not necessarily applicable when the cash flows are uncertain. We shall therefore examine the basic concepts and measures related to uncertainty and risk in order to determine the conditions under which the criteria of maximizing expected profits are applicable, and to examine other factors which are pertinent. Because of the complexities of the underlying issues, we shall attempt to introduce the key ideas with a minimum amount of detail.

15.2 EXPECTED VALUES AND VARIANCES

We first introduce mathematical descriptions of expected returns and the uncertainty associated with these returns on the basis of probability theory. We shall confine our discussion to the most elementary level since no prior knowledge of probability theory is assumed.

Let $x_1, x_2, \ldots, x_n$ be n mutually exclusive possible outcomes of an uncertain event X, and let $p_1, p_2, \ldots, p_n$ be the probabilities of their occurrence, respectively, such that

$$p_1 + p_2 + \cdots + p_n = 1$$

The *expected value* of event X is denoted by $E[X] = \mu$ and is defined as

$$\mu = p_1x_1 + p_2x_2 + \cdots + p_nx_n \tag{15.1}$$

To see that $E[X] = \mu$ represents a forecast of an average, suppose that m observations of the event X have been recorded in the past, with values x'_1, $x'_2, \ldots, x'_m$. The average of these observations is given by

$$\mu = \frac{1}{m}(x'_1 + x'_2 + \cdots + x'_m)$$

Suppose that among these values, x_1 occurs m_1 times, x_2 occurs m_2 times, . . . , and x_n occurs m_n times, where n is the number of distinct observed values $x_1, x_2, \ldots, x_n$. Then, we can combine the common values to get

$$\begin{aligned}\overline{x} &= \frac{1}{m}(m_1x_1 + m_2x_2 + \cdots + m_nx_n)\\ &= \frac{m_1}{m}x_1 + \frac{m_2}{m}x_2 + \cdots + \frac{m_n}{m}x_n\end{aligned}$$

If m is large, the relative frequencies $m_1/m, m_2/m, \ldots, m_n/m$ approach the probabilities $p_1, p_2, \ldots, p_n$, respectively. Hence, the expected value of uncertain event X may be interpreted as the weighted average of the relative frequencies of the occurrence of the outcomes $x_1, x_2, \ldots, x_n$. The expected value μ is identical to the *mean* value of the outcomes.

The *variance* of an uncertain event X is defined as the expected value of the function $(X - \mu)^2$, whose possible outcomes are $(x_1 - \mu)^2$, $(x_2 - \mu)^2, \ldots$, and $(x_n - \mu)^2$ with probabilities of occurrence $p_1, p_2, \ldots, p_n$, respectively. The variance of event X is denoted by $V[X] = \sigma^2$ which is numerically equal to

$$\begin{aligned}\sigma^2 &= p_1(x_1 - \mu)^2 + p_2(x_2 - \mu)^2 + \cdots + p_n(x_n - \mu)^2\\ &= p_1(x_1^2 - 2x_1\mu + \mu^2) + p_2(x_2^2 - 2x_2\mu + \mu^2) + \cdots + p_n(x_n^2 - 2x_n\mu + \mu^2)\\ &= (p_1x_1^2 + p_2x_2^2 + \cdots + p_nx_n^2) - 2\mu(p_1x_1 + p_2x_2 + \cdots + p_nx_n)\\ &\qquad + \mu^2(p_1 + p_2 + \cdots + p_n)\\ &= (p_1x_1^2 + p_2x_2^2 + \cdots + p_nx_n^2) - 2\mu^2 + \mu^2\end{aligned}$$

Hence,

$$\sigma^2 = \sum_{i=1}^{n} p_ix_i^2 - \mu^2 \tag{15.2}$$

Note that the variance of the uncertain event may be interpreted as the weighted average of $(x_1 - \mu)^2$, $(x_2 - \mu)^2, \ldots$, and $(x_n - \mu)^2$. The values $(x_1 - \mu)$, $(x_2 - \mu), \ldots$, and $(x_n - \mu)$ represent the deviations of the possible outcomes $x_1, x_2, \ldots$, and x_n from the expected value μ. Although the deviations may be positive or negative and thus cancel each other if they are added algebraically, the squares of these quantities are always positive and the sum of the squares is an indication of the magnitudes of the deviations. Consequently, the variance σ^2 is a measure of the dispersion of the outcomes but by no means the only measure of all the characteristics of uncertainty.

The quantity σ is referred to as the *standard deviation* and has the same unit as μ. Thus, it is a more convenient measure of the dispersion and its value is given by

$$\sigma = \sqrt{\sum_{i=1}^{n} p_i x_i^2 - \mu^2} \tag{15.3}$$

Since x_i can be expressed in any unit, but p_i is dimensionless, σ always has the same unit as μ. For example, if $x_1, x_2, \ldots,$ and x_n represent the predicted net benefits in dollars under different conditions of the economy, both μ and σ will be expressed in terms of dollars.

Another quantity for measuring dispersion is the *coefficient of variation*, which is defined as

$$C = \frac{\sigma}{\mu} \tag{15.4}$$

This coefficient gives the variability of event X in dimensionless form and indicates the dispersion relative to the expected value.

Therefore, an uncertain event X with a number of possible outcomes is referred to as a *random variable*. The probabilistic description of a random variable X is defined by all possible outcomes $x_1, x_2, \ldots, x_n$ and the corresponding probabilities of their occurrence $p_1, p_2, \ldots, p_n$. The function $p_i = f(x_i)$, as schematically shown in Fig. 15.1, is called the probability distribution of X. When the outcomes of X represent a set of discrete values, X is said to be a *discrete random variable*, and the probability distribution of X is referred to as the *probability mass function* of X.

In some situations, X may be idealized as a *continuous random variable* which can assume all values of x within a specified interval. Then, the probability distribution of X can be represented by the function $p = f(x)$, which is referred to as the *probability density function*. The *normal probability distribution* shown in Fig. 15.2 is an example of a continuous probability distribution in which the ordinate $f(x)$ represents the *probability density* for a given value of x and the *probability* is defined only for an interval of x.

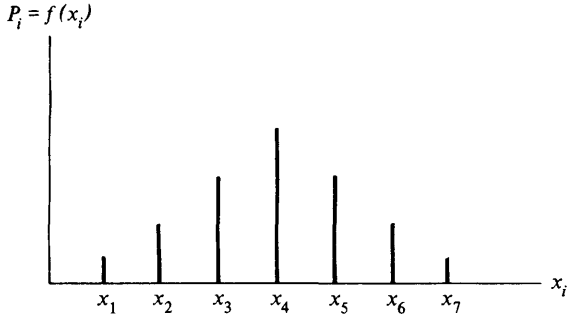

Figure 15.1 Probability mass function.

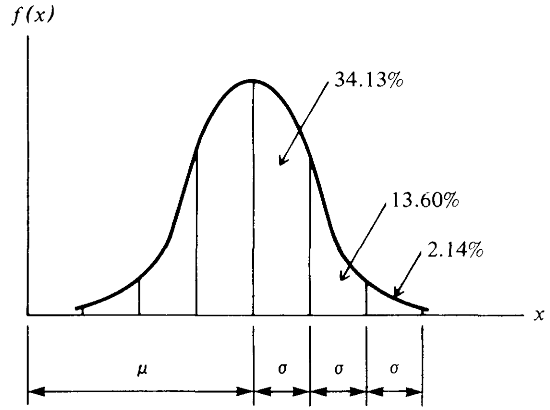

Figure 15.2 Normal Probability density function.

The normal distribution is characterized by a bell-shaped curve that is symmetrical with respect to $x = \mu$ and has points of inflection at $x = \mu \pm \sigma$. Mathematically, it is represented by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \tag{15.5}$$

where exp is an exponential function of the natural base e of the logarithm, and x covers the range from $-\infty$ to $+\infty$. The probabilities of occurrence at intervals of σ are represented by the respective areas under the curve for such intervals. Hence, the probabilities that a normally distributed random variable X will have a value with σ, 2σ, or 3σ on either side of the expected value μ are approximately 68.3%, 95.5%, and 99.7%, respectively.

The expected value and the standard deviation are the two most important parameters describing the central tendency and the dispersion of a probability distribution, respectively. In comparing random variables with different expected values, the coefficients of variation of the variables indicate the relative dispersions of their probability distributions. We shall limit our discussion to these parameters.

In statistics, we often conduct observations on a small sample of outcomes of a random variable and make statistical inferences about its probability distribution as if we were able to observe all possible outcomes. We shall not consider the implications of statistical sampling. Hence, Eqs. (15.1) and (15.2) will be used for computing the expected value and the variance regardless of the number of observations. Then, the standard deviation and the coefficient of variation can also be computed from Eqs. (15.3) and (15.4) accordingly.

Example 15.1

Find the expected value and the standard deviation of each investment in Table 15.1 if the occurrence of each outcome is equally likely, i.e., $p_1 = p_2 = p_3 = \frac{1}{3}$.

For the investment in household goods, x_1 = \$1,200, x_2 = \$1,000, and x_3 = \$800. Hence,

$$\mu = (\tfrac{1}{3})(1{,}200 + 1{,}000 + 800) = \$1{,}000$$

and

$$\sigma = \sqrt{(\tfrac{1}{3})[(1{,}200)^2 + (1{,}000)^2 + (800)^2] - (1{,}000)^2}$$
$$= \sqrt{26{,}667} = \$163$$

For the investment in recreational supplies, x_1 = \$2,000, x_2 = \$1,050, and x_3 = −\$50. Then

$$\mu = (\tfrac{1}{3})(2{,}050 + 1{,}000 - 50) = \$1{,}000$$

and

$$\sigma = \sqrt{(\tfrac{1}{3})[(2{,}050)^2 + (1{,}000)^2 + (-50)^2] - (1{,}000)^2}$$
$$= \sqrt{735{,}000} = \$857$$

It can be seen that while μ is identical for both cases, the value of σ for the second case is greater than that for the first case. Consequently, the uncertainty concerning the second investment is greater than that of the first.

15.3 RISK EXPOSURES AND ATTITUDES TOWARD RISK

The return from an investment under uncertainty may be regarded as a *random variable* with a number of possible outcomes whose probabilities of occurrence may be estimated on the basis of historical observations or subjective judgment. The expected value of the return is a measure of its *expected profitability,* whether the return is expressed in a dollar amount or in an annual percentage rate. The dispersion or variability of the return reflects the *degree of uncertainty* to the investor and is a measure of the *risk,* since the return of an investment proposal with a small standard deviation indicates a high degree of confidence in its outcome and a high standard deviation indicates otherwise. The coefficient of variation, which is the ratio of the standard deviation and the expected value of the probability distribution for the return, is a measure of the *variability* in dimensionless form.

Although the standard deviation is a useful measure of uncertainty and risk, it indicates only the exposures of an individual or organization to risk. However, people and organizations may react differently in confronting risk exposures under the same or different circumstances. Their attitudes may range from aversion, neutrality, or preference toward risk. Generally, below a threshold level of investment, people may prefer risk taking by choosing options that offer significant rewards even though the odds of winning are unfavorable. The popularity of lotteries to the general public and the spirit of entrepreneurial oil drillers attest to their risk preference.[1]

[1] For further discussion, see Ref. 15.3, pp. 354–360.

However, above a threshold level, the attitude of decision makers toward risk can be observed by how they value tradeoffs or choices between a payoff with certainty and the expected payoff in a gamble with uncertain outcomes but known probabilities. People are said to be risk taking if they are willing to accept a gamble whose expected payoff increases more slowly relative to the increase from the certain payoff, and risk-neutral if they accept a gamble whose expected payoff increases proportionally to the certain payoff. People are said to be risk-averse if they prefer a gamble only if its expected payoff increases faster relative to the increase from the certain payoff. As the stakes become higher, risk-averse investors will be unwilling even to accept an investment with a positive expected return if the magnitude of possible loss is large or if the probability of any loss is large. This risk-averse attitude among most investors is evidenced by the large number of persons who are eager and willing to buy insurance to avoid the possibility of financial disaster.

Since the standard deviation is a measure of risk, it is desirable to reduce the magnitude of a risk through diversification. That is, instead of investing a sum of money in a single project, we may invest the same amount in a group of smaller projects whose returns are not affected by the return from any other project in the group. Because of risk sharing, the standard deviation of all of the projects in the group theoretically tends to decrease as the number of projects in the group is increased. In practical terms, it can be seen that if the returns from a group of projects are independent of each other, any adverse or beneficial effect to one of the projects generally will not affect the others. Thus, by undertaking a large number of independent projects of relatively small risk, the total expected return resulting from all the projects is fairly close to certainty since the variability of the outcomes is greatly reduced.

In general, it is useful to break down the risk from a group of projects undertaken by an organization into two components: *unsystematic* risk and *systematic* risk. The unsystematic risk refers to the risk that is caused by some unique and uncontrollable factors such as a technological breakthrough or a natural disaster which

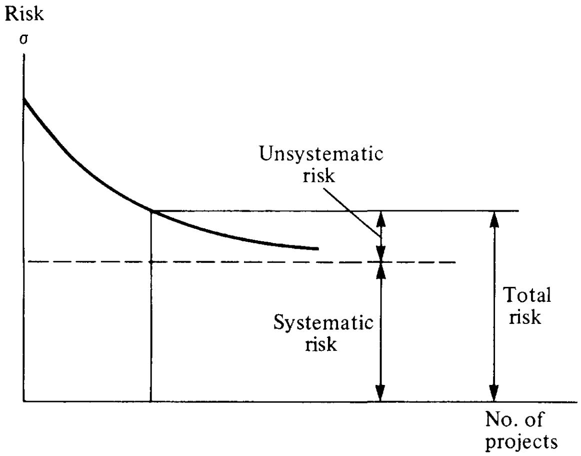

Figure 15.3 Systematic and unsystematic risk.

is independent of other projects and can be eliminated by diversification. The systematic risk refers to the risk that is still present after the elimination of unsystematic risk because the returns from different projects are correlated with one another to some extent and tend to fluctuate more or less in unison under the influence of environmental factors. Unsystematic risk and systematic risk are also referred to as *diversifiable* risk and *nondiversifiable* risk, respectively. The reduction of unsystematic risk through diversification is shown schematically in Fig. 15.3.

15.4 EXPECTED VALUE DECISION CRITERION

Other things being equal, the expected net present values of mutually exclusive proposals may be used to rank the merits of such proposals. Let us consider two proposals whose net present values (NPV) are represented by random variables X and Y. Suppose that these random variables have independent and identical probability distributions except that the expected values $E[X] = \mu_x$ and $E[Y] = \mu_y$ are different, as shown by their probability density functions in Fig. 15.4. Although not every possible outcome of the random variable X is necessarily higher than that of the random variable Y, on the average in the long run, the NPV of X is higher than the NPV of Y. Since the expected net present value of proposal X is higher than that of proposal Y, X is preferred over Y. The decision rule that is based on the expected NPV of the proposals is referred to as the *expected value decision criterion.*[2]

Before formalizing this decision criterion for project selection, we examine several underlying assumptions to determine under what conditions this criterion is applicable. For example, what do we mean by "independent and identical probability distributions" or "on the average in the long run" in real-world situations? Which

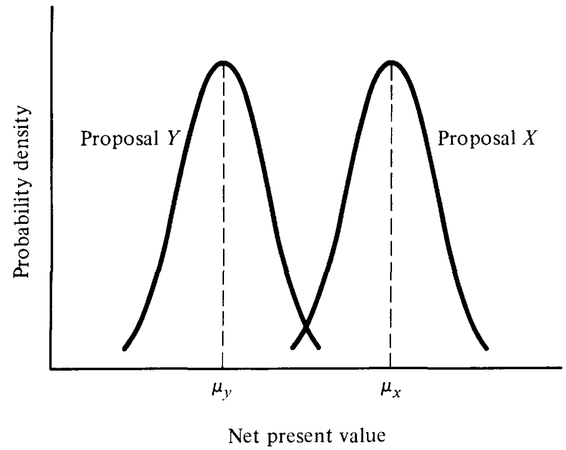

Figure 15.4 Preference of proposal X over proposal Y.

[2] The theoretical basis for this criterion and its application is discussed in detail in Ref. 15.1, pp. 164–197.

parameters in a merit measure should be selected as random variables? Can other merit measures of investment proposals besides the NPV be used to rank the proposals on the basis of the expected value decision criterion?

First, let us consider the issues related to probability distributions. Suppose that we plan to construct a vacation home on a flood plain and want to determine the probability of having a flood in any given year that will reach the level of the house floor. In reviewing the historical flood records, we may be able to find the highest stage of the annual flood in the last 100 years. If there were 5 out of 100 years during which the flood stage reached the level of the planned house floor, we would conclude that on the average in the long run, there is a 1 in 20 chance (or a probability of 5%) of flooding at the house floor level. But what do we know about the probability distributions of the annual floods? Really not very much. However, it is plausible to assume that the probable outcomes of the flood level in any given year is not affected by those in previous years, and that the probable outcomes in different years follow a common pattern. Hence, the distributions representing these probable outcomes are said to be independent and identical. We want to be sure of the plausibility of the basic assumptions because the mathematical relationships leading to the expected value criterion are based on such assumptions.

The net present value of an investment proposal is a direct merit measure and is given by

$$\text{NPV} = \sum_{t=0}^{n} \frac{A_t}{(1+i)^t} = \sum_{t=0}^{n} (B_t - C_t)(1+i)^{-t}$$

If the discount rate i is a random variable, then $X = 1 + i$ is also a random variable with the expected value $E[X] = E[1 + i]$ and $E[1/X] = E[1/(1 + i)]$. Since the expected value of the sum of random variables equals the sum of the expected values of these variables, the expected value of the NPV is

$$E[\text{NPV}] = \sum_{t=0}^{n} A_t \left(E\left[\frac{1}{X}\right] \right)^t$$

Similarly, the net future value is also a direct measure and is given by

$$\text{NFV} = \sum_{t=0}^{n} A_t (1+i)^{n-t}$$

Hence, the expected value of the NFV is

$$E[\text{NFV}] = \sum_{t=0}^{n} A_t (E[X])^{n-t}$$

However, $E[1/X] \neq 1/E[X]$; i.e., the expected value of the reciprocal of a random variable does not equal the reciprocal of the expected value of that random variable. Therefore, E[NFV] cannot be obtained by multiplying E[NPV] by a factor of $X^n = (1 + i)^n$ if i is a random variable. Fortunately, the discount rate is usually selected as a matter of policy and not by a random process. On the other hand, the benefits B_t and cost C_t in periods $t = 1, 2, \ldots, n$ may indeed be treated as random variables since each may be subject to uncertain outcomes beyond the control of the decision maker.

We can illustrate why, in general, $E[1/X] \neq 1/E[X]$ with a simple example.[3] Suppose that the time required to complete a unit of work is a random variable T which depends on the weather. Then, $X = 1/T$ is a random variable representing the units of work completed per unit time, which is sometimes referred to as productivity. If $T = \frac{1}{6}$ day in good weather and $T = \frac{1}{2}$ day in bad weather, and if it is equally likely to have good or bad weather, then $E[T] = (\frac{1}{2})(\frac{1}{6}) + (\frac{1}{2})(\frac{1}{2}) = \frac{1}{3}$ day. On the other hand, since $X = 6$ units per day in good weather and $X = 2$ units per day in bad weather, $E[X] = (\frac{1}{2})(6) + (\frac{1}{2})(2) = 4$ days. Thus, $E[T] = E[1/X] = \frac{1}{3}$ day while $1/E[X] = \frac{1}{4}$ day.

With a fixed discount rate i, the expected value decision criterion for selecting a proposal j among m mutually exclusive proposals is given by

$$\max_{j \in m} E[\text{NPV}_j] = \sum_{t=0}^{n} (E[B_t] - E[C_t])(1 + i)^{-t} \tag{15.6}$$

where B_t and C_t are random variables. For problems involving minimization of costs, the criterion for selecting an alternative j among m mutually exclusive proposals is

$$\min_{j \in m} E[\text{CPV}_j] = \sum_{t=0}^{n} E[C_t]\,(1 + i)^{-t} \tag{15.7}$$

Since the benefit-cost ratio (BCR), the internal rate of return (IRR), and the overall rate of return (ORR) are indirect merit measures and cannot be used to rank mutually exclusive proposals for decision under certainty, their expected values also cannot be used to rank mutually exclusive proposals for decision under the conditions of uncertainty and risk.

15.5 DECISION TREE

The basic structure of a decision problem under uncertainty can be represented by a decision tree which is made up of nodes and branches. A typical decision tree includes three types of nodes: the decision nodes, the chance nodes, and the consequence nodes. At each decision node, the emerging branches represent the possible courses of action that can be selected by the decision maker. At each chance node, the emerging branches represent the chance outcomes that cannot be controlled by the decision maker. A consequence node is the end point of a sequence of branches representing a combination of a possible action and a chance outcome arranged in the proper chronological order. Although a decision tree can be constructed for

[3] This example is taken from a discussion of estimation of productivity in Ref. 15.4, pp. 282–285.

fairly complex problems, we shall confine our discussion to trees with a very simple structure as illustrated by the following example.

Suppose that a government agency is considering building a number of breakwater projects at different locations along the Gulf Coast as havens for small boats during hurricanes. Without a breakwater, the total damage at a particular location during a hurricane season is estimated to be D dollars if it is struck by hurricanes. With the protection of a breakwater, the total damage due to hurricanes is reduced to qD dollars where q is a fraction less than unity, i.e., $0 < q < 1$. If no hurricane occurs, then of course there is no loss, whether or not a breakwater is constructed. The initial cost of constructing a breakwater to provide the required protection is C_0 dollars and will be amortized to an equivalent uniform annual cost of C_u over its useful life of n years at a discount rate i. Hence, $C_u = C_0(U \mid P, i, n)$. The problem confronting the government agency is to decide whether a breakwater should be built at each of these locations.

Let us isolate two separate issues in this problem. The fact that the agency wants to build a number of breakwater projects in scattered locations in the region is at least in part motivated by the diversification of unsystematic risk. The proper allocation of available funds among various projects at different locations to maximize the protection is not an issue we wish to address here. Instead, we are interested only in the question of whether a breakwater should be built on the basis of the expected value criterion once a particular location is selected for consideration. In this regard, we need to know from historical data the likelihood that this location may be struck by hurricanes in each season, and the consequences of the alternative actions as expressed in annual costs in Table 15.2. Then, the problem is to decide what action to take for this particular location in the light of available information.

Since hurricanes that occur each year can be assumed to be independent events, the average annual cost over a number of years will be approximately equal to the expected value of the annual cost if the probability of being struck by hurricanes at this particular location can be determined. Suppose that the probability of being struck is p and the probability of no strike is $(1 - p)$. Furthermore, let the cost associated with a decision be denoted by a random variable X and the expected annual cost be denoted by $E[X]$. Then, the government agency should choose the alternative that will minimize the expected annual cost.

TABLE 15.2 CONSEQUENCES OF ALTERNATIVE ACTIONS (ANNUAL COSTS)

	Outcome	
Decision	Hurricane strikes	No hurricane strikes
Do not build protection	D	0
Build a breakwater	$C_u + qD$	C_u

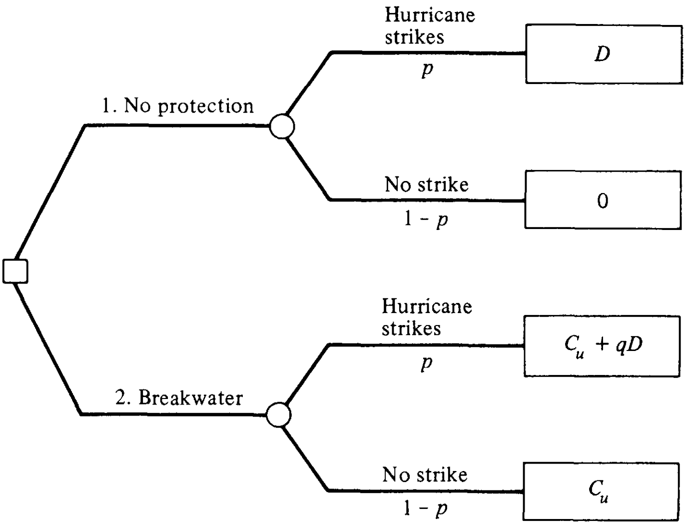

Figure 15.5 Decision tree for construction of a breakwater.

A decision tree showing the combinations of actions and outcomes in deciding whether to construct a breakwater is shown in Fig. 15.5. The decision node is indicated by a square, the chance nodes are denoted by circles, and the consequence nodes are represented by the rectangular boxes which contain the annual costs resulting from various possible combinations of decisions and chances. The relative merits of the consequences from various combinations in Fig. 15.5 are identical to those shown in Table 15.2. The probability of occurrence or nonoccurrence of the hurricane strikes is also noted on the appropriate branch of the decision tree.

Then, according to the definition of expected values, the expected annual costs for the two alternatives in the decision tree can be computed as follows:

1. No protective action

$$E_1[X] = pD$$

2. Constructing a breakwater

$$E_2[X] = p(C_u + qD) + (1 - p)C_u = pqD + C_u$$

The relationship between p and $E[X]$ for these two alternatives is plotted in Fig. 15.6. It is apparent from the figure that if $p < \hat{p}$, $E_1[X] < E_2[X]$; and if $p > \hat{p}$, $E_1[X] > E_2[X]$. In other words, if $p < \hat{p}$, the construction of a breakwater is not justified since the expected annual cost with no protective action is smaller; otherwise, it is justified. Since $\hat{p}$ is the value of p at the intersection of the two straight lines, we can find p by equating $E_1[X]$ and $E_2[X]$ as follows:

$$pD = pqD + C_u$$

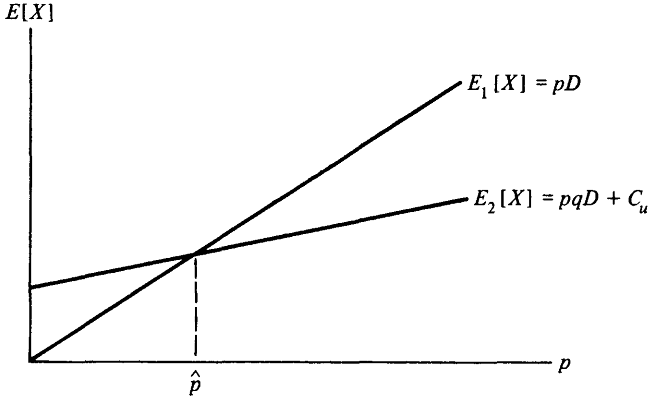

Figure 15.6 Expected value versus probability of occurrence.

Hence, the result becomes

$$\hat{p} = \frac{C_u}{D(1 - q)} \tag{15.8}$$

Therefore, the construction of a breakwater at this particular location is economically justified if

$$p > \frac{C_u}{D(1 - q)}$$

Example 15.2

In the decision problem concerning the construction of a breakwater at a particular location described in this section, the annual damages without and with protection are estimated to be $D = \$100{,}000$ and $qD = \$40{,}000$. The initial construction cost of the breakwater is \$54,446.50, which is expected to last 25 years with no salvage value, and the interest rate for amortization is 10%. Determine the value p above which the construction is justified.

Since the initial construction cost C_0 of the breakwater is amortized for 25 years at an interest rate of 10%, we have

$$\begin{aligned} C_u &= C_0(U \mid P, 10\%, 25) \\ &= (54{,}446.50)(0.1102) = 6{,}000 \end{aligned}$$

Also, $q = 40{,}000/100{,}000 = 0.4$. Then, from Eq. (15.6)

$$\hat{p} = \frac{6{,}000}{(100{,}000)(1 - 0.4)} = 0.1$$

Hence, the construction is justified if $p > 0.1$.

15.6 REGRET OR OPPORTUNITY LOSS

Since a decision maker cannot predict exactly outcomes related to chance in advance, the decision based on the available information involving chance outcomes may turn out to be less than perfect when the decision is reconsidered in retrospect after the chance outcomes are unfolded. The *regret* or *opportunity loss* for each combination of action and outcome is defined as the difference between value of the consequence for that combination and the best that could have been achieved by considering all possible actions and the same outcome.

To illustrate this definition, let us consider the previous example concerning the construction of a breakwater for hurricane protection as represented by the decision tree in Fig. 15.5. Since the regret depends on the occurrence or nonoccurrence of hurricane strikes in retrospect, we shall first determine the best value that could have been achieved for each of the possible chance outcomes. Thus, if we can be certain before the decision that there will be hurricane strikes, the best action is to construct the breakwater and the cost corresponding to that action is C_u; on the other hand, if we are certain that there will be no hurricane strike, the best action is to provide no protection and the cost corresponding to that action is 0. Therefore, the regret L for each combination of action and outcome can be determined as follows:

1. If no protection is provided and the hurricane strikes,

   $$L = D - (C_u + qD) = D(1 - q) - C_u$$

   That is, this amount could have been saved if the decision maker had had the foresight of knowing that the hurricane would strike.
2. If no protection is provided and there is no hurricane strike,

   $$L = 0 - 0 = 0$$

   In this case, the decision maker has made the right choice with no regret.
3. If a breakwater is constructed and hurricanes strike,

   $$L = (C_u + qD) - (C_u + qD) = 0$$

   Again, the decision maker has made the right choice with no regret.
4. If a breakwater is constructed and no hurricane strikes,

   $$L = C_u - 0 = C_u$$

   In this case, the amount C_u could have been saved if it had been known in advance that there would be no hurricane strike.

Consequently, the regret or opportunity loss can be summarized in Table 15.3.

Note that the regret L is a random variable because it depends on the occurrence or nonoccurrence of hurricane strikes. That is, if no protection is provided,

TABLE 15.3 REGRET OR OPPORTUNITY LOSS

Decision	Outcome: Hurricane strikes	Outcome: No hurricane strikes
Do not build protection	$D(1 - q) - C_u$	0
Build a breakwater	0	C_u

$$L = \begin{cases} D(1 - q) - C_u & \text{with probability } p \\ 0 & \text{with probability } (1 - p) \end{cases}$$

and if a breakwater is constructed,

$$L = \begin{cases} 0 & \text{with probability } p \\ C_u & \text{with probability } (1 - p) \end{cases}$$

The expected value of L is called the *cost of uncertainty* because it represents the long-run average cost which results from having less than perfect information. Thus, if there is no protection that will be justified for $0 \leq p \leq \hat{p}$, the expected value of L is given by

$$E[L] = [D(1 - q) - C_u]p + (0)(1 - p) = [D(1 - q) - C_u]p \tag{15.9a}$$

and if there is a protective breakwater that is justified for $\hat{p} \leq p \leq 1$, then

$$E[L] = (0)(p) + C_u(1 - p) = C_u(1 - p) \tag{15.9b}$$

Example 15.3

Using the numerical data in Example 15.2, determine the regret and the cost of uncertainty for the decision problem concerning the construction of a breakwater for hurricane protection.

The regret L for each combination of action and outcome can be obtained as follows:

1. If no protection is provided $(0 \leq p \leq 0.1)$

$$L = \begin{cases} (100{,}000)(1 - 0.4) - 6{,}000 = 54{,}000 & \text{with probability } p \\ 0 & \text{with probability } (1 - p) \end{cases}$$

2. If a breakwater is constructed $(0.1 \leq p \leq 1)$

$$L = \begin{cases} 0 & \text{with probability } p \\ 6{,}000 & \text{with probability } (1 - p) \end{cases}$$

Hence, the cost of uncertainty can be obtained from Eqs. (15.9). That is, for $0 \leq p \leq 0.1$,

$$E[L] = 54{,}000p$$

and for $0.1 \leq p \leq 1$,

$$E[L] = 6{,}000(1 - p)$$

The decision tree is shown together with the regrets in Fig. 15.7.

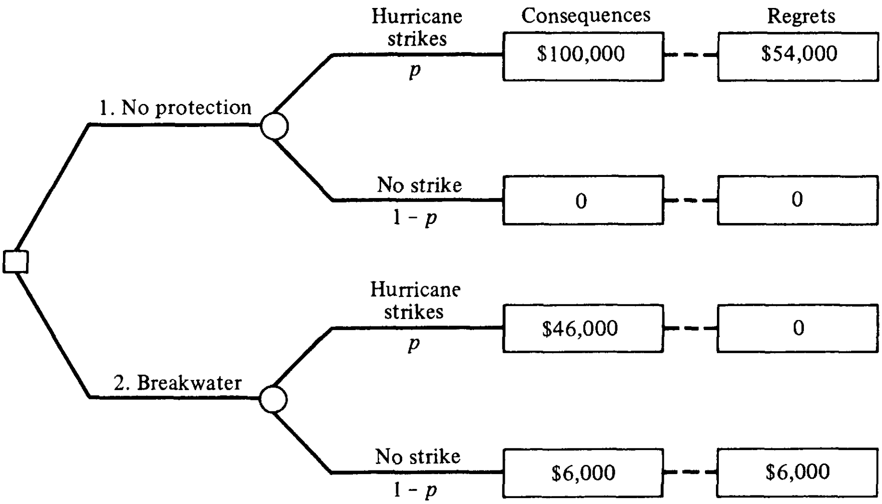

Figure 15.7 Regrets of decision under uncertainty.

15.7 VALUE OF INFORMATION FOR REDUCING UNCERTAINTY

The cost of uncertainty or expected regret provides a useful basis for evaluating the value of information developed to reduce uncertainty. Suppose that a new technology has been developed to provide a better forecast of hurricane strikes. If the cost of uncertainty can be reduced from $E[L]$ to $E'[L]$ because of the use of the new forecasting technology, then the average benefit derived from using the information in the long run can be expressed by the expected benefit $E[B]$ as follows:

$$E[B] = E[L] - E'[L] \qquad (15.10)$$

Suppose that the new technology can forecast exactly the locations of hurricane strikes several days in advance. Then the boats in the areas anticipating hurricane strikes can be removed to other havens where no strike will occur, and the cost of uncertainty or expected regret will be reduced to zero. Thus, for $E'[L] = 0$,

$$E[B] = E[L]$$

However, if the new technology can produce the same forecast only several hours in advance, then it is likely that only some of the damages can be avoided, and $E'[L]$ will be less than $E[L]$ but not equal to zero.

The expected benefit $E[B]$ is a useful measure against which a decision maker may judge the value of information to reduce uncertainty. A decision maker would be willing to pay for such information only if the cost does not exceed the expected benefit.

Example 15.4

Suppose that a new system of forecasting hurricane strikes is available and the adoption of this system will reduce the damage D in Example 15.2 from \$100,000 to \$50,000 while all other quantities remain the same. Determine the value of the information obtained from this new forecasting system.

The consequences of using the new forecasting system are computed and shown in the consequence nodes of the decision tree in Figure 15.8. The expected annual costs for the two alternative actions are

$$E_1[X] = 50{,}000p$$

$$E_2[X] = 26{,}000p + (6{,}000)(1 - p) = 6{,}000 + 20{,}000p$$

The value of $\hat{p}$ at which the decision maker will be indifferent to either action is obtained from Eq. (15.8) as

$$\hat{p} = \frac{6{,}000}{(50{,}000)(1 - 0.4)} = 0.2$$

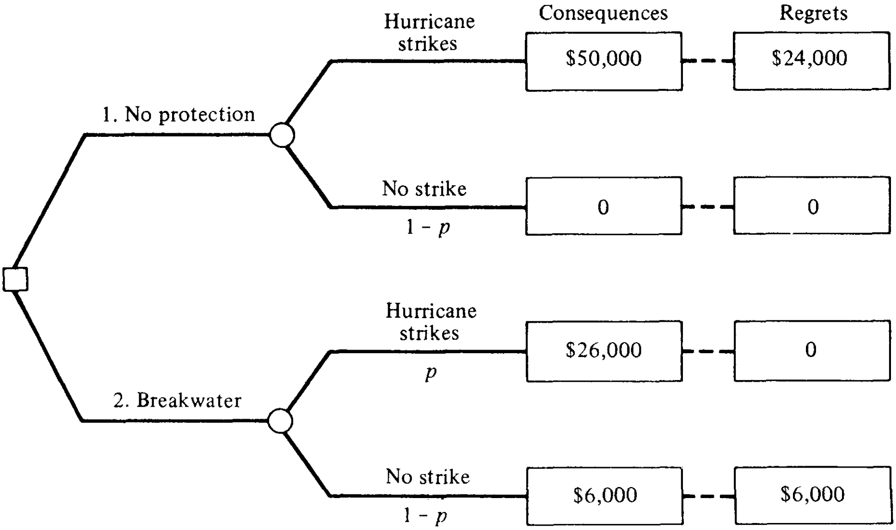

Figure 15.8 Regrets with new information for reducing uncertainty.

The regret L for each combination of action and outcome for using the new forecasting system can be obtained by using Eqs. (15.9). Thus, if there is no protection, which should be the case for $0 \leq p \leq 0.2$,

$$L = \begin{cases} 24{,}000 & \text{with probability } p \\ 0 & \text{with probability } (1 - p) \end{cases}$$

and if there is a protective breakwater, which would be for $0.2 \leq p \leq 1$,

$$L = \begin{cases} 0 & \text{with probability } p \\ 6{,}000 & \text{with probability } (1 - p) \end{cases}$$

Hence, the cost of uncertainty with the new forecasting system is reduced to

$$E'[L] = \begin{cases} 24{,}000p & \text{for } 0 \leq p \leq 0.2 \\ 6{,}000(1 - p) & \text{for } 0.2 \leq p \leq 1 \end{cases}$$

In view of the value of the cost of uncertainty in Example 15.3 when the new forecasting technology is not available, the expected benefit $E[B]$ from the new forecasting technology can be computed according to Eq. (15.10) as follows:

1. For $0 \leq p \leq 0.1$,
$$E[B] = 54{,}000p - 24{,}000p = 30{,}000p$$
2. For $0.1 \leq p \leq 0.2$,
$$E[B] = (6{,}000)(1 - p) - 24{,}000p = 6{,}000 - 30{,}000p$$
3. For $0.2 \leq p \leq 1$,
$$E[B] = (6{,}000)(1 - p) - (6{,}000)(1 - p) = 0$$

The relationship between $E[B]$ and p is shown in Fig. 15.9. It can be seen that the value of the information obtained from the new forecasting system is dependent upon p, which is a measure of the statistical regularity of the occurrence of hurricane strikes based on the historical record.

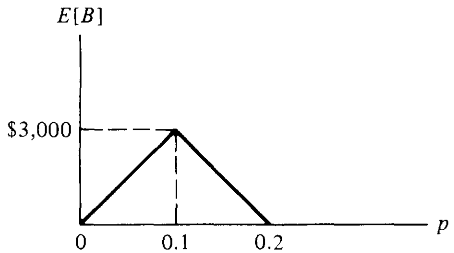

Figure 15.9 Expected benefit from use of new information.

15.8 GAMBLER'S INDIFFERENCE MAP

When we encounter problems involving random variables that do not exhibit the property of independent and identical probability distributions except for different expected values, the expected value decision criterion may not be valid. For example, the probability distributions of the two random variables X and Y shown in Fig. 5.10 have different means and variances. While X has a larger expected value, it also has larger variability, whereas Y has a smaller expected value and small variability. Hence, a risk-averse decision maker who fears very low outcomes of NPV associated with X, no matter how unlikely, might prefer Y despite the fact that the expected NPV of Y is lower. Under such circumstances, the consequences of individual decisions must be considered in the light of these probability distributions.

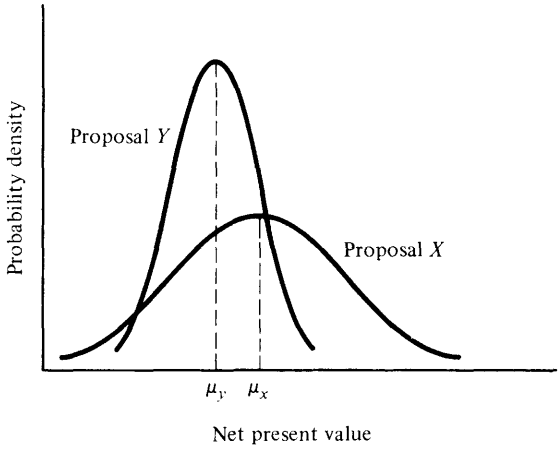

Figure 15.10 Means and variances of two proposals.

In situations where each of the probability distributions is completely specified by its mean μ and variance σ^2, the merit measure R for each alternative as a decision process can be expressed in some function of μ and σ. That is, each alternative can be characterized by a point (μ, σ) for some function expressed in the form

$$R = f(\mu, \sigma) \tag{15.11}$$

Then, the decision can be made by ranking the R values resulting from the specified sets of values of (μ, σ) for various alternatives. The decision criterion is either to maximize net benefit or to minimize the cost, as the case may be.

Each function for R can be represented by a curve in a two-dimensional space, and such a curve is referred to as an *indifference curve* because all points on the curve are equally appealing on the basis of the merit measure R. A set of indifference curves representing different values of R is called a *gambler's indifference map*.

Example 15.5[4]

Suppose that a commuter can travel to work each morning by either one of two alternative routes. From previous experience, he has found that his travel times for the two routes are normally distributed, the first with a mean and a standard deviation of $\mu_1 = 40$ minutes and $\sigma_1 = 8$ minutes, and the second with a mean of $\mu_2 = 45$ minutes and a standard deviation $\sigma_2 = 2$ minutes. Thus, for route 1, the mean travel time is smaller but the larger standard deviation indicates a much larger variation in travel time from day to day.

The importance of the additional uncertainty for route 1 would depend on the nature of the commuter's job. Suppose that he is a self-employed business man who is not required to start work at any particular time each day. This being the case, he would have little concern for the day-to-day variation in travel time and would choose route 1 because of the smaller mean value.

However, if the commuter is employed at a somewhat more restrictive job which requires that he be at work every morning at 8.00 A.M., the uncertainty in travel time as expressed by the variance would be important. Since late arrivals might result in his being fired, he must start early enough in the morning to be reasonably sure that he will arrive at work by 8.00 A.M. Consequently, he wishes to choose the alternative that permits him to leave at the latest possible time and still arrive at work by 8.00 A.M.

Suppose that, so long as he is on time at least 99% of the time, he is assured of keeping his job. We find from Appendix E for the standard normal distribution that $z = 2.33$ for $N(z) = 0.99$; hence, the time that he must allow to get to work is given by $R = \mu + 2.33\sigma$, from which we obtain $\sigma = 0.433(R - \mu)$. If he takes route 1,

$$R_1 = \mu_1 + 2.33\sigma_1 = 59 \text{ min}$$

whereas if he takes route 2, because of the smaller variance,

$$R_2 = \mu_2 + 2.33\sigma_2 = 50 \text{ min}$$

Thus, even though the mean travel time for route 1 is smaller, the larger variance makes alternative 2 the more attractive choice. The commuter in this case sacrifices additional travel time over the long run to ensure that he is more likely to be at work on time. In effect, he is paying a premium in travel time to "buy insurance" against being fired from his job.

The gambler's indifference map shown in Fig.15.11 is for the commuter problem just described, each line corresponding to a particular value of R, with μ and σ being related by the expression $\sigma = 0.433(R - \mu)$. As we move

[4]This example, including Fig. 15.11, is reproduced with permission from Au, T., R. M. Shane, and L. A. Hoel, *Fundamentals of Systems Engineering, Probabilistic Models*, pp. 186–187. Reading, MA: Addison-Wesley, 1972.

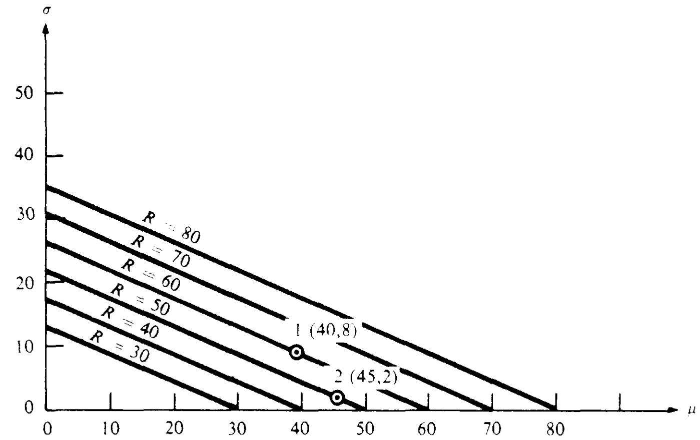

Figure 15.11 Gambler's indifference map.

along any one of these lines, the reduction in the standard deviation σ is just sufficient to offset the increase in the mean μ and produce the same value of R. Eventually each line intersects the μ axis. At this point, $\sigma = 0$ and the corresponding value of μ is called the "certainty equivalent" for all points along the line. That is, for $\sigma = 0$, the travel time is known exactly and is equal to $\mu = R$. The commuter then would be indifferent between an alternative with a travel time R without day-to-day variation and any other alternative along the indifference line, since he has to be R minutes early in leaving home anyway. The two alternatives discussed above are indicated on the map in Fig. 15.11. Alternative 2, of course, falls on an indifference line corresponding to a smaller value of R; hence, it is clearly the better alternative.

15.9 APPLICATION OF CERTAINTY EQUIVALENT CONCEPT TO MULTIPERIOD INVESTMENTS

In Section 15.4, we stated that for a multiperiod investment, the discount rate is usually selected as a matter of policy while the cash flow in each period may be random variables. Then, an attempt may be made to determine the "certainty equivalent" of the random cash flow in each period. Let Y_t (for $t = 1, 2, \ldots, n$) be a stream of uncertain returns for an initial investment Y_0. The idea is that at a given period t, the decision maker is willing to exchange an uncertain return with several possible outcomes for a certain return of a_t, Y_t, where a_t is a factor less than one. If we know how to determine a_t (for $t = 1, 2, \ldots, n$), then we can essentially remove the uncertainty and use the risk-free rate r_f for discounting. That is,

$$[\text{NPV}]_{r_f} = Y_0 + \frac{a_1 Y_1}{1 + r_f} + \frac{a_2 Y_2}{(1 + r_f)^2} + \frac{a_n Y_n}{(1 + r_f)^n} \qquad (15.12)$$

The difficulty in using this approach is that the factors a_t (for $t = 1, 2, \ldots, n$) cannot be determined easily except for relatively simple situations.

The concept of certainty equivalent can also be formulated by considering the problem of cost minimization when the benefits of various alternatives are identical. Let X_t (for $t = 1, 2, \ldots, n$) be the cost profile of uncertain future expenses for an initial investment X_0. Then,

$$[\text{CPV}]_{r_f} = X_0 + \frac{b_1 X_1}{1 + r_f} + \frac{b_2 X_2}{(1 + r_f)^2} + \cdots + \frac{b_n X_n}{(1 + r_f)^n} \tag{15.13}$$

where b_t is a factor greater than one for $t = 1, 2, \ldots, n$. That is, the decision maker is willing to pay a higher cost $b_t X_t$ for a certain benefit in lieu of an uncertain benefit with several possible outcomes.

15.10 OPERATING RISK VERSUS FINANCIAL RISK

So far we have emphasized only the problems of uncertainty and risk related to operation or production. The acceptance of a capital project and its concomitant financing plan also incurs financial risk. The discussion of uncertainty and risk would be amiss without also some understanding of financial risk. Furthermore, since the application of the concept of certainty equivalent is quite restricted, an attempt is made to assess approximately the operating risk of a project through the study of similar projects undertaken by firms whose securities are traded publicly in financial markets.

There are striking parallels in the treatment of uncertainty and risk in operation and in financing. For example, the concept of risk diversification and the expected value decision criterion are useful in making financial investments. The value of information is recognized all too well in the financing community as evidenced by the large number of published financial news and reports. The advantage of insider information for reducing uncertainty is so overwhelming that this form of dissemination is branded a crime!

However, it is important to distinguish the risk of a project from that of a firm. Project risks must be determined individually by considering all relevant factors. For a firm whose securities are traded publicly in capital markets, risks of the firm can be observed through the collective action of investors. However, the risk of a firm's common stock is influenced by its mix of equity and debt. The use of debt is said to create a *financial leverage*. Hence, it is necessary to know the asset value and the capital structure of a firm in order to separate its operating risk from financing risk.

The market value of a firm is represented by the package consisting of its assets, liabilities, and stockholders' equity. When a firm accepts a new project, the action changes the value of the firm since new value is created by the potential net benefit of the new project. This change will be reflected in the market price of the firm over the long run when it is recognized by the investors.

We can look at the market value of a firm as if we were looking at both sides of the balance sheet of the firm. This is purely an analogy since the book value in accounting is different from the market value of the firm. However, such an analogy is desirable in order to examine some of the most important issues in corporate finance. On the left side of the balance sheet is a collection of assets from which the value of the firm can be determined, regardless of the nature of the claims against it. On the right side is a mix of debt and equity which represents the capital structure of the firm and the claims of its debtholders and stockholders.

Let us first consider the market value of the firm from its assets without worrying about where the money for acquiring these assets comes from. Then, the market value of a firm can be represented by the net present value of a stream of expected cash flows which may be negative or positive over the planning horizon of the firm. The net present value of the firm can also be obtained by summing up the net present values of the expected cash flow profiles of all ongoing projects over the same planning horizon, plus the present value of intangible assets if applicable. Intangible assets may include items such as patent rights or well-recognized management talents which are sought after in the market for a price.

On the other hand, the stream of expected cash flows of a firm may be thought of as being split into two streams on the basis of debt and equity. Thus, the stream of expected cash flows for debt will be claimed by the debtholders, and the stream for equity will be claimed by the stockholders. By borrowing money, a firm can acquire more assets and thus increase its value. Since the debtholders have the first claim to the assets in case of financial distress, the expected rate of return to the debtholders is smaller than that for the shareholders because of the difference in their risks. In addition, the interest payment on debt is tax deductible, an advantage resulting from debt financing that is referred to as the *tax shield*. Thus, the expected rate of return for the stockholders increases as the firm borrows more money to acquire profitable assets, provided that the firm does not encounter any costly financial distress which will negate the increase in profit from operation.

15.11 RISK MEASUREMENTS IN CAPITAL MARKETS

The risk characteristics of a collection of financial assets may be examined in the context of expected returns. The collection of financial assets amassed by an investor is referred to as an *investment portfolio*. Investors who set prices in capital markets will take risky investments only if they are compensated for the risk they bear. Consequently, the greater the perceived risk, the higher must be the expected return. The collective action of investors in capital markets provides a standard in measuring the trade-off between expected return and risk.

For example, if an investor puts all his or her money in default-free government bonds, the expected return from the portfolio can be regarded as risk-free. On the other hand, if the investor puts the money in the stock of only one firm, the portfolio reflects both the expected return and risk associated with the stock of this firm. Generally, an investor chooses a variety of financial assets in the portfolio for the purpose of diversification.

The risk of a portfolio is measured by the standard deviation of its expected return. It is useful to break down the risk of a portfolio into *unsystematic* risk and *systematic* risk. Unsystematic risk is unique to a particular security and can be reduced by holding a large number of diversified securities. Systematic risk is related to the overall market risk caused by factors such as changes in the state of economy and energy prices, since most securities are affected by these factors to some degree. Since the unsystematic risk of a security can be reduced through diversification of the holdings in the portfolio, only the systematic risk is significant in the analysis of the risk characteristics of a security.

Suppose that an investor includes in a portfolio every common stock or its equivalent available in the market and that this investor buys the same proportion of each stock based on its current market value. Then the resulting portfolio will be a miniature of the capital market and is referred to as the *market portfolio*. For example, a portfolio which includes all of the stocks in the Standard and Poor's 500 stock average and in the same proportion is a close approximation of the market portfolio. Because of the diversification of the market portfolio, unsystematic risk has been reduced to a minimum and only systematic risk remains.

For a risk-free portfolio consisting entirely of default-free government bonds, there is no systematic risk associated with its expected return. For the market portfolio, the systematic risk associated with its expected return is measured by its standard deviation and is referred to as the *market risk*. The market risk is regarded as the standard against which the systematic risk of an individual security or portfolio is measured.

The systematic risk of a security can be compared to the systematic risk of the market portfolio by examining available historical data. For example, the rates of return based on Standard and Poor's 500 stock index versus those for a security over a large number of time periods may be plotted against each other as shown schematically in Fig. 15.12. Then, a theoretical line, referred to as the *regression line,* can be fitted to the scattered points by statistical techniques. The dispersion of the scattered points about the regression line is a measure of the unsystematic risk of the se-

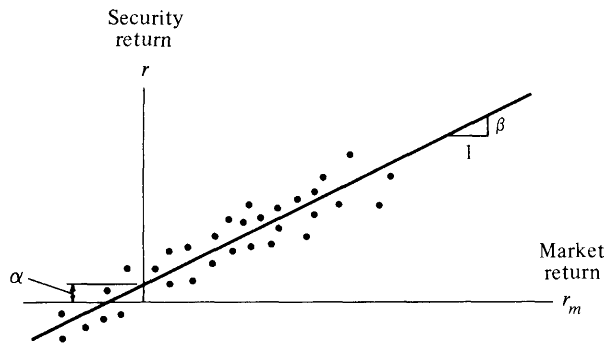

Figure 15.12 Determination of beta coefficient of a security.

curity. The intercept of the regression line with the vertical axis, denoted by α, is the average value of the rates of return of the security due to unsystematic risk over time and in theory tends to approach zero. The slope of the regression line, denoted by β, is called the *beta coefficient*, which represents the systematic risk of a security *relative to* that of the market portfolio. A value of $\beta = 1$ indicates that both have the same risk, $\beta > 1$ indicates that the security is riskier, and $\beta < 1$ indicates that it is less risky.

The type of stock with $\beta > 1$ is often referred to as an aggressive investment, and the type with $\beta < 1$ is called a defensive investment. Several financial organizations regularly publish beta values for actively traded stocks in the financial markets.[5] The beta coefficients of most stocks range from 0.7 to 1.4, although they may fall outside of this range for some stocks.

15.12 THE CAPITAL ASSET PRICING MODEL

Since securities can readily be exchanged in organized capital markets with relatively small transaction costs, there is clearly a criterion based on which values of the securities are evaluated by the financial community. Aside from short-run fluctuations of the prices of stocks and bonds which can be caused by a variety of factors, the value of a security is recognized and reflected by its market price in a long run. Given the risk-averse attitude of most investors in a competitive financial market, the price of a security reflects both its expected return and its risk.

The relationship between the expected return and the systematic risk of a security can be established by the *capital asset pricing model* (CAPM) which is based on simple economic principles.[6] The underlying theory is that assets with the same risk should have the same rate of return. Thus, the prices of assets in the capital markets should be adjusted until equivalent risk assets have identical expected returns. We shall not examine the theoretical basis of the CAPM but will only discuss briefly its practical implications.

The capital asset pricing model states that the expected rate of return on a portfolio should exceed the risk-free rate by an amount which is proportional to the beta coefficient of the portfolio. Supposing that an investor selects only a single security, the relationship between return and risk is given by

$$r_e = r_f + \beta_e(r_m - r_f) \tag{15.14}$$

where r_f = the risk-free rate of return
r_m = the expected rate of return on the market portfolio
r_e = the expected rate of return on a security
β_e = the beta coefficient for that security

[5] The beta values for actively traded stocks are published weekly in the *Value Line Investment Survey* by Value Line, Inc.

[6] This was developed during the mid 1960s simultaneously by several researchers. See Refs. 15.6 and 15.11 for their original contributions.

The linear relationship between r_e and β_e in Eq. (15.14) is referred to as the *security market line,* which is graphically represented in Fig. 15.13. It is noted that $r_e = r_m$ for $\beta_e = 1$, and that $r_e > r_m$ for $\beta_e > 1$, and $r_e < r_m$ for $\beta_e < 1$. The factor $(r_m - r_f)$ represents the market's return-risk trade-off rate in determining the relative risk.

The expected rate of return r_e for a security in Eq. (15.14) may be interpreted as the sum of a risk-free rate r_f reflecting the time value of money and a risk component r_p representing an adjustment of the rate of return for the risk of the security. That is,

$$r_e = r_f + r_p \tag{15.15}$$

where r_e is referred to as the *risk-adjusted rate of return* for a security, and r_p is called the *risk premium.*

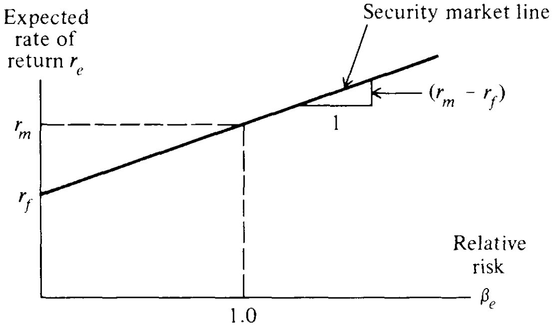

Figure 15.13 Security market line.

In comparing Eq. (15.15) with Eq. (15.14), it is obvious that the risk premium is given by

$$r_p = \beta_e(r_m - r_f) \tag{15.16}$$

The beta coefficient for a stock or equity is designated as β_e in Eqs. (15.14) and (15.16) to emphasize that the stock of a company carries its operating and financial risks as reflected by its mix of equity and debt. It is referred to as the *levered beta* of the company since the debt provides a financial leverage to the company's assets. If a company's assets are all financed by equity, the beta coefficient is designated by β_a and is referred to as the *unlevered beta.* Because of the limited data and simplifying assumptions used to compute a company's levered beta β_e, it is important to recognize the approximate nature of the CAPM.

Note that the systematic risk β of a security relative to that of the market portfolio is the value of the slope of the regression line which can best be fitted to the data points over a large number of time periods. Suppose that the points in Fig. 15.12 represent monthly data compiled over a period of 5 years. How would this beta value compare with another one obtained by using twice as many points over a period of 10 years? The answer depends on a number of factors. If a company is engaged in the same type of business with the same systematic risk (both operating and financial risks) over the 10-year period from which the data were taken, the beta value tends to be more reliable as the number of data points increases. On the other hand, if a company is engaged in one type of business using a specific capital structure in the first 5 years but suddenly changes its business risk characteristics and/or its capital structure in the second 5 years, then the data for the two periods should not even be mixed together. In other words, a beta value should be determined for each of the two 5-year periods to reflect different risk characteristics of the company during these two periods. What happens if the company changes its risk characteristics gradually by replacing its assets periodically? In that case, not only the beta value representing the slope of the regression line will change as new monthly data are added to replace the oldest ones, but the intercept of the regression line on the vertical axis also will not tend to be zero. This intercept, denoted by alpha (α), represents the rate of price appreciation of the security per time period of change, i.e., per month if the regression line is recomputed monthly. A positive alpha value indicates the rate of price appreciation earned by the holders of this security while the investors in the market portfolio receive no price appreciation at all. In general, a large alpha value indicates that the beta value of the company is unstable during the period.

There are many types of debt instruments by which a company can borrow money. Not all of these debt instruments are risk-free, and their expected rates of return are different. If the expected rates of return of U.S. Treasury bills is regarded as the risk-free rate r_f, then the expected rates of return for other types of debt instruments, such as high-grade corporate bonds, are expected to be higher. However, because the differences between such rates and the risk-free rates are small, we often speak of the debts of a company in a general sense and treat all debts as risk-free for the sake of simplicity.

Example 15.6

The expected rate of return on U.S. Treasury bills is currently 10%, and the expected rate of return on the market portfolio is 14%. We are interested in following two stocks which have beta coefficients as indicated:

1. An airline stock with $\beta = 1.5$
2. A gold-mining stock with $\beta = 0.5$

Determine the expected rate of return for each of these stocks.

The expected rate of return for each stock can be obtained from Eq. (15.14) by noting that $r_f = 10\%$ and $r_m = 14\%$. For case 1

$$r_e = 10\% + (1.5)(14\% - 10\%) = 16\%$$

For case 2

$$r_e = 10\% + (0.5)(14\% - 10\%) = 12\%$$

Since case 1 represents an aggressive stock with a high systematic risk, $r_e = 16\%$ is higher than that for the market portfolio. Conversely, because case 2 is a defensive stock with a low systematic risk, $r_e = 12\%$ is lower than that for the market portfolio.

15.13 DETERMINATION OF ASSET RISK OF A FIRM

The systematic risk β_e of a security reflects the operating risk of the real assets held by a firm that usually is levered by its debts issued to finance its capital investments. For this discussion, the financial structure of a firm as represented by its balance sheet can be simplified as shown in Table 15.4.

TABLE 15.4 A SIMPLIFIED VIEW OF THE FINANCIAL STRUCTURE OF A FIRM

Asset value (working capital & fixed assets)	Debt value (D) Equity value (E)
Total asset value	Total firm value (V)

Let β_a be the beta coefficient of a firm's assets corresponding to all-equity financing and thus reflecting only the operating risk, and let β_d be the beta coefficient reflecting the risk of the firm's financing instrument. Suppose that you own this firm by acquiring 100% of its equity and 100% of its debt. Ignoring the effects of corporate income taxes, the asset risk of the firm is given by

$$\beta_a = \beta_e\left(\frac{E}{E + D}\right) + \beta_d\left(\frac{D}{E + D}\right) \tag{15.17}$$

Solving for β_e from Eq. (15.17), we get

$$\beta_e = \beta_a + (\beta_a - \beta_d)D/E \tag{15.18}$$

Since the interest payment on debt is tax deductible, the effect of an income tax rate X has the effect of reducing the size of debt subject to risk from D to $D(1 - X)$. Substituting $D(1 - X)$ for D in Eqs. (15.17) and (15.18) and simplifying, we have

$$\beta_a = \beta_e\left[\frac{1}{1 + (1 - X)D/E}\right] + \beta_d\left[\frac{(1 - X)D/E}{1 + (1 - X)D/E}\right] \tag{15.19}$$

and

$$\beta_e = \beta_a + (\beta_a - \beta_d)(1 - X)(D/E) \tag{15.20}$$

where D/E is referred to as the *debt-to-equity ratio*.

For well-established companies, the financial risk on their debt instruments are often very low, and can be assumed to be risk-free, i.e., $\beta_d = 0$. For example, long-term high-grade corporate bonds historically have only slightly higher risks than those of long-term government bonds. Then, Eqs. (15.17) through (15.20) can be simplified by letting $\beta_d = 0$. This does not apply to low-grade corporate bonds, which are popularly referred to as *junk bonds*.

Since the capital asset pricing model is based on a single-period investment, many attempts have been made to apply the methodology to long-lived capital assets.[7] It is possible to derive suitable formulas for the evaluation of capital investment projects on the basis of the CAPM, provided that further assumptions are introduced. We will not consider the mathematical derivations of these approaches, but will discuss only their implications on the evaluation criteria.

In a world of certainty with perfect capital markets, the criteria for evaluating investment projects are well understood. In a world of uncertainty, it is plausible enough to replace the known cash flows with expected cash flows and to add a risk premium to the discount rate, but these modifications lack vigorous support.

Nevertheless, we will follow through the concept of CAPM to see how it can be applied to project evaluation. Given the nature of the operation or production for a proposed project, we can find the average industry-wide unlevered beta, β_a, in the risk class corresponding to this project. Since $(r_m - r_f)$ in Eq. (15.12) depends only on the market rate of return and the risk-free rate, it can be applied to any firm whether it is levered or unlevered. Hence, the expected rate of return of an unlevered firm is

$$r_a = r_f + \beta_a(r_m - r_f) \tag{15.21}$$

If the uncertainty is realistically tied to a constant rate over time only, then the risk-adjusted rate of return r_a for an all-equity financed project can be used as the discount rate to take into account the effects of time and operating risk on the present value of a stream of uncertain returns. Let Y_t (for $t = 1, 2, \ldots, n$) be a stream of expected cash flows of an investment; then

$$[\text{NPV}]_{r_a} = Y_0 + \frac{Y_1}{1 + r_a} + \frac{Y_2}{(1 + r_a)^2} + \cdots + \frac{Y_n}{(1 + r_a)^n} \tag{15.22}$$

In this approach, it is assumed that our estimates of cash flows become progressively more uncertain as we look further into the future. Since we have no other knowledge about the uncertainty, this approach is reasonable although its outlook is conservative.

It can easily be seen that Eq. (15.22) may be regarded as a special case of Eq. (15.12) for the certainty equivalent approach in Section 15.9 if

$$a_t = \left(\frac{1 + r_f}{1 + r_a}\right)^t \tag{15.23}$$

for $t = 1, 2, \ldots, n$. In other words, if the factor a_t becomes progressively smaller

[7] See Refs. 15.8 and 15.9 for detailed discussions.

according to power t of the ratio $(1 + r_f)$ to $(1 + r_a)$, then Eq. (15.12) will be reduced to Eq. (15.22). Thus, when we use the risk-adjusted discount rate r_a, we are implicitly making a special assumption about a_t as stated in Eq. (15.23).

It has been clearly demonstrated that the CAPM approach and the certainty equivalent approach can each be applied to specific situations. If the operating characteristics representing the certainty equivalent factors of a production process over the life of the project can be predicted in advance, the certainty equivalent approach is desirable since time and risk can be treated as separate parameters. However, when the certainty equivalent factors cannot be realistically determined, the CAPM approach provides at least an approximation in assessing the effects of operating risk on project evaluation.

Example 15.7

The Kantaka Company has a debt-to-equity ratio of 0.2 and the beta of the security is 1.1. The long-term bonds of this company can be assumed to be risk-free. Find the unlevered beta coefficient of this company if (a) income taxes are neglected and (b) an income tax rate of 34% is included.

Since $\beta_d = 0$ for this company, the unlevered beta coefficient can be computed by noting $\beta_e = 1.1$ and $D/E = 0.2$. (a) From Eq. (15.17),

$$\beta_a = \beta_e\left(\frac{1}{1 + D/E}\right) = (1.1)\left(\frac{1}{1 + 0.2}\right) = 0.917$$

(b) From Eq. (15.17),

$$\beta_a = (1.1)\left[\frac{1}{1 + (1 - 0.34)(0.2)}\right] = 0.972$$

Example 15.8

Suppose that the market risk-free rate r_f in 1987 is 7.5%, and the differential rate $(r_m - r_f)$ may be assumed to be 8.3%. Determine the unlevered risk-adjusted rate of return for the Kantaka Company in Example 15.7.

Since the unlevered beta is found to be $\beta_a = 0.917$ when income taxes are neglected and $\beta_a = 0.972$ when an income tax rate of 34% is included, we find for the case of neglecting income taxes,

$$r_a = 7.5 + (0.917)(8.3) = 15.1\%$$

and for the case including an income tax rate of 34%,

$$r_a = 7.5 + (0.972)(8.3) = 15.7\%$$

Example 15.9

Suppose that an investment project is to be discounted by a risk-adjusted rate of $r_a = 15\%$ for its operating cash flow profile according to CAPM. If the risk-free rate is $r_f = 6.2\%$, what is the equivalent factor a_t in computing in certainty equivalent so that the results from both approaches will be identical?

Using Eq. (15.23), we obtain for $t = 1, 2, \ldots, n$,

$$a_t = \left(\frac{1 + 0.062}{1 + 0.05}\right)^t = (0.9235)^t$$

Note that $a_1 = 0.9235$ (<1) and the subsequent values of a_t will decrease as t increases.

15.14 ESTIMATION OF MARKET EXPECTED RATES OF RETURN

The capital asset pricing model provides the theoretical framework for the analysis of historical data on market expected rates of return. However, the use of historical information requires proper interpretation. The market rates of return for portfolios of common stocks, high-grade bonds, and other securities over long periods of time have been collected and organized in useful form by Ibbotson and Sinquefield.[8] Some of the data are summarized in Tables 15.5 through 15.7.

In these tables, the arithmetic mean $\bar{r}$ of annual returns is the simple average of annual rate r_t over a period of n years. That is,

$$\bar{r} = \frac{1}{n}\sum_{t=1}^{n} r_t \tag{15.24}$$

where $t = 1$ refers to year 1926 and $t = n$ refers to year 1987 with $n = 62$ years. The standard deviation is given by

$$\sigma_r = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n} (r_t - \bar{r})^2} \tag{15.25}$$

TABLE 15.5 ANNUAL RETURNS OF FINANCIAL ASSET CLASSES (1926–1987)

Financial asset classes	Arithmetic mean	Standard deviation
Common stocks	12.0%	21.1%
Long-term corporate bonds	5.2%	8.5%
Long-term government bonds	4.6%	8.5%
U.S. Treasury bills	3.5%	3.4%
Inflation rates	3.2%	4.8%

Source: Ibbotson, R. G., and R. A. Sinquefield, *Stocks, Bonds, Bills and Inflation: Historical Returns 1926–1987*. Charlottesville, VA: Financial Analysts' Research Foundation, 1989.

[8] See Ref. 15.5, especially pp. 69–78.

TABLE 15.6 RISK PREMIUM OF ANNUAL RETURNS (1926–1987)

Risk premium	Arithmetic mean	Standard deviation
Equity risk premiums (stocks − T bills)	8.3%	21.1%
Default premiums (LT Corp. − LT govt. bonds)	0.7%	3.0%
Horizon premiums (LT govt. bonds − T bills)	1.1%	8.0%

Source: Ibbotson, R. G., and R. A. Sinquefield, *Stocks, Bonds, Bills and Inflation: Historical Returns 1926–1987.* Charlottesville, VA: Financial Analysts' Research Foundation, 1989.

TABLE 15.7 INFLATION-ADJUSTED ANNUAL RETURN (1926–1987)

Financial asset classes	Arithmetic mean	Standard deviation
Common stocks	8.8%	21.2%
Long-term corporate bonds	2.3%	10.0%
Long-term government bonds	1.7%	10.2%
U.S. Treasury bills	0.5%	4.4%

Source: Ibbotson, R. G., and R. A. Sinquefield, *Stocks, Bonds, Bills and Inflation: Historical Returns 1926–1987.* Charlottesville, VA: Financial Analysts' Research Foundation, 1989.

The annual returns of financial asset classes in Table 15.5 include an inflation component, while those in Table 15.7 are inflation adjusted. The risk premiums in Table 15.6 are based on the annual returns in Table 15.5. While the equity risk premiums between common stocks and U.S. Treasury bills in Table 15.6 are significant, other premiums between long-term corporation bonds and long-term government bonds, or between long-term government bonds and U.S. Treasury bills are quite small. These figures seem to justify the common practice of assuming all debt instruments, with the exception of low-grade corporate bonds, to be risk-free. It should be noted that if the risk premiums are computed from the inflation-adjusted annual returns in Table 15.7, the results will be essentially the same as those in Table 15.6. Since the risk premiums are obtained from the difference in returns from two financial asset classes, the inflation component will be canceled out during the subtraction.

Example 15.10

Using the equity risk premium (r_m − r_f269 8.3% as obtained from the arithmetic mean in Table 15.6, find the expected rate of return from the stock of Kantaka Company in Example 15.7 if the r_f in 1987 is 7.5% and the income tax rate is 34%.

Since $\beta_e = 1.1$ for the security of the company, and $X = 34\%$, then, for $r_f = 7.5\%$ and $(r_m - r_f) = 8.3\%$, we obtain from Eq. (15.12),

$$r_e = 7.5\% + (1.1)(8.3\%) = 16.63\%$$

15.15 SUMMARY AND STUDY GUIDE

In this chapter, we have discussed the concepts of uncertainty and risk and how they affect investment decisions. We have introduced a bare minimum of terms in probability and statistics and have avoided the mathematical proofs of various methods and procedures in dealing with uncertainty and risk. However, we have placed great emphasis on the development of useful insights through the logical presentation of topics leading to the resolution of the dilemma of uncertainty and risk confronting a decision maker.

We have examined individual attitudes toward uncertainty and risk, and possible actions to be taken so that the unsystematic risk is diversified while only the systematic risk remains. We introduced the use of the decision tree and the expected value decision criterion where applicable. We discussed the meaning of regret or opportunity loss under uncertainty and the value of information for reducing uncertainty.

In dealing with problems involving probability distributionss that do not exhibit the property of stochastic dominance, the expected value decision criterion is no longer valid. In some situations where each of the probability distributions representing a random variable can be completely specified by its mean and variance, we introduced the concept of indifference curves resulting from the specified sets of mean and variance values for various alternatives.

Then, we turned to the risk measurements in capital markets because they are of great importance in laying the groundwork for determining the minimum attractive rate of return in project evaluation. We introduced the capital asset pricing model (CAPM) for assessing systematic risk and examined the risk premium for both operating risk and financial risk. Using CAPM as a theoretical framework, we considered the estimation of expected market rates of returns of securities of companies and the various problems associated with the estimation because of the approximate nature of the CAPM. Finally, we considered the application of the idea of the CAPM and the "certainty equivalent" approach to the evaluation of capital projects.

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PROBLEMS

P15.1 Suppose that four possible outcomes of the uncertain return to an investment are \$5,000, \$6,000, \$7,000, and \$9,000, with the probabilities of their occurrence being 0.2, 0.3, 0.3, and 0.2, respectively. Determine the expected value and the standard deviation of the return.

P15.2 The possible damages to an area due to seasonal flooding are estimated to be \$8,000, \$4,000, and \$3,000 with probabilities of 0.2, 0.3, and 0.5, respectively. Find the expected value and the standard deviation of the damage.

P15.3 A contractor having a sizable capital, equipment, and labor force is confronted with the decision whether it should submit a bid for a new construction project. In order to have a chance of getting the contract for this project, it must submit a detailed bid document that costs C_1 dollars. If it submits a bid and wins the contract, it will make a net profit of C_2 after deducting the bid cost as well as other costs. If it submits a bid and loses the contract, it will suffer a loss by incurring the bidding cost.

On the other hand, if it makes no attempt to bid for the contract, it will neither gain a profit nor incur a loss. Construct a decision tree for this situation.

P15.4 A stretch of the parkway in the downtown area of a city is located on a flood plain. Each spring, there is the possibility that a flood may force the closing of the parkway for several days, and the costs of inconvenience to the travelers are estimated to be \$100,000 when it occurs. If a protective wall is built, the probability of flooding for this stretch of parkway is reduced from 0.4 to 0.1, and the costs will also be reduced to \$60,000 because a smaller area will be involved. The amortized uniform annual cost for building the protective wall is \$10,000. Draw the decision tree for this situation and determine whether the protective wall should be built on the basis of the expected value criterion.

P15.5 A contractor specializing in the construction of highway bridges is confronted repeatedly with the problem of building a cofferdam for the protection of the working space at sites for bridge piers at river crossings. If it builds a very high cofferdam at an amortized uniform annual cost of \$5,000, it is certain that there will be no possibility of flooding during the construction period. If it builds a low cofferdam at an amortized uniform annual cost of \$4,000, flooding may occur with a probability p; and the total annual cost including the original construction and the restoration of the damaged site to working conditions due to flooding is \$14,000.

(a) Using an expected value decision criterion, determine the probability p such that the contractor will be indifferent between building a high cofferdam and a low cofferdam.

(b) If perfect information about floods is available before the construction of each cofferdam so that the state of flooding for the low cofferdam can be ascertained in advance, what is the expected value of the benefit of this information?

P15.6 A farmer is planning to plant one of two crops in his field in the next year. The cost of planting the first crop is \$3,000 per acre and the revenue depends on the chance outcome of three possibilities: 50% chance of getting \$8,200, 40% chance of getting \$7,300 and 10% chance of getting \$1,400. On the other hand, the cost of planting the second crop is \$2,500 per acre and the revenue per acre depends on the chance outcome of another three possibilities: 60% chance of getting \$8,000, 20% chance of getting \$9,700 and 20% chance of getting \$400. If the expected net present value criterion is applicable, which alternative should be the choice?

P15.7 A highway contractor maintains \$250,000 worth of heavy equipment, which he uses during the warm weather portion of the year. During the winter months, the equipment is to be stored in one of two locations. One storage area is located on the flood plain of a stream, which costs him \$10,000 to store the equipment for the season, but may cause total damage to his equipment if it is overrun by flood. The other storage area can be rented for \$20,000 for the season, and cannot be reached by the flood.

(a) Using the expected value criterion, determine the probability of occurrence of a flood on the flood plain such that the contractor will be indifferent in choosing either area. Draw the decision tree and the graph showing the expected value versus flood probability.

(b) Find the opportunity loss for each combination of decision and outcome. Also determine the cost of uncertainty.

(c) Suppose that the contractor can remove \$50,000 worth of equipment if he is warned of an impending flood 3 hours in advance. Determine the expected benefits derived from the flood warning system. Show a graph of the expected benefits versus flood probability.

P15.8 A concrete supplier for a large building construction project in an urban area promises to deliver fresh concrete on time every morning over a period of 6 months. The supplier can select one of the two routes for the delivery truck to travel from the mixing plant to the construction site. From available traffic information, the travel times for the two routes are normally distributed, the first with a mean $\mu_1 = 60$ minutes and a standard deviation $\sigma_1 = 16$ minutes, and the second with a mean $\mu_2 = 70$ minutes and a standard deviation $\sigma_2 = 4$ minutes. As long as the delivery is on time at least 95% of the time, there will be no penalty ($z = 1.65$ for $N(z) = 0.95$ using the standardized normal distribution in Appendix E). The supplier wants to avoid the penalty by scheduling the delivery truck to leave the mixing plant with sufficient lead times.

(a) Determine the lead time required for each of the two routes.

(b) Construct a "gambler's indifference map" corresponding to the lead times of 60, 70, 80, and 90 minutes.

P15.9 If the expected rate of return on U.S. Treasury bills is currently 8.2% and the expected rate of return on the market portfolio is 15%, determine the expected rate of a security that has a beta coefficient of 1.2.

P15.10 The security of Alfred Corporation has a beta coefficient of 1.2, which reflects a debt-to-equity ratio of 0.4. The tax rate of the company is 36%. Suppose the market risk-free rate r_f is 8%, and the differential rate of the market portfolio and the risk-free bonds $(r_m - r_f)$ is 8.3%. Determine the unlevered beta and the unlevered risk-adjusted rate of return of the company.

P15.11 The Benjamin Company has a debt-to-equity ratio of 0.25 and has a company beta value of 1.15 according to the latest published reports. The market risk-free rate is 8%, and the differential rate of the market portfolio and the risk-free securities $(r_m - r_f)$ is 8.3%. Assuming perfect capital markets and a tax rate of 34% for the company, determine the expected rate of return on the stock of this levered firm and its unlevered risk-adjusted rate of return.

P15.12 The cash flow profiles to two mutually exclusive proposals are given below. The certainty equivalent factors for proposal 1 are $a_1 = 0.95$, $a_2 = 0.91$, and $a_3 = 0.86$, and those for proposal 1 are $a_1 = 0.97$, $a_2 = 0.95$, and $a_3 = 0.90$. If the risk-free rate is 8%, determine which proposal should be selected.

Year	Proposal 1		Proposal 2	
t	$B_{t,1}$	$C_{t,1}$	$B_{t,2}$	$C_{t,2}$
0	0	\$15,000	0	\$14,000
1	\$8,000	1,500	\$8,000	2,000
2	8,500	2,000	8,500	2,500
3	9,000	2,500	9,000	3,000

CHAPTER

16 Capital Investment Decisions in Private Firms

16.1 IMPORTANT FACTORS FOR CONSIDERATION

In the evaluation of investment projects in the private sector, it is tacitly assumed that if a project is economically worthwhile, the project will contribute to the value of a firm that invests in it. However, the acceptance of an investment project and its concomitant financing plan may also change the financial structure of the firm. From the standpoint of the stockholders of a firm, the crucial question for accepting a proposed project is whether the project, as financed, will increase the market value of shares of the firm to the maximum extent possible given the available investment opportunities.

It is impossible to address this question without examining the interactions of investment and financing decisions in a firm, and the relationship between opportunity cost of a proposed project and the cost of capital of a firm. In this regard, the risk in operating the assets associated with the project and the risk through financial leverage play an important role in shaping the decision in capital budgeting. The decision is often subjective, with only a thin thread to rational analysis which must necessarily be based on simplified assumptions.

Some of the most important factors related to the complex and sometimes controversial issues in capital budgeting include:

1. The selection of the minimum attractive rate of return (MARR) in the evaluation of an investment project.
2. The determination of a single internal rate of return reflecting the combined effects of operation and financing.

3. The determination of an overall rate of return from a project if a unique value of the internal rate of return does not exist.
4. The effect of a reinvestment rate on the evaluation if it is different from the MARR of the investment.

We shall discuss these pertinent issues to highlight the basic principles often used to guide the actions of decision makers.

16.2 THE COST OF CAPITAL FOR AN INVESTMENT PROJECT

The *opportunity cost of capital* of an investment project is defined as the rate of return that the foregone opportunity will earn. The term *cost of capital* is often used as an abbreviation in this context. In a world of certainty with a perfect capital market, we can use a risk-free rate as the opportunity cost for discounting both the operating cash flow profile and the financial cash flow profile of a project as discussed in Chapter 12. In a world of uncertainty and imperfections, we must explore other avenues to see how to account for the effects of both operational and financial risks on project evaluation.

However, the term *cost of capital* has different meanings in other contexts. It may mean the discount rate for discounting the future values of a firm's assets to a present value. It may also mean the expected rate of return for the equity (stock) or that for the debt (bond) of a firm in attracting and retaining capital in a competitive environment. In the real world, there is no such thing as a perfectly competitive capital market in which one can lend or borrow freely at the same rate, as has been assumed in earlier chapters. Instead, there are many capital markets, such as the stock market and the bond market. Each is competitive within its own boundary, but its efficiency is retarded by factors such as transaction costs. Therefore, it would be more precise to use terms such as "cost of equity capital," "cost of debt capital," and "cost of capital of the firm" in dealing with such specific situations.

The estimation of the opportunity cost of capital is a forecast to discern the expected rate of return of an investment. In Chapter 15, we identified two basic concepts in which different expected rates of return are used to discount the operating cash flows of a project when the returns are uncertain. Except for simple situations, the certainty equivalent concept cannot be easily applied for lack of sufficient information. The capital asset pricing model (CAPM) is more commonly used by the business community, although it is by no means the only alternative available. Even if we accept the CAPM and its implicit assumptions, we are still confronted with a number of variations in its application.

Consequently, we must look deeper into the interactions between investment and financing decisions in the face of uncertain returns and potential risks in operation and financing. We attempt to provide a general understanding without getting into the specifics of many aspects of corporate finance.

16.3 THE MM MODEL OF CORPORATE FINANCE AND INVESTMENT

Based on the assumption of perfect capital markets, Modigliani and Miller[1] derived two basic propositions with respect to the valuation of the securities of companies with different capital structures, and their theory is often referred to as the *MM model*. In spite of their assumptions which cause some discrepancies between the theory and real-world observations, it is important to understand the conditions under which the MM model is valid so that we know what to expect under different conditions of market imperfections.

Proposition I of the MM model states that *the market value of a firm is independent of its capital structure and is given by the expected return from its assets at a rate r_a appropriate to its risk class*. This proposition allows the complete separation of the capital investment decision and the use of financial leverage. In the MM model, the firms are divided into "equivalent return classes" with similar operating characteristics, meaning that firms having similar unlevered relative systematic risks belong to the same risk class. As long as the rate of return r_a used to compute the net present value of the assets of a firm corresponds to that of other firms belonging to the same risk class, then we can ignore the financing decisions because no single combination of debt and equity is better than any other combination. If the MM model's proposition I holds, the simplest financing plan is to use all-equity financing. Hence, the market value of the firm can be determined by the net present value of the stream of cash flows from operations using the firm's assets.

Although borrowing by a firm increases the expected rate of return on shareholders' investment, it also increases the firm's risk. The MM model postulates that the increase in the expected return by holding shares of a levered firm is offset exactly by the increase in its financial risk. In perfect capital markets, an investor can borrow as freely and at the same rate as a firm. Therefore, an investor who wishes to assume a higher risk through financial leverage to get a higher expected return can obtain a personal loan and buy more shares of an unlevered firm to achieve the same result since the expected return of an unlevered firm is lower than that of a levered firm in the same risk class. Thus, it makes no difference whether the investor assumes the financial risk indirectly by holding the shares of a levered firm or directly by obtaining a personal loan to buy more shares of an unlevered firm.

Proposition II of the MM model states that the *expected rate of return* r_e *on the stock of a levered firm is equal to the expected rate of return* r_a *on a firm's assets corresponding to the firm's risk class with all-equity financing, plus a financial risk premium equal to the debt-to-equity ratio times the spread between* r_a *and* r_d*, where* r_d *is the expected rate of return on debt*. That is,

$$r_e = r_a + \frac{D}{E}(r_a - r_d) \tag{16.1}$$

[1] See Refs. 16.6 and 16.7 for the original presentation of the theory.

where D and E are market values of debt and equity, respectively. The above equation is based on the assumption of no tax as well as the assumption of perfect capital markets. This proposition indicates that the expected rate of return on the stock of a levered firm increases in proportion to the debt-to-equity ratio, as shown in Fig. 16.1.

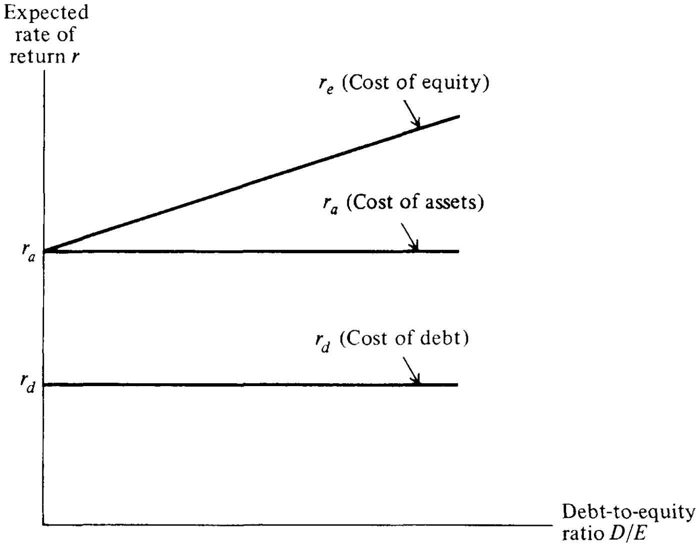

Figure 16.1 Expected rate of return versus debt-to-equity ratio.

Solving for r_a from Eq. (16.1), the expected return on a firm's assets with all-equity financing is

$$r_a = \frac{E}{D + E}r_e + \frac{D}{D + E}r_d \tag{16.2}$$

If an investor's portfolio consists of all of a firm's stock and debt, he or she is entitled to all of the firm's operating income generated by its assets. Then, the expected return on the portfolio is equal to the weighted average of the expected rate of return on its stock r_e and the expected rate of return on its debt r_d, with the proportions of equity and debt respectively as weights. Consequently, r_a is sometimes referred to as the *weighted average cost of capital.*

The MM model does include a provision to account for the effects of corporate income taxes. Let L be the target proportion of debt and τ be the effective tax rate that is shielded by the debt.[2] Then, the expected rate of return on the firm's assets including its financial leverage and tax shields is given by

$$r_0 = r_a(1 - \tau L) \tag{16.3}$$

[2] See Refs. 16.4 and 16.5 for detailed discussions on dividend policies and taxes.

The expected rate of return r_0 is referred to as the *adjusted cost of capital* of a firm. The target proportion of debt L represents a project's marginal contribution to the firm's debt capacity as a proportion of the project's present value, and is expressed as a percentage of the asset value of the project. Thus, the debt capacity L is a debt policy that the firm chooses, not the absolute limit that the firm can borrow. The effective tax rate τ is lower than the marginal corporate tax rate X because the tax shield from corporate debt is reduced by the personal income tax of the debt holder benefited by the corporate tax shield.[2] There is no agreement on what τ should be, and for all practical purposes X is often used instead of τ in Eq. (16.3).

The MM model is conceptually significant because it captures the major determinants of the cost of capital of a firm. However, in the real world with taxes, transaction costs, and other market imperfections, discrepancies exist between theory and practice.

Example 16.1

Company A, which has a market value of $50,000 for its equity and no debt, is expected to yield a rate of return of 20%. Company B, which has a market value of $40,000 for its equity and $10,000 for its debt, is expected to yield a rate of return of 22%. In a perfect capital market, the borrowing rate is 12% and there is no tax on the return. Suppose that you have $5,000 and are considering one of the following two options: (a) buy 10% of the stock of Company A, and (b) buy 10% of the stock and 10% of the bonds (a debt instrument) of Company B. Which option is better?

If you buy 10% of the stock of Company A as represented by its equity of $50,000, which has a market rate of return of 20%, your expected return from the stock is

$$(0.10)(50{,}000)(0.20) = \$1{,}000$$

If you buy 10% of both the stock and the bonds of Company B, your return is

From stock: $(0.10)(40{,}000)(0.22) = \$880$

From bonds: $(0.10)(10{,}000)(0.12) = \$120$

Total return: $880 + 120 = \$1{,}000$

It can be seen that both options lead to the same return on your investment, even though the financial structures of the two companies are different. Since Company A is unlevered, it has no financial risk. Consequently, its market rate of return of 20% is lower than that of Company B, which is 22%. If you do not want to take the additional risk of buying the stock of Company B only, you can protect yourself by buying a suitable proportion of its bonds which have a lower risk and also a lower market rate of return. In fact, you can invest $1,000 (10% of Company B's debt) in any debt instrument that pays 12% return in order to offset the additional risk of investing $4,000 in Company B's stock. Therefore, an investor can exercise the same method to accomplish what a firm can do in increasing or decreasing the financial leverage and its associated financial risk, provided that he or she can borrow freely and on the same terms as a firm.

Example 16.2

Determine the expected rate of return r_a for a firm under all-equity financing which has the same operating risk as Company B in Example 16.1.

For D = \$10,000, E = \$40,000, $r_d = 12\%$, and $r_e = 22\%$, we obtain from Eq. (16.2)

$$r_a = \frac{40{,}000}{50{,}000}(22\%) + \frac{10{,}000}{50{,}000}(12\%) = 20\%$$

It can be seen that Company B has the same operating risk as Company A in Example 16.1 since $r_a = 20\%$ for both companies.

16.4 INTERACTIONS OF INVESTMENT AND FINANCING DECISIONS

The MM model suggests that, theoretically at least, the value of an investment project is independent of its concomitant financing plan. This approach allows the separate treatment of the operating cash flow profile and its financial side effects, and is referred to as the *adjusted present value approach*. That is,

$$\text{APV} = [\text{NPV}]_{i_a} + [\text{FPV}]_{i_f} \tag{16.4}$$

where $[\text{NPV}]_{i_a}$ is the net present value of the after-tax operating cash flow profile discounted at the risk-adjusted rate i_a according to its asset risk, and $[\text{FPV}]_{i_f}$ is the present value of all financial side effects whether they are positive (such as tax shields and government subsidized loans), or negative (such as transaction costs of issuing securities), discounted at a risk-free rate i_f. The treatment of transaction costs associated with financing was discussed in Section 9.7. Using Eqs. (9.9), (9.10) and (9.11), we can incorporate the transaction costs into Eq. (16.4) whenever necessary.

With the recognition of the conditions in the real world, it is generally conceded that the debt policy of a firm does matter. The important question is how financial leverage would affect the return of an investment. According to Eq. (16.1), the expected rate of return r_e on equity increases linearly with D/E as long as r_d is constant, as shown in Fig. 16.1. As a firm invests in a new project, it may borrow more and at some point must pay a higher expected rate of return r_d on its debt. Then, according to Eq. (16.1), the rate of increase of r_e tends to slow down as r_d increases at a higher debt-to-equity ratio. Following this reasoning, the conventional wisdom seems to indicate that there is an optimum debt-to-equity ratio at which the market value of a firm is maximum, as shown in Fig. 16.2.

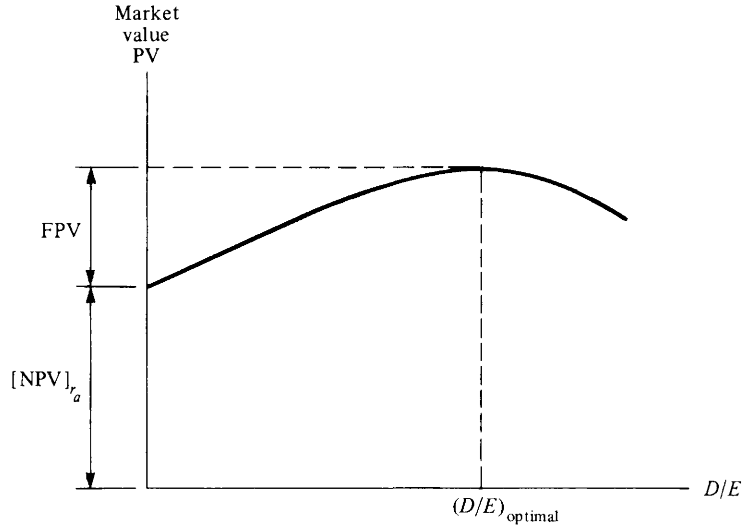

Figure 16.2 Optimal D/E for maximizing market value.

The adjusted present value approach requires a considerable amount of computation effort, including the determination of the opportunity cost of capital corresponding to the risk class of a project's assets. Recalling that in Section 15.13, the market rate of return of a project with an operating risk r_a can be estimated approximately from the industry-wide average for operating similar assets, it is logical to assume that the opportunity cost i_a equals r_a. However, the market rate based on the industry-wide average changes with the times since the beta coefficients of all companies in the industry follow market valuation of their securities. If the debt incurred for a project is tied to an irrevocable agreement such that the tax shield is guaranteed, then, the after-tax financing cash flows should be discounted at an after-tax risk-free rate, i.e., $i_f = r_f(1 - X)$ in which r_f is the before-tax risk-free rate and X is the marginal corporate income tax rate. However, if the debt is not tied to an irrevocable agreement, the tax shield is by no means certain, and cannot be counted on. Then, r_f should be used for i_f to reflect that the tax shield is just as risky as the interest payments for borrowing.

Since the MM model also allows the consideration of tax shields in the form of Eq. (16.3), it is theoretically possible to determine the value of an investment project and the effect of its concomitant financing plan in a single step. This approach is referred to as the *adjusted cost of capital approach*. However, the adjusted cost of capital r_0 does not include other financial side effects. Because the size of tax shields is questionable in view of personal income taxes on the gains from corporate tax shields, the effects of the financial cash flows are ignored in this approach. Thus, only the after-tax cash flows for operation are discounted to find the present value of the project. Then,

$$\text{APV} = [\text{NPV}]_{r_0} \qquad (16.5)$$

The adjusted cost of capital approach presents great difficulties in application. Since Eq. (16.3) assumes a constant debt proportion for the firm, r_0 should be adjusted to reflect this assumption when the firm makes an investment and expands its debt obligation.[3] In any case, the value of r_0 in Eq. (16.3) decreases as the target proportion of debt increases. However, as the debt increases beyond a certain proportion, r_0 is expected to rise because of financial side effects. Again, conventional wisdom seems to indicate that there is an optimum debt-to-equity ratio at which the adjusted cost of capital is maximum, as shown in Fig. 16.3.

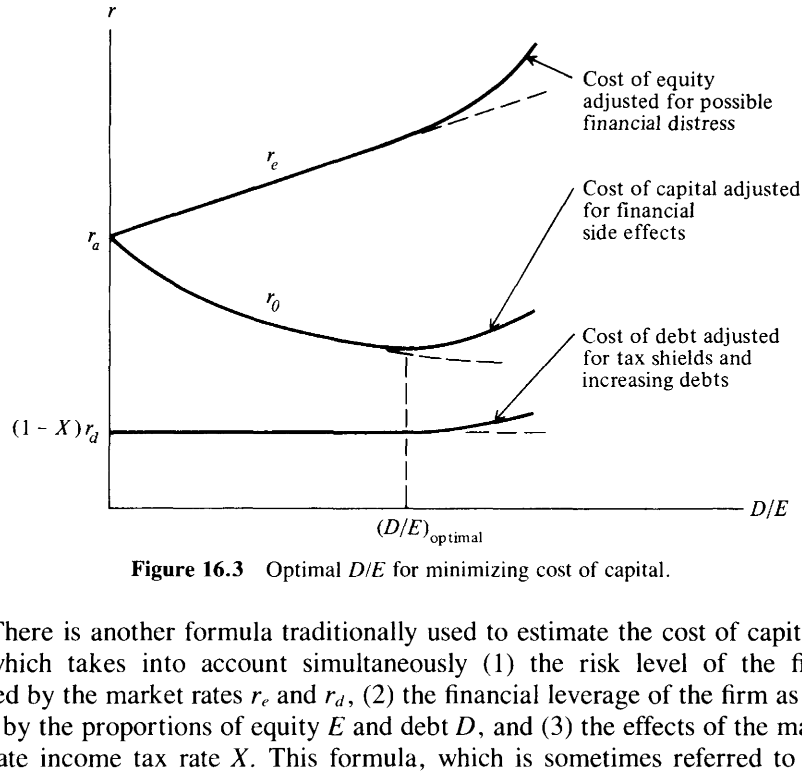

Figure 16.3 Optimal D/E for minimizing cost of capital.

There is another formula traditionally used to estimate the cost of capital of a firm which takes into account simultaneously (1) the risk level of the firm as reflected by the market rates r_e and r_d, (2) the financial leverage of the firm as represented by the proportions of equity E and debt D, and (3) the effects of the marginal corporate income tax rate X. This formula, which is sometimes referred to as the *traditional weighted average cost of capital formula,* yields the weighted average cost of capital of a firm as follows[4]:

$$r_w = \frac{E}{D + E} r_e + \frac{D}{D + E}(1 - X) r_d \qquad (16.6)$$

In an idealized world with no taxes, this equation leads to the same result as r_a in Eq. (16.2). If a project has the same risk characteristics and financial leverage as the

[3] See Ref. 16.2 for further refinement of r_0.

[4] For detailed discussion of this formula, see Ref. 16.3, pp. 451–455.

firm as a whole, it is reasonable to use r_w instead of r_0 for the adjusted cost of capital. Then,

$$\text{APV} = [\text{NPV}]_{r_w} \tag{16.7}$$

The traditional weighted average cost of capital r_w in Eq. (16.6) is often used as a benchmark MARR value for projects with average risk. Since r_w is the firm's cost of capital, it is reasonable to assume that a project with average risk resembles the characteristics of the operating and financial risks of the firm. However, if the firm embarks on a new venture, the operating risk for such projects will be higher than the existing line of work, or if the firm undertakes safe projects that make use of improvement of known technology, the operating risk is lower. Such variations will not be reflected by the firm's cost of capital.

Example 16.3

A private firm is considering a pilot project for solar energy development and is using the adjusted present value approach for project evaluation. Since this is a highly risky project in view of the uncertain returns, the firm has chosen an appropriate r_a for the project. The firm also intends to finance the project through borrowing from one of the two following sources:

(a) A bank that wants to sign an irrevocable agreement on the loan and repayment schedule at the market interest rate of 12%.
(b) A government agency that promises to provide the loan at a subsidized interest rate of 3% even though the market risk-free rate is 12%. However, this loan arrangement can be canceled if the crude oil price falls below a specified level.

Assuming that there are no transaction costs and other financial side effects and the marginal corporate income tax rate is 34%, what is the opportunity cost that should be used for discounting the after-tax financing cash flows?

In case (a), the tax shield is quite certain. Hence, the discount rate for the financial cash flow profile should be $i_f = r_f(1 - X) = (12\%)(1 - 0.34) = 7.92\%$. In case (b), the government loan at a subsidized rate of 3% will generate a tremendous positive side effect. However, since this side effect as well as the tax shield is uncertain, a discount rate of $i_f = r_f = 12\%$ should be used.

Example 16.4

The sources of capital of a firm together with their expected market rates of return before tax are shown in Table 16.1. If the tax rate of the firm is 36%, determine the cost of capital of the firm, using the traditional weighted average of cost of capital formula.

The first two items in Table 16.1 represent the equity and the last two represent the debt of the firm. The total market value of debt and equity of the firm is $20 million. Hence,

TABLE 16.1 SOURCES OF CAPITAL AND EXPECTED RATES OF RETURN

Sources of capital	Market value ($ million)	Before-tax expected rate of return
Common stock	8	24%
Retained earnings	4	18%
Bonds	6	10%
Long-term notes	2	14%

$$r_w = \frac{8}{20}(24\%) + \frac{4}{20}(18\%) + \left[\frac{6}{20}(10\%) + \frac{2}{20}(14\%)\right](1 - 0.36)$$

$$= 9.6\% + 3.6\% + (3\% + 1.4\%)(0.64) = 16\%$$

If we attempt to find r_e and r_d separately, these values can be obtained from their respective weighted averages as follows:

$$r_e = \frac{8}{12}(24\%) + \frac{4}{12}(18\%) = 22\%$$

and

$$r_d = \frac{6}{8}(10\%) + \frac{2}{8}(14\%) = 11\%$$

These values will also lead to $r_w = 16\%$ according to Eq. (16.6).

16.5 SELECTION OF APPROPRIATE MINIMUM ATTRACTIVE RATES OF RETURN

In the economic evaluation of capital investment opportunities, the minimum attractive rate of return (MARR) specified by an organization is critically important in determining whether an investment proposal is worthwhile from the standpoint of the organization. We emphasized in earlier chapters (particularly in Chapters 5 and 6) that because the MARR of an organization often cannot be measured accurately, it is advisable to use several values of the MARR for discounting in order to assess the sensitivity of the profit potential of an investment proposal with respect to variations in the MARR. After having examined the problems of uncertainty and risk in investments and the use of financial leverage by private firms, we now have a better appreciation why the determination of the MARR for project evaluation is such a difficult and inexact task.

Not every project undertaken by a firm should be treated with the same degree of sophistication. The scale of a project may vary from a routine maintenance program requiring an investment of less than $100,000 to an exceptionally large and complex project such as the Alaskan Pipeline project which cost almost $10 billion at the time of its completion. Most projects have a scope and expenditure somewhere in between. The selection of an appropriate MARR should be based on the nature and scope of a project.

For small routine projects, the expected returns from operation may be regarded as safe cash flows. The financing cash flows from debt, if any, are also safe when there is a contract requiring payments of principal and interest in fixed nominal sums regardless of future inflation. The principles and procedures developed in Chapter 12 can be applied without any modification. Then, the MARR should be based on the after-tax market rate of borrowing since the opportunity cost is the lending rate of the firm which is assumed to be the same as the borrowing rate in a perfectly competitive capital market.

For projects of average risk, i.e., projects whose operating and financing risk characteristics resemble those of the firm, Eq. (16.7) is applicable. However, the expected returns from operation may no longer be regarded as safe cash flows, and the tax shields are as safe as the anticipated interest payments. The rate r_e in Eq. (16.6) is intended to take into consideration the operating risk and the borrowing rate r_d is modified by a factor $(1 - X)$ to account for the tax shields. The rationale for Eq. (16.7) is that a new project is acceptable if the income from operation is profitable enough to pay at least the after-tax interest on the debt used to finance it as well as the expected return on the equity in it. Thus, the MARR for this project is r_w as represented by Eq. (16.6).

If the project risk is different from the average risk of a firm, i.e., if a firm intends to embark on a new project that has different risk characteristics, then the traditional weighted average of cost of capital r_w cannot be used. Theoretically, the adjusted cost of capital r_0 in Eq. (16.3) can be applied to discount the expected return from operation as shown in Eq. (16.5). In that case, the MARR for the project is r_0; however, as explained earlier, there are practical difficulties in determining r_0.

When the project risk differs from the average risk of a firm, it is most logical to use the adjusted present value approach in which an opportunity cost r_a is used as the MARR for operation corresponding to its risk and a market borrowing rate r_f is used as the MARR for the use of financing leverage. For large new ventures, not only are the operating cash flows uncertain, but tax shields resulting from interest payments cannot be counted on as safe. Consequently, the market borrowing rate before tax, which can be approximated by the risk-free rate of government securities, is usually assumed in discounting the financial cash flows.

The additional advantage of using the adjusted present value approach is that the transaction costs of financing can be accounted for as noted in Eq. (9.11) as follows:

$$\text{APV} = [\text{NPV}]_{r_a} + [\text{PVT}]_{r_f} - \text{PVQ} \tag{16.8}$$

where PVQ denotes the up-front transaction costs in financing.

Recognizing that different projects undertaken by a firm may have different operating risks, many firms require different rates of return from different categories. In the past, firms supplying public utilities under government regulations came as close to have the risk characteristics of a new project resembling those of the firm as ever existed. However, even such firms realize the changing risk introduced by new technologies. Consequently, the use of the adjusted present value approach is on the rise, although its adoption is far from universal. Because many firms are interested in diversification of business, there are many investment opportunities with different levels of operating risks for consideration. Hence, it is not practical to determine the market rate of return corresponding to the operating risk of each project according to the CAPM. The adoption of different values of MARR for several categories of investment seems most common in practice. Besides a benchmark MARR for the expansion of existing operation, there are progressively higher values of MARR for new products and speculative ventures, and lower values of MARR for operations making use of efficiency improvement of known technology. Thus, an appropriate r_a value may be chosen to reflect the opportunity cost of an investment.

The difference between using cost of capital r_w of a firm as the MARR for all new projects and adopting a different expected market rate of return r_e of a security corresponding to the risk characteristics of each new project as the MARR in project evaluation is illustrated in Fig. 16.4. The use of a single cost of capital of a firm as the MARR for project evaluation has the same effect as selecting a decision criterion as represented by the horizontal line, whereas the adoption of the market expected risk-adjusted rates of return for different levels of risk corresponds to a decision criterion as represented by the inclined line. For projects *A* and *B*, both criteria give the same signal of acceptance or rejection of the projects, i.e., the expected rate of return on *A* is higher than the MARR specified, and the expected rate of return on *B* is lower than the MARR specified. For projects *C* and *D*, however, the two criteria give different signals, i.e., project *C* should be accepted according to the market expected risk-adjusted rate of return but rejected according to the cost of capital of the firm, and the situation is reversed for project *D*. Therefore, the use of the cost of

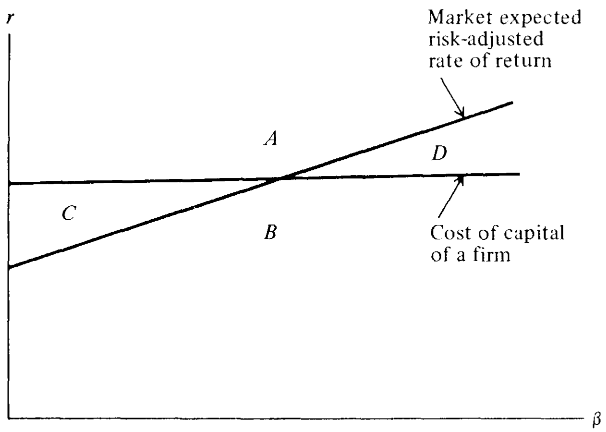

Figure 16.4 Comparison of opportunity costs used in project evaluation.

capital as the decision criterion will lead to the wrong decisions to reject C and accept D.

Example 16.5

A company has three divisions, each of which is assigned to operate a different class of project. Division A handles "safe" projects with well-established production processes and consumer markets. Division B handles relatively routine projects with "average" risk. Division C handles projects which are highly innovative and are regarded as somewhat risky. The company has decided to use the values of unlevered beta β_a of 0.3, 0.9, and 1.5 for divisions A, B, and C, respectively. If the risk-free rate r_f is 9% and the differential rate of the market portfolio versus risk-free securities $(r_m - r_f)$ is 8.3%, determine the MARR for evaluating the projects with all-equity financing for each of these three divisions.

The opportunity cost of capital $r_{a,x}$ for each division x is based on the expected market rate of return on projects with operating risk similar to that of the divisional unlevered $\beta_{a,x}$. Hence, the MARR for each division can be obtained from Eq. (15.21). Let x = A, B, and C denote divisions A, B, and C, respectively. Then, the MARR for each of these divisions is given by

$$r_{a,\text{A}} = 9\% + (0.3)(8.3\%) = 11.49\%$$

$$r_{a,\text{B}} = 9\% + (0.9)(8.3\%) = 16.47\%$$

$$r_{a,\text{C}} = 9\% + (1.5)(8.3\%) = 21.45\%$$

Example 16.6

A pilot plant for coal gasification requires an initial investment of $800 million and is expected to generate a uniform after-tax cash income of $150 million per year for 15 years, and there is no salvage value at the end. The opportunity cost of capital for the project based on its operating risk is 18%. The firm undertaking this project intends to secure a loan of $200 million, which will be repaid at the end of 15 years but the interest at a rate of 12% will be paid annually. The combined federal and state tax rate of this firm is 46%. The firm also plans to issue new stock to obtain another $300 million, and the transaction costs amount to 6% of the gross proceeds of the stock. Is this project worthwhile?

In using the adjusted net present value approach, the terms on the right side of Eq. (16.8) can be obtained separately as follows: The net present value of the project from all-equity financing in millions of dollars is

$$[\text{NPV}]_{18\%} = -800 + (150)(P \mid U, 18\%, 15)$$
$$= -800 + (150)(5.0916) = -36.26$$

It is assumed that tax shields are as risky as the projected interest payments. Hence, the market borrowing rate of 12% before tax is used for discounting $\bar{Y}_t$.

The present value of tax shields for the annual interest in millions of dollars over 15 years for a similar case has been worked out in Example 12.5 according to the formulas in Chapter 12. When the loan will be repaid at the end of 15 years with only interest to be paid annually and only tax shields are considered, the solution can be simplified as follows:

$$[\text{PVT}]_{12\%} = (200)(0.12)(0.46)(P \mid U, 12\%, 15)$$
$$= (11.04)(6.8109) = 75.19$$

The transaction costs of issuing the new stock are paid to underwriters, lawyers, and others at the time of initiating the project, and are obtained in millions of dollars as

$$\text{PVQ} = \frac{300}{(1 - 0.06)} - 300 = 19.15$$

Consequently, according to Eq. (16.8), the adjusted net present value in millions of dollars is

$$\text{APV} = -36.26 + 75.19 - 19.15 = +19.78$$

Thus, the project is worthwhile since the APV is \$19.78 million after the inclusion of the tax shields.

Example 16.7

Suppose that the pilot plant for coal gasification in Example 16.6 is qualified for a low-interest loan from a federal government program for encouraging the development of synthetic fuel projects. The loan of \$200 million is subsidized at an interest rate of 3%, repayable in a series of uniform annual payments of \$16.76 million each year over 15 years, covering both principal and interest. Hence, the firm will no longer try to obtain a commercial loan at 12% in the capital markets. What is the adjusted net present value of this project?

The only difference between this example and Example 16.6 is the term PVT in the computation of the adjusted net present value APV because of the government loan of \$200 million at a subsidized interest rate of 3%. We must first determine the portions of principal and interest payments in each year from the uniform annual payments in order to assess the tax shield. However, these annual interest payments have been obtained in Example 12.3 in Chapter 12.

Therefore, the after-tax financial stream of cash flows can be obtained from Table 16.2 as follows:

$$\text{At } t = 0, \bar{Y}_0 = \bar{A}_0$$
$$\text{At } t = 1 \text{ to } 15, \bar{Y}_t = \bar{A}_t + X_t I_t$$
$$= (Q_t - I_t) + X_t I_t$$
$$= Q_t - (1 - X_t) I_t$$

TABLE 16.2 COMPUTATION OF QUANTITIES LEADING TO PVT

t	$\bar{Y}_t = Q_t - (1 - X_t)I_t$	$(P \mid F, 12\%, t)$	$\bar{Y}_t(P \mid F, 12\%, t)$
0	+200.00	1.0000	+200.0000
1	−14.00	0.8929	−12.5006
2	−14.15	0.7972	−11.2804
3	−14.30	0.7118	−10.1787
4	−14.46	0.6355	−9.1893
5	−14.62	0.5674	−8.2954
6	−14.79	0.5066	−7.4926
7	−14.96	0.4523	−6.7664
8	−15.14	0.4039	−6.1150
9	−15.32	0.3606	−5.5244
10	−15.51	0.3220	−4.9942
11	−15.70	0.2875	−4.5138
12	−15.90	0.2567	−4.0815
13	−16.11	0.2292	−3.6924
14	−16.32	0.2046	−3.3391
15	−16.54	0.1827	−3.0219
			Total = +99.0143

where $X = 46\%$ is the income tax rate. Consequently,

$$\text{PVT} = \bar{A}_0 + \sum_{t=1}^{15} [Q_t - (1 - X_t)I_t](P \mid F, 12\%, t)$$

Note that the discount rate of 12% is used to obtain the present value because the company can earn 12% at the capital markets even though it only pays a subsidized interest rate of 3% to the government. The computation of PVT can be simplified by tabulating the results of the product terms as shown in Table 16.2. Noting that $A_0 = 200$, we obtain

$$\text{PVT} = 200 - 100.9857 = 99.0143$$

The remaining terms for computing APV can be taken from Example 16.6. That is,

$$\begin{aligned} \text{APV} &= [\text{NPV}]_{18\%} + [\text{PVT}]_{12\%} - \text{PVQ} \\ &= -36.26 + 99.01 - 19.15 = +\$43.60 \text{ million} \end{aligned}$$

Hence, the project is worthwhile if this financing scheme is used.

16.6 THE INTERNAL RATE OF RETURN AS A MERIT MEASURE

The adoption of the adjusted present value approach for making investment decisions does not naturally lend itself to a single MARR so that it can be compared to the internal rate of return (IRR) of the combined operating and financing cash flows.

This does not diminish the fact that the adjusted cost of capital approach is not practical in application because (1) r_0 is difficult to pinpoint, (2) financial side effects other than tax shields are not taken into account in r_0, and (3) r_w can only be used as a benchmark discount rate for projects with average risk in a firm. As long as the adjusted present value approach can give a more reliable investment decision, we should use it to make the best choice. Once a project is selected on that basis, however, there is no reason why we cannot try to satisfy those who want to look at the merit measure in the traditional way.

Let Y_t be the after-tax operating cash flow for period t (for $t = 0, 1, 2, \ldots, n$). Then Eq. (16.4) can be written as

$$\text{APV} = Y_0 + \sum_{t=1}^{n} Y_t(1 + r_a)^{-t} + [\text{FPV}]_{r_f} \tag{16.9}$$

For the sake of simplicity, assume that the after-tax operating cash flow Y_t is uniform for $t = 1, 2, \ldots, n$. Then Eq. (16.9) becomes

$$\text{APV} = Y_0 + Y_t(P \mid U, r_a, n) + [\text{FPV}]_{r_f} \tag{16.10}$$

For a project $x = 1$, the adjusted present value APV_1 corresponding to a given opportunity cost r^* can be obtained by substituting $r_a = r^*$ in Eq. (16.10), and the resulting r^* is shown schematically in Fig. 16.5.

Let us consider another hypothetical project $x = 0$ which has the same initial investment Y_0 but has a uniform operating cash flow Y_t^* for $t = 1, 2, \ldots, n$ such that APV in Eq. (16.10) becomes 0. That is,

$$Y_0 + Y_t^*(P \mid U, r_a, n) + [\text{FPV}]_{r_f} = 0 \tag{16.11}$$

If the financing plan of the project will not be changed, i.e., the term $[\text{FPV}]_{r_f}$ will remain constant, and the opportunity cost of project $x = 0$ is also r^*, then we can substitute $r_a = r^*$ into Eq. (16.11) and solve for Y_t^*. The resulting Y_t^* (for $t = 1, 2, \ldots, n$) is the minimum acceptable uniform after-tax return from operation such that the project will break even, i.e., $\text{APV}_0 = 0$ at $r_a = r^*$ as indicated in Fig. 16.5.

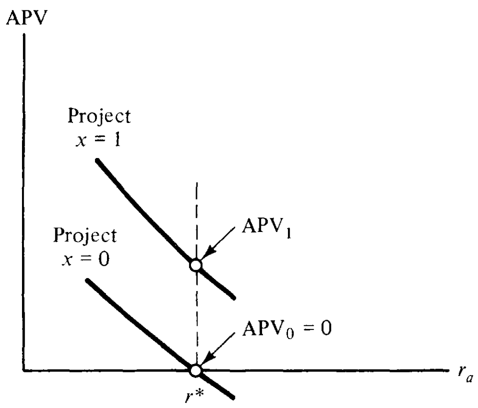

Figure 16.5 Adjusted present value versus risk-adjusted rate of return.

If we look at the project from the perspective of the adjusted cost of capital approach as stated in Eq. (16.5), the break-even case for which APV = 0 is given by

$$Y_0 + Y_t^*(P \mid U, r_0, n) = 0 \tag{16.12}$$

where r_0 is the adjusted cost of capital. Solving for r_0 in Eq. (16.12), this r_0 value is equivalent to the internal rate of return (IRR) of the project since APV is zero. Consequently, we can find a single IRR reflecting the combined effects of operation and financing of project $x = 1$ from Eq. (16.12).

Example 16.8

Referring to Example 16.6, find the IRR for the combined operation and financing cash flows of the project.

The APV of \$19.78 million for this project is based on $r_a = 18\%$ and $r_f = 12\%$. For the given financing plan, FPV = 75.19 − 19.15 = \$56.04 million. Then, the Y_t^* (in \$ million) for the break-even case can be obtained by Eq. (16.11) as follows:

$$-800 + Y_t^*(P \mid U, 18\%, 15) + 56.04 = 0$$

from which

$$Y_t^* = \frac{800 - 56.04}{5.0916} = 146.12$$

Viewing from the perspective of the adjusted cost of capital approach, we can find r_0 by setting APV = 0. Thus,

$$-800 + (146.12)(P \mid U, r_0, 15) = 0$$

from which we find $r_0 = 16.39\%$ using an electronic spreadsheet. This is the IRR of the project reflecting the combined effects of operation and financing.

16.7 THE OVERALL RATE OF RETURN FOR UNCONVENTIONAL CASH FLOW PROFILES

The overall rate of return (ORR) was introduced as a profit measure in Section 4.8 because the IRR of a project generally does not tell the whole story about the net gain or loss of a project at the end of project life since the interim receipts and disbursements of a cash flow profile will also contribute to the overall gain or loss. However, for the sake of simplicity, we confined our discussion in Section 4.8 to finding the ORR of conventional cash flow profiles which contain only one sign change. We now generalize the procedure for determining the overall rate of return and the overall rate of borrowing (or overall rate of return to the lender) of unconventional cash flow profiles. The transition point of an unconventional cash flow profile is the time period at which the profile is split into two separate parts: one consisting of a pure borrowing cash flow stream that is financially self-sufficient and another one consisting of a pure investment cash flow stream that is economically self-sufficient. Thus, we can obtain an investment base and a resultant return in the case of an investment opportunity, or a borrowing base and a resultant repayment in the case of a borrowing opportunity.[5]

Let $A_0, A_1, A_2, \ldots, A_n$ be an n-period cash flow profile of an investment or a borrowing opportunity. For this profile of A_t ($t = 0, 1, 2, \ldots, n$), a transition point occurs at a time period $t = j$ (for $0 \leq j < n$) where A_j and A_{j+1} differ in sign. When A_j is followed by zero coefficients from A_{j+1} to A_{j+c}, then A_j and A_{j+c+1} must differ in sign. Since an unconventional cash flow profile has two or more sign changes in A_t, there are two or more potential transition points, and the above condition regarding a transition point is necessary but not sufficient. To state the sufficient condition, we first define the cumulative present value of the cash flows from $t = 0$ to $t = j$ as

$$P_j = \sum_{t=0}^{j} A_t(1 + i^*)^{-t} \tag{16.13}$$

and define the cumulative future value of the cash flows from $t = j + 1$ to $t = n$ as

$$F_j = \sum_{t=j+1}^{n} A_t(1 + i^*)^{n-t} \tag{16.14}$$

The transition point of an investment project is the earliest time $t = e$ after which the remaining cash flow stream of the project is economically self-sufficient with the following characteristics:

1. A time period $t = j$ is a potential transition point if $A_j < 0$ and $A_{j+1} > 0$. When $A_j < 0$ is followed by zero coefficients from A_{j+1} to A_{j+c}, then it is necessary that $A_{j+c+1} > 0$.
2. Among the time periods $t = j$ that satisfy the above condition, the transition point $t = e$ ($0 \leq e < n$) is that point at which P_j computed by Eq. (16.13) is minimum. As a result, F_j computed by Eq. (16.14) for $t = e$ is maximum. Then, $-P_e = -P_j$ for $j = e$ is the investment base and $F_e = F_j$ for $j = e$ is the resultant return of the investment project.
3. The transition point $t = e$ is the period at which the investment base reserved for the project at $t = 0$ will be exhausted as a result of disbursements from $t = 0$ to $t = e$.

The transition point of a borrowing opportunity is the earliest time $t = e$ after which the remaining cash flow stream of the opportunity is financially self-sufficient with the following characteristics:

[5] The concept was first introduced in Ref. 16.1 and is adapted for this section. The authors are indebted to R. G. Beaves for further clarifications through private communication.

1. A time period $t = j$ is a potential transition point if $A_j > 0$ and $A_{j+1} < 0$. When $A_j > 0$ is followed by zero coefficients from A_{j+1} to A_{j+c}, then it is necessary that $A_{j+c+1} < 0$.
2. Among the time periods $t = j$ that satisfy the above condition, the transition point $t = e$ $(0 \le e < n)$ is that point at which P_j computed by Eq. (16.13) is maximum. As a result, F_j computed by Eq. (16.14) for $t = e$ is minimum. Then, $P_e = P_j$ for $j = e$ is the borrowing base and $-F_e = -F_j$ for $j = e$ is the resultant repayment for the borrowing opportunity.
3. The transition point $t = e$ is the period at which the borrowing base designated for the project at $t = 0$ will be replenished as a result of receipts from $t = 0$ to $t = e$.

Once the transition point at $t = e$ is determined for an unconventional cash flow profile, P_e at $t = 0$ and F_e at $t = n$ for a MARR equal to i^* are obtained as follows:

$$P_e = \sum_{t=0}^{e} A_t(1 + i^*)^{-t} \tag{16.15}$$

$$F_e = \sum_{t=e+1}^{n} A_t(1 + i^*)^{n-t} \tag{16.16}$$

The overall rate of return i^0 can be obtained from the following relationship:

$$P_e(1 + i^0)^n + F_e = 0 \tag{16.17}$$

Hence,

$$i^0 = \left(-\frac{F_e}{P_e}\right)^{1/n} - 1 \tag{16.18}$$

The ORR can also be obtained by interpolation from the discount amount factor in Appendix A by noting the following relation

$$(P \mid F, i^0, n) = -\frac{P_e}{F_e} \tag{16.19}$$

Consequently, the overall rate of return may also be used as an indirect merit measure for an unconventional cash flow profile. However, the decision rule for an investment opportunity is different from that for a borrowing opportunity because the ORR refers to the overall rate of return in the former, but refers to the overall rate of borrowing in the latter. The set of decision rules associated with the ORR criterion for accepting or rejecting a project can be stated as follows:

1. For an investment opportunity, accept if ORR $\ge$ MARR; reject otherwise.
2. For a borrowing opportunity, accept if ORR $\le$ MARR; reject otherwise.

Example 16.9

For the unconventional investment cash flow profile in Example 4.9, find the transition point and the overall rate of return if the MARR is 8%.

It is noted that $t = 3$ is the only time period at which the sign of A_t changes from negative to positive in the next period, i.e., $A_3 < 0$ and $A_4 > 0$. From Eqs. (16.15) and (16.16)

$$P_e = 18 + (10)(P \mid F, 8\%, 1) - (40)(P \mid F, 8\%, 2) - (60)(P \mid F, 8\%, 3)$$

$$= -54.665$$

$$F_e = (30)(F \mid P, 8\%, 1) + 50 = 84.2$$

Hence, the investment base is $-P_e = 54.665$ (\$ million) and the resultant return is $F_e = 84.2$ (\$ million).

The overall rate of return is found to be 8.56% from Eq. (16.18). Since the ORR is higher than the MARR = 8%, this investment project is acceptable.

Example 16.10

Consider the unconventional cash flow profile of an investment project in Example 4.13. If the MARR is 8%, find the transition point and the overall rate of return.

There are three potential transition points $t = j$ that satisfy the conditions $A_j < 0$ and $A_{j+1} > 0$. They are $t = 0$, $t = 2$, and $t = 5$. From Eq. (16.13),

$$t = 0,\ P_0 = -20{,}000$$

$$t = 2,\ P_2 = -20{,}000 + (12{,}000)(P \mid F, 8\%, 1) - (19{,}000)(P \mid F, 8\%, 2)$$

$$= -25{,}718$$

$$t = 5,\ P_5 = -20{,}000 + (12{,}000)(P \mid F, 8\%, 1) - (19{,}000)(P \mid F, 8\%, 2) + (23{,}000)(P \mid F, 8\%, 4) - (15{,}900)(P \mid F, 8\%, 5)$$

$$= -19{,}634$$

Since $P_2 = -25{,}718$ is minimum, $t = 2$ is the transition point. Hence, the investment base is $-P_e = 25{,}718$. The resultant return F_e is obtained by Eq. (16.16) as follows:

$$F_e = (23{,}000)(F \mid P, 8\%, 3) - (15{,}900)(F \mid P, 8\%, 1) + 30{,}800$$

$$= 28{,}973 - 17{,}172 + 30{,}800 = 42{,}601$$

The overall rate of return is found to be 9.17% from Eq. (16.18). Since the ORR is greater than the MARR = 8%, this investment project is acceptable.

Example 16.11

Consider the unconventional cash flow profile for a borrowing opportunity as follows: $A_0 = +20{,}000$, $A_1 = -12{,}000$, $A_2 = +19{,}000$, $A_3 = -23{,}000$, $A_4 = 0$, $A_5 = +15{,}900$, and $A_6 = -30{,}800$. If the MARR is 12%, find the transition point and compute the overall rate of return.

There are three potential transition points $t = j$ that satisfy the conditions $A_j > 0$ and $A_{j+1} < 0$. They are $t = 0$, $t = 2$, and $t = 5$. Using Eq. (16.13), it is found that $P_2 = 24{,}433$ is maximum among these time periods. From Eqs. (16.15) and (16.16),

$$P_e = 20{,}000 - (12{,}000)(P \mid F, 12\%, 1) + (19{,}000)(P \mid F, 12\%, 2)$$

$$= 20{,}000 - 10{,}715 + 15{,}147 = 24{,}432$$

$$F_e = (-23{,}000)(F \mid P, 12\%, 3) + (15{,}900)(F \mid P, 12\%, 1) - 30{,}800$$

$$= -32{,}318 + 17{,}808 - 30{,}800 = -45{,}305$$

Hence, the borrowing base is $P_e = 24{,}432$ and the resultant repayment is $F_e = -45{,}305$. The overall rate of return to the lender (or the overall rate of borrowing) is found to be 10.86% from Eq. (16.18). Since the ORR is less than MARR = 12%, this borrowing opportunity is acceptable.

Example 16.12

An unconventional cash flow profile is given as follows: $A_0 = +1{,}000$, $A_1 = -6{,}600$, $A_2 = 0$, $A_3 = +17{,}000$, $A_4 = 0$, and $A_5 = -3{,}000$. If the MARR is 10% for lending or borrowing, find the overall rate of return if the profile is proposed (a) as an investment project and (b) as a borrowing opportunity.

It has been found that no IRR exists in the positive range of i for this unconventional cash flow profile since the NPV is negative for that range. Hence, the ORR for this cash flow profile can still be found whether it is offered as an investment project or a borrowing opportunity.

If this cash flow profile is offered as an investment project, we look for the sign change of A_t from negative to positive. Hence, $t = 1$ is the transition point since it is the only point that satisfies the required condition; i.e., $A_1 < 0$ and $A_3 > 0$, while $A_2 = 0$. Then, $P_e = -5{,}000$ and $F_e = -5{,}470$ are found from Eqs. (16.15) and (16.16). The overall rate of return for the investment project is found to be 1.81% from Eq. (16.18). Since this ORR is far below MARR = 10%, the investment project is unacceptable.

However, if the same cash flow profile is offered as a borrowing opportunity, we look for the sign change of A_t from positive to negative. Both $t = 0$ and $t = 3$ are potential transition points since we have not only $A_0 > 0$ and $A_1 < 0$ but also $A_3 > 0$ and $A_5 < 0$, while $A_4 = 0$. Then, from Eq. (16.13), we get

$t = 0,\ P_0 = 1{,}000$

$t = 3,\ P_3 = 1{,}000 - (6{,}600)(P \mid F, 10\%, 1) + (7{,}000)(P \mid F, 10\%, 3)$
$= 259$

Since $P_0 = 1{,}000$ is maximum, $t = 0$ is the transition point. Then, $P_e = 1{,}000$ and $F_e = -4{,}193$ are found from Eqs. (16.15) and (16.16). The overall rate of return to the lender (or the overall rate of borrowing) is found to be 33% from Eq. (6.18). Since this ORR is far above MARR = 10%, the borrowing opportunity is unacceptable.

16.8 EXTERNAL BALANCES FOR UNCONVENTIONAL CASH FLOW PROFILES

For an unconventional investment cash flow profile, the investment base may be treated as an external balance in a debit account for a project from which all disbursements are drawn and to which all receipts are deposited. Similarly, for an unconventional borrowing cash flow profile, the borrowing base may be treated as an external balance in a credit account to which all receipts are deposited and from which all disbursements are drawn. In either case, the external balance of a project at the end of period j (for $j = 0, 1, 2, \ldots, n$) is defined as

$$R'_j = \sum_{t=0}^{j} A_t(1 + i^*)^{j-t} - P_e(1 + i^*)^j \tag{16.20}$$

Then, for $j = 0$,

$$R'_0 = A_0 - P_e \tag{16.21}$$

At the transition point $j = e$, it is seen from the relation in Eq. (16.15) that

$$R'_e = 0 \tag{16.22}$$

and for $j = n$,

$$R'_n = \sum_{t=e+1}^{n} A_t(1 + i^*)^{n-t} = F_e \tag{16.23}$$

It is noteworthy that the investment base $-P_e$ is analogous to a project debit account with an initial reserve of $-P_e$ prior to $t = 0$. Then, starting at $t = 0$, we can also find the external balances successively at subsequent periods from $t = 1$ to $t = n$. The account balance (for $t = 1, 2, \ldots, n$) is given by the following relations

$$R'_j = A_j + R'_{j-1}(1 + i^*) \tag{16.24}$$

where i^* is the discount rate on the external balance for each period. This equation can also be used to find the external balances successively for a credit account.

Example 16.13

For the unconventional investment cash flow profile in Example 16.10, find the external balances in successive periods for a MARR of 8%.

Noting that $P_e = -25{,}178$ in Example 16.10 and using Eqs. (16.21) and (16.24), the external balances R'_t are computed as follows:

$$t = 0,\ R'_0 = -20{,}000 + 25{,}178 = +5{,}178$$

$$t = 1,\ R'_1 = 12{,}000 + (5{,}178)(1 + 0.08) = +17{,}592$$

$$t = 2,\ R'_2 = -19{,}000 + (17{,}592)(1 + 0.08) = 0$$

$$t = 3,\ R'_3 = 23{,}000 + 0 = +23{,}000$$

$$t = 4,\ R'_4 = 0 + (23{,}000)(1 + 0.08) = 24{,}840$$

$$t = 5,\ R'_5 = -15{,}900 + (24{,}840)(1 + 0.08) = +10{,}927$$

$$t = 6,\ R'_6 = 30{,}800 + (10{,}927)(1 + 0.08) = +42{,}601$$

Note that the external balance at the transition point is zero and the balance at the end is the resultant return; i.e., $F_e = R'_6 = 42{,}601$.

Example 16.14

For the unconventional borrowing cash flow profile in Example 16.11, find the external balances in successive periods for a MARR of 12%.

Noting that $P_e = 24{,}432$ in Example 16.11 and using Eqs. (16.21) and (16.24), the external balances R'_t are computed as follows:

$$t = 0,\ R'_0 = 20{,}000 - 24{,}432 = -4{,}432$$

$$t = 1,\ R'_1 = -12{,}000 - (4{,}432)(1 + 0.12) = -16{,}964$$

$$t = 2,\ R'_2 = 19{,}000 - (16{,}964)(1 + 0.12) = 0$$

$$t = 3,\ R'_3 = -23{,}000 + 0 = -23{,}000$$

$$t = 4,\ R'_4 = 0 - (23{,}000)(1 + 0.12) = -25{,}760$$

$$t = 5,\ R'_5 = 15{,}900 - (25{,}760)(1 + 0.12) = -12{,}951$$

$$t = 6,\ R'_6 = 30{,}800 - (12{,}951)(1 + 0.12) = -45{,}305$$

Note that the external balance at the transition point is zero and the balance at the end is the resultant repayment which is $-F_e = -R'_6 = 45{,}305$.

16.9 EFFECTS OF REINVESTMENT POLICIES ON PROJECT EVALUATION

Up to this point, we have assumed that, without explicit policies to the contrary, early returns from an investment project will be reinvested at the MARR. In this section, we examine the effects on evaluation of an investment project if the reinvestment rates are set at a level different from the MARR. This course of action may be taken by an organization if the returns in interim periods from a project are reinvested in opportunities where risks are different.

In deriving the relationships based on a new policy of specifying a reinvestment rate that is different from the MARR, we introduce the following notation under the new reinvestment policy:

i^* = the minimum attractive rate of return (MARR) of an investment project
r = the reinvestment rate for returns from interim periods
$\overline{\text{NFV}} = \text{NFV}(r)$ = net future value at $t = n$
$\overline{\text{NPV}} = \text{NPV}(i^*, r)$ = net present value at $t = 0$
$i_r^0 = \text{ORR}(i^*, r)$ = overall rate of return
$\bar{P}_j = P_j(r)$ = cumulative present value of cash flows from $t = 0$ to $t = j$
$\bar{F}_j = F_j(r)$ = cumulative future value of cash flows from $t = j + 1$ to $t = n$

For a conventional cash flow profile $A_0, A_1, \ldots, A_n$ with outlay at $t = 0$ only, the transition point is at $t = 0$. Then, the net future value and net present value under the new policy are respectively given by

$$\overline{\text{NFV}} = \sum_{t=0}^{n} A_t(1 + r)^{n-t} \tag{16.25}$$

and

$$\overline{\text{NPV}} = A_0 + (1 + i^*)^{-n} \sum_{t=1}^{n} A_t(1 + r)^{n-t} \tag{16.26}$$

Note that $\overline{\text{NFV}}$ does not involve i^*. Here, only $\overline{\text{NPV}}$ should be used as a decision criterion for project evaluation.

In general, for an unconventional investment cash flow profile, the transition point can be obtained by first finding

$$\bar{P}_j = \sum_{t=0}^{j} A_t(1 + r)^{-t} \tag{16.27}$$

$$\bar{F}_j = \sum_{t=j+1}^{n} A_t(1 + r)^{n-t} \tag{16.28}$$

The conditions for locating the transition point of a cash flow profile remain the same as stated in Section 16.7 except that P_j and F_j in Eqs. (16.13) and (16.14) should be replaced by $\bar{P}_j$ and $\bar{F}_j$ in Eqs. (16.27) and (16.28), respectively. At the transition point $j = e$, $\bar{P}_j = \bar{P}_e$ and $\bar{F}_j = \bar{F}_e$. Then,

$$\overline{\text{NPV}} = \bar{P}_e + (1 + i^*)^{-n}\bar{F}_e \tag{16.29}$$

or

$$\overline{\text{NPV}} = \sum_{t=0}^{e} A_t(1 + r)^{-t} + (1 + i^*)^{-n} \sum_{t=e+1}^{n} A_t(1 + r)^{n-t} \tag{16.30}$$

To find the overall rate of return, we set $\overline{\text{NPV}} = 0$ upon letting $i_r^0 = i^*$ in Eq. (16.29). Thus,

$$\bar{P}_e + (1 + i_r^0)^{-n}\bar{F}_e = 0 \tag{16.31}$$

from which the ORR $(i^*, r) = i_r^0$ can be obtained by direct solution or interpolation in a manner similar to Eqs. (16.18) and (16.19) for the special case of $i^* = r$.

When the reinvestment rate is different from the MARR for an investment project, the decision criterion for accepting an independent project based on the net present value is

$$\overline{\text{NPV}} \geq 0 \tag{16.32}$$

The same decision criterion may be applied to a borrowing opportunity for which the MARR refers to the maximum acceptable rate of borrowing (or the minimum attractive return of return to the lender).

Example 16.15

An unconventional investment cash flow profile (in \$ million) is given as follows: $A_0 = +18$, $A_1 = +10$, $A_2 = -40$, $A_3 = -60$, $A_4 = +30$, and $A_5 = +50$. If the MARR of the project is specified at 8% and the reinvestment rate at 9%, find the net present value and the overall rate of return of this cash flow profile.

To find the net present value, we can locate the transition point by noting that there is only one sign change from negative to positive, i.e., from $A_3 < 0$ to $A_4 > 0$, in this investment cash flow profile. Hence, the transition point is at $t = 3$.

First, we find the cumulative present values at potential transition points from Eqs. (16.27) and (16.28), as follows:

$$\bar{P}_e = 18 + (10)(1.09)^{-1} - (40)(1.09)^{-2} - (60)(1.09)^{-3} = -52.824$$

$$\bar{F}_e = (30)(1.09) + 50 = 82.7$$

The net present value is given by Eq. (16.29):

$$\overline{\text{NPV}} = -52.824 + (1.08)^{-5}(82.7) = 3.462 \text{ (\$ million)}$$

The overall rate of return can be computed by using Eq. (16.31). Solving the equation by interpretation, we get

$$(P \mid F, i_r^0, 5) = \frac{P_0}{F_n} = \frac{52.824}{82.7} = 0.6387$$

from which $i_r^0 = 9.39\%$.

Example 16.16

Consider the unconventional investment cash flow profile in Example 16.10. If the MARR is specified as 8% and the reinvestment rate is 9%, find the net present value and the overall rate of return of this cash flow profile.

First, we find the cumulative present values at potential transition points as follows:

$$t = 0, \bar{P}_0 = -20{,}000$$

$$t = 2, \bar{P}_2 = -20{,}000 + (12{,}000)(1.09)^{-1} - (19{,}000)(1.09)^{-2}$$
$$= -24{,}983$$

$$t = 5, \bar{P}_5 = -24{,}983 + (23{,}000)(1.09)^{-3} - (15{,}900)(1.09)^{-5}$$
$$= -17{,}556$$

Since $\bar{P}_2$ is minimum, $t = 2$ is the transition point and $\bar{P}_e = -24{,}983$. From Eq. (16.28),

$$\bar{F}_e = (23{,}000)(1.09)^3 - (15{,}900)(1.09) + (30{,}800) = 43{,}255$$

Then, from Eq. (16.29)

$$\overline{\text{NPV}} = 24{,}983 + (1.08)^{-6}(43{,}255) = 2{,}276$$

Furthermore, from Eq. (16.31),

$$(P \mid F, i_r^0, 6) = \frac{\bar{P}_e}{\bar{F}_e} = \frac{24{,}983}{43{,}255} = 0.5776$$

from which $i_r^0 = 9.59\%$.

16.10 EVALUATION OF MUTUALLY EXCLUSIVE PROPOSALS USING DISTINCT REINVESTMENT RATE

It was shown in the previous section how a reinvestment rate that is different from the MARR affects the merit measures that are used as criteria for the evaluation of independent projects. However, the evaluation of mutually exclusive proposals is complicated by the question whether a common reinvestment rate should be used for all proposals if the MARR for these proposals are different. This situation arises because the risks for the operation of the competing proposals may be different and the MARR for these projects are based on different risk-adjusted discount rates. For example, if we compare a proposal of constructing a chemical processing plant versus that of building a steel mill, the risks for the operation of these two proposals are anticipated. In reality, however, even if a corporation operates both types of business, it is likely that they are run by two separate divisions and that decisions for the acquisition of new facilities of each type will be handled separately. Hence, the use of different MARR for competing proposals is not common.

Generally, as long as the same MARR is used for the competing proposals, a common reinvestment rate that is different from the MARR can be used for project selection. Even if an organization uses different MARR for comparison of competing projects in different risk categories, a distinct common reinvestment rate should still be used since returns from interim periods will be reinvested at a specified opportunity cost regardless of which proposal is selected.

Example 16.17

The net annual cash flow profiles generated by three machines are shown in Table 16.3. Determine which machine should be selected on the basis of the net present value criterion if the MARR is specified to be 12% and the reinvestment rate to be 15% for all machines.

TABLE 16.3 CASH FLOW PROFILES FOR THE MACHINES

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−10,000	−10,000	−1,500
1–2 (each)	3,000	5,500	1,000
3–5 (each)	3,000	500	1,000
6	3,000	5,500	1,000

For each of the cash flow profiles, the transition point is $t = 0$. The net present values can be computed according to Eq. (16.30) as follows:

$$\overline{\text{NPV}}_1 = -10{,}000 + (P \mid F, 12\%, 6)(3{,}000)(F \mid U, 15\%, 6)$$
$$= -10{,}000 + (0.5066)(3{,}000)(8.7537) = 3{,}304$$

$$\overline{\text{NPV}}_2 = -10{,}000 + (P \mid F, 12\%, 6)[(5{,}500)(F \mid U, 15\%, 6) - (5{,}000)(F \mid U, 15\%, 3)(F \mid P, 15\%, 1)]$$
$$= -10{,}000 + (0.5066)[(5{,}500)(8.7537) - (5{,}000)(3.4725)(1.15)] = 4{,}275$$

$$\overline{\text{NPV}}_3 = -1{,}500 + (P \mid F, 12\%, 6)(1{,}000)(F \mid U, 15\%, 6)$$
$$= -1{,}500 + (0.5066)(1{,}000)(8.7537) = 2{,}935$$

Since proposal 2 has the highest $\overline{\text{NPV}}$, it is the best alternative.

16.11 SUMMARY AND STUDY GUIDE

This chapter has presented an overview of the impact of capital investment decisions on the long-range objectives of private firms. It has highlighted some pertinent issues in corporate finance in order to explain the criterion for accepting or rejecting an investment project which is consistent with the objective of maximizing the value of a firm.

We first looked at the market value of a firm from its assets without worrying about where the money for acquiring these assets comes from. Then, we looked separately at the stream of expected cash flows for debt that will be claimed by the debtholders and at the stream for equity that will be claimed by the stockholders. On the basis of this understanding, we considered the MM model of corporate finance and investment as a theoretical framework to provide useful insights into some important issues related to the interactions of investment and financing decisions. We also discussed some practical aspects of corporate finance and investment beyond the theoretical framework and examined qualitatively the conventional wisdom concerning the optimal financial structure of a firm.

After providing a general background on the adjusted cost of approach and the adjusted present net present value approach for project evaluation we addressed the crucial issue of selecting the appropriate minimum attractive rates of return (MARR) under such context. We also introduced the traditional weighted average of cost of capital of a firm as the benchmark cost of capital of a project with average risk. The choice of a MARR for projects with various categories of risk was also discussed.

We then turned to the question of the internal rate of return (IRR) of an investment project with a given financing plan. Since the IRR cannot be used as a profit measure if a unique value does not exist, we introduced the overall rate of return as an alternative merit measure for unconventional cash flow profiles. The procedure is equally applicable to an unconventional borrowing opportunity as well as to an investment opportunity.

Finally, we examined the effects of reinvestment policies on project evaluation. When the investment rate is different from the MARR, the same procedure of project evaluation can be applied with only slight modification, whether the cash flow profile is conventional or unconventional. In the evaluation of mutually exclusive proposals including those in different risk categories, a distinct common reinvestment rate should be used for all proposals in making investment decisions.

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PROBLEMS

P16.1 Redspot Company has a market value of \$60,000 for its equity and \$40,000 for its debt. Its stock has an expected rate of return of 20%, and its bonds (for the debt) have an expected return of 12.5%. The sole stockholder of the company, Ms. Redspot, can afford to pay off the debt of \$40,000 by withdrawing the amount from bank deposits that currently earn a market interest rate of 12.5%. The expected rate of return of the stock of the company is 17% when the market value of its equity becomes \$100,000 without debt. Assuming no income tax, compare the annual returns for Ms. Redspot in the following situations: (a) Do not pay off the debt and keep the cash in deposits. (b) Pay off the debt with the cash.

P16.2 Green Manufacturing Company has a 20% expected return on its assets which are currently valued at \$1 million of equity and no debt. However, the company plans to borrow another \$1 million for expansion at the market interest rate of 15%. What is the expected rate of return on the stock after incurring this debt? If the beta of the assets (with all-equity financing) is 0.9, what is the beta of the stock after the expansion? Assume that there is no income tax.

P16.3 Given that $r_a = 15\%$, $r_d = 10\%$, $D/E = 0.4$, and $X = 36\%$ for a firm, find the adjusted average cost of capital of the firm according to the MM model, assuming $L = D/(D + E)$ and $\tau = X$.

P16.4 Given that $r_e = 17\%$, $r_d = 10\%$, $D/E = 0.4$, and $X = 36\%$ for a firm, find the weighted average cost of capital of the firm. Use the traditional formula.

P16.5 The expected rate of return on a firm with all-equity financing is 25% and the expected rate of return of its debt is 12%. If the target debt proportion is 1/3 and the effective tax rate is 34%, find the adjusted cost of capital according to the MM model.

P16.6 The expected rate of return on a firm with all-equity financing is 25% and the expected rate of return of its debts is 12%. If the debt-to-equity ratio is 0.5 and the marginal tax rate is 34%, find the weighted average cost of capital according to the traditional formula.

P16.7 The Wexford Corporation has four divisions, each of which is assigned to operate projects with different risk characteristics. The divisions *A*, *B*, *C*, and *D* are estimated to have unlevered beta values 0.5, 0.8, 1.1, and 1.4, respectively. The market risk-free rate r_f is 8% and the differential rate $(r_m - r_f)$ is 8.3%. Determine the

MARR for evaluating capital projects with all-equity financing for each of these divisions.

P16.8 Company A, which specializes in electronics, has a debt-to-equity ratio of 0.2 and a levered beta value of 1.2. Company B, which specializes in electric appliances, has a debt-to-equity ratio of 0.8 and a levered beta value of 1.6. Company C has a division for electronics and another division for electric appliances, among other divisions. Company C believes that Company A and Company B are more representative of its corresponding divisions than the industry-wide averages. All three companies have a tax rate of 36%. The market risk-free rate r_f is 12% and the differential rate $(r_m - r_f)$ is 8.3%. What should be the MARR selected for each of these two divisions of Company C for evaluating capital projects with all-equity financing?

P16.9 Suppose that the pilot plant for the coal gasification in Example 16.6 in this chapter needs a commercial loan of \$300 million at 12% interest. The loan will be repaid in a series of uniform annual payments of \$44.04 million per year over 15 years, covering both principal and interest. On the other hand, it will issue new stock to obtain another \$200 million, and the transaction costs of issuing the new stock amount to 7% of the gross proceeds of the stock. Is the project worthwhile under the new financing plan?

P16.10 A manufacturer of solar energy collectors is considering an investment in a new plant that requires an initial amount of \$3 million and is expected to generate a uniform stream of after-tax cash incomes of \$1 million per year for the next 5 years with no salvage value at the end. The opportunity cost of capital for the project based on its operating risk is 15%. The company is assured of a government-subsidized loan of \$1 million for which annual interest is charged at a rate of 4% at the end of each year with the repayment of principal at the end of 5 years. The market interest rate for borrowing and lending is 8%. The income tax rate of this company is 34%. The company will also issue a new stock to obtain another \$1 million, and the transaction costs of the new issue amount to 5% of the proceeds of stock. Is this new plant worthwhile?

P16.11 For the project in Problem P16.10, find the minimum acceptable uniform after-tax annual return such that it will break even, assuming that the financing plan remains unchanged. Also find the adjusted cost of capital that is equivalent to the internal rate of return (IRR) of the project.

P16.12 A solid waste disposal operator is considering a 5-year investment project that will have substantial net cash inflows in the first three years but large outflows later for meeting the environmental regulations. The net cash flows in millions of dollars are as follows: $A_0 = -57.1$, $A_1 = +18$, $A_2 = +68$, $A_3 = +88$, $A_4 = -2$, and $A_5 = -130$. If the MARR of this operation is 20%, find the overall rate of return of this project, assuming that the reinvestment rate is the same as the MARR.

P16.13 If the reinvestment rate of the operator in Problem P16.12 is 15% while all other data remain unchanged, find the overall rate of return of the project.

P16.14 The Canfield Coal Company is considering a strip-mining project that has the following cash flow profile for the 5 years of its operation: $A_0 = -38.3$, $A_1 = A_2 = A_3 = A_4 = 28$, and $A_5 = -80$, all in millions of dollars. If the MARR of this operation is 20%, find the overall rate of return of this project, assuming that the reinvestment rate is the same as the MARR.

P16.15 If the reinvestment rate of Canfield Coal Company in Problem P16.14 is 15% while all other data remain unchanged, find the overall rate of return of the project.

P16.16 The net cash flow profiles of three mutually exclusive investment proposals are given below. Suppose that the MARR is 10% and the reinvestment rate is also 10%. Find the net present values and the overall rate of returns of these proposals, and determine which proposals should be selected.

t	$A_{t,1}$	$A_{t,2}$	$A_{t,3}$
0	−$100,000	−$100,000	−$80,000
1	+20,000	+25,000	+60,000
2	−24,200	−20,000	+25,000
3	+10,000	+50,000	−90,000
4	+75,000	+50,000	+100,000
5	+90,000	+56,189	+27,241

P16.17 For each of the proposals in Problem P16.16, find the external balances at $t = 0, 1, 2, 3, 4$, and 5. The MARR and the reinvestment rate remain unchanged.

P16.18 Suppose that the MARR in Problem P16.16 is 10% but the reinvestment rate is 12%, while all other data remained unchanged. Find the net present values and the overall rate of returns of the three proposals and determine which proposal should be selected.

P16.19 The cash flow profile of a borrowing opportunity is given as follows: $A_0 = \$9{,}000$, $A_1 = -\$2{,}000$, $A_2 = \$3{,}500$, $A_3 = -\$1{,}000$, $A_4 = -\$8{,}500$, and $A_5 = -\$8{,}500$. If the MARR and the reinvestment rate both equal 12%, find (a) the overall rate of return of this opportunity, and (b) the external balances at $t = 0, 1, 2, 3, 4$, and 5.

P16.20 The cash flow profile of a borrowing opportunity is given as follows: $A_0 = \$8{,}000$, $A_1 = -\$5{,}000$, $A_2 = -\$2{,}000$, $A_3 = \$7{,}000$, $A_4 = -\$9{,}000$, and $A_5 = -\$3{,}000$. If the MARR and the reinvestment rate both equal 12% find (a) the overall rate of return of this opportunity, and (b) the external balances at $t = 0, 1, 2, 3, 4$, and 5.

CHAPTER

17 Resource Allocation Decisions for Public Projects

17.1 IMPORTANT FACTORS FOR CONSIDERATION

In the evaluation of public investment projects, we are often confronted with broad social and political issues which we have avoided thus far. Significant interactions among economic, social, and political concerns are tacitly, if not explicitly, considered in the acceptance and selection of public projects. The allocation of scarce resources among alternative uses is said to be *economically efficient* when the total amount of benefits received by members of society is maximized under the prevailing income distribution. In a mixed economy where the private market is subject to government intervention, the crucial question is how to assess properly that a proposed public project will maximize the improvement of social welfare.

To address this question adequately, we cannot ignore the reasons for government intervention in the market economy. Some of the issues involving government intervention are complex and controversial, and they rightfully belong to more advanced courses in benefit-cost analysis, regulatory economics, and public finance. We are only interested in bridging the gap between the engineers and social planners in a decentralized capital budgeting process.

Some of the most important factors that often cause debates on the allocation of resources between private and public sectors are:

1. The inefficient allocation of resources because of private market failures
2. The estimation of costs and benefits of public projects that accrue to the public in general, and the fairness of their distribution

3. The determination of the minimum attractive rate of return for public projects, referred to as the *social discount rate*

We shall consider these factors in general terms in order to understand some of the basic issues that cause many of the controversies. We cannot expect to discuss in detail many of the subtle points supporting various positions. Because government intervention in the private market economy may be based on a number of factors, it is advisable to consider briefly the role of government as reflected in its fiscal policy objectives.

17.2 FISCAL POLICY OBJECTIVES FOR THE PUBLIC SECTOR

The fiscal policy objectives for the public sector generally include the following major functions: (1) the allocation function, (2) the distribution function, and (3) the stabilization function.[1] Regulatory policies, which may be regarded as a part of the allocation function, are not directly related to the fiscal policy.

The allocation function refers to the process by which total resource use is divided between private and public goods, and by which the mix of public goods is chosen. The objective is to achieve an economically efficient allocation of resources where *economic efficiency* is defined as the maximum improvement in social welfare available under the prevailing income distribution. It does not address the basic social issues of income distribution and redistribution.

The question of fair distribution of wealth involves considerations of social philosophy and value judgments. Distribution problems are and will continue to be a vital factor in public policy formulation. Even though policy measures may not be concerned primarily with distributional objectives, they often have distributional repercussions. For example, in reducing expenditures on various types of public projects, the reductions for some types of projects have more effects on the urban poor while others will have more effects on rural and farming populations. On the other hand, different distributional changes may also distort economic efficiency in different ways. Finally, a standard of distributive justice or fairness should be applied in making distributional changes. Among various fiscal devices, redistribution is implemented most directly by (1) a tax scheme combining progressive income taxes for high-income households with a subsidy to low-income households, (2) publicly financed programs which benefit low-income households, and (3) taxes on luxury goods and subsidies of other goods which are used chiefly by low-income consumers. Because expenditures on public projects are related to the tax revenues, the distributional function is an important issue in public policy.

Another objective of budget policy is to achieve the goals of high employment and price stability. The overall level of employment and prices in the market econ-

[1] This classification follows that suggested in Ref. 17.6, pp. 1–21.

omy depends upon the level of aggregate demand of millions of consumers whose decisions in turn depend on many factors such as income, wealth, credit availability, and expectations. At any given time, the level of expenditures may be insufficient to maintain full employment, and stimulating policies to raise aggregate demand may be introduced. On the other hand, expenditures may exceed the available output under conditions of high employment and then cause inflation, and restrictive conditions are needed to reduce demand. Thus, the stabilization function is often a part of the budget policy.

Public policy must also be introduced to influence the rate of growth of potential output as determined by market forces. Since growth depends on the rate of capital formation, the division of output between present consumption and future consumption is crucial in the choice of a rate of growth. Thus, the decisions affecting the rate of growth serve the allocation function more than the stabilization function.

In the United States, public expenditures and revenues are divided among the various levels of government. The basic functions of budget policy may not be well coordinated at different levels of government. Furthermore, at each level, budget planning often does not permit evaluation of various objectives on their own merits, and one objective may be achieved at the expense of another. Nevertheless, it is important to realize the conflicts of these objectives which eventually shape the investments in both public and private sectors of the economy.

17.3 THE PROVISION OF PUBLIC PROJECTS

A normative view of a market economy is based on the premise that the composition of output should satisfy the preferences of individual consumers and that a market system which establishes prices competitively according to consumer preferences is the most efficient way of allocating resources. What then are the reasons for government intervention in the market economy?

One of the reasons for government intervention is the provision of *pure public goods,* as distinguished from private goods. They refer to commodities which provide benefits not only to the consumers who wish to purchase them, but also to others who do not. The consumption of such products by various individuals is not "rival" in the sense that accruing of benefits to an individual does not reduce the benefits available to others. Furthermore, it is often impossible or impractical to exclude others from deriving benefits provided by such goods, or it is undesirable to exclude others even if it can be done. National defense and weather forecasting are examples of pure public goods. Hence, pure public goods refer only to those commodities which are nonrival and nonexcludable, and the government has an obligation to provide them for the benefit of society since individual consumers have no reason to offer voluntary payments for them.

Aside from the provision of pure public goods, government intervention may take many forms, depending on the circumstances under which the private market mechanism fails. For example, because of *imperfect competition* in the private market (e.g., monopoly), the government may choose to invoke the antitrust laws to break up the monopoly, or to grant the exclusive right to a "natural monopoly" under the government regulation as it has been done for most public utilities. Another example of private market failure concerning *externalities* (e.g., benefits accruing to or costs inflicted on society instead of the producer of goods) can be dealt with either through government regulation of the private enterprise or by government takeover of that enterprise. The form of government intervention sometimes depends on historical precedents as well as economic efficiency. Hence, the political process may become a substitute for preferences of the individuals of the community.

It is important to distinguish the provision of *public projects* and the *public production*. These two concepts are different and should not be confused with each other. Public projects may be undertaken by private firms and paid for by the government, or carried out directly under public management. Private goods may also be produced either by private firms or by public enterprises. Thus, public projects refer to government investments in goods and services in the public sector. In the United States, most public projects and practically all private goods are produced by private firms; only a small fraction of public projects are produced by public agencies and managed by civil servants or public enterprises under the auspices of local governments and independent authorities. For example, a very large percentage of the gross national product is allocated annually for national defense by the federal government; however, the contractors which produce the defense equipment are private corporations. At the state and local levels, most public projects utilize the expertise of private firms under the supervision or control of the appropriate government units. Hence, in a mixed economy, a large share of resources may be allocated for the public sector even though the private firms may be heavily engaged in activities related to the provision of public projects.

17.4 ECONOMIC EFFICIENCY AND EXTERNALITIES

Since economic efficiency is defined as the maximum improvement of social welfare under the prevailing income distribution, the allocation of available resources is said to be efficient when the total amount of benefits received by members of society from the consumption of all commodities produced from the resources is maximized. It is important to realize that when a decision is made to produce more of a commodity, fewer resources are available for the production of others. Hence, the *social benefits* gained from the production of the former are accompanied by the loss of potential benefits in foregoing the opportunity of producing the latter. The benefits thus foregone are often referred to as opportunity costs, which represent the *social costs*.

Given society's limited resources, the criterion of economic efficiency in the production of a commodity may be stated as the maximization of its *net social benefit* (NSB), which refers to the *difference* between *social benefits* (SB) and *social costs*

(SC) resulting from the allocation of resources for producing this commodity. Alternately, the criterion of economic efficiency may be restated in terms of *marginal social benefit* (MSB) and *marginal social cost* (MSC) as follows: resources are efficiently allocated to the production of a commodity when the marginal social benefit equals the marginal social cost. In analogy to the discussions in Section 13.7 (Chapter 13), the relationship between these two different representations of the economic efficiency criterion is shown in Fig. 17.1.

In the private market system, individual consumers base their choice of commodities on the benefits they expect from consuming these commodities and the costs of these commodities as determined by their market prices. The benefits of consumption to individuals who purchase these commodities are said to be "internalized" because their consumption excludes the consumption by others. Hence, the benefits of consuming privately purchased goods are referred to as *private benefits* (PB) and the costs of purchasing such goods are referred to as *private costs* (PC). If the private market mechanism is perfect, the market price of a commodity is determined by a demand schedule and a supply schedule of the commodity under the pre-

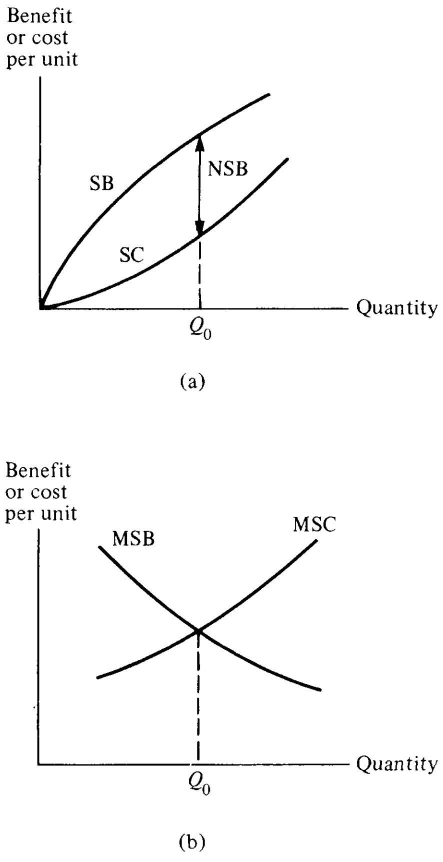
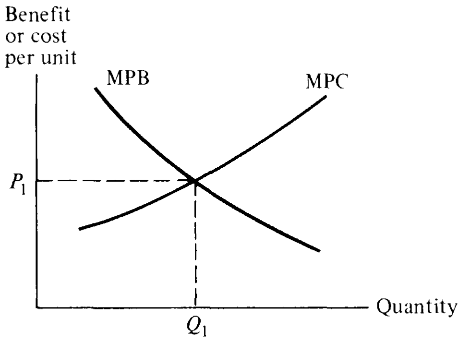
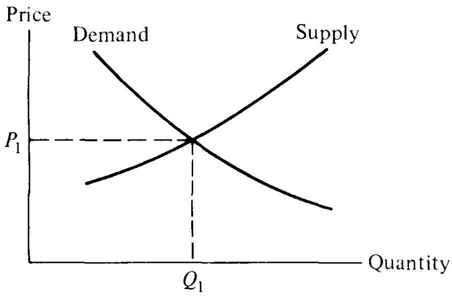


Figure 17.1 Marginal social benefits and marginal social costs.

vailing income distribution of the consumers. Hence, the marginal private benefit (MPB) is equivalent to the demand for a commodity and the marginal private cost (MPC) is equivalent to the supply for a commodity as the consumers tend to maximize their net private benefits and the producers to maximize their profits. Because the private market system reaches the equilibrium when the quantity demanded equals the quantity supplied, this quantity also represents the amount at which the marginal private benefit equals the marginal private cost as shown in Fig. 17.2. Hence, resources are efficiently allocated to the production of a commodity in a perfectly competitive private market system since under such conditions, the marginal private benefit equals the marginal social benefit (MPB = MSB) and the marginal private cost equals the marginal social cost (MPC = MSC) as shown in Fig. 17.3.



(a)



(b)

Figure 17.2 Marginal private benefits and marginal private costs.

However, because of market imperfections, the goal of achieving economic efficiency is often elusive. More specifically, externality is usually related to economic inefficiency of capital investment projects. Instead of the perfect fit of curves of MSC and MPC and those of MSB and MPB as shown in Fig. 17.3, the two sets of curves representing the public view and the private view in Figs. 17.2(a) and (b),

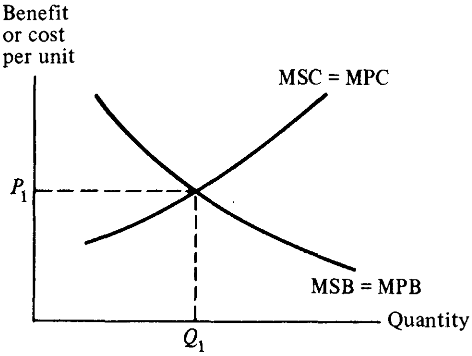


Figure 17.3 Efficient allocation of resources.

respectively, may be far apart when externality exists. To simplify the discussion, we will confine our discussion to two cases: (1) MSC = MPC but MSB and MPB are different, and (2) MSB = MPB but MSC and MPC are different.

Externality is defined as a benefit or cost in production or consumption that does not accrue to the producer or consumer of the commodity. It is an indirect benefit or cost to society. The marginal indirect benefit (MIB) is the difference between the marginal social benefit (MSB) and the marginal private benefit (MPB), and the marginal indirect cost (MIC) is the difference between the marginal social cost (MSC) and marginal private cost (MPC). That is, given MSC = MPC,

$$\text{MIB} = \text{MSB} - \text{MPB} \tag{17.1}$$

Similarly, given MSB = MPB,

$$\text{MIC} = \text{MSC} - \text{MPC} \tag{17.2}$$

The indirect benefit is referred to as a *positive externality* and the indirect cost is referred to as a *negative externality*. The terms MIB and MIC are refered to as *positive marginal externality* and *negative marginal externality*, respectively. Thus, a positive externality exists when the social benefit exceeds the private benefit, and a negative externality exists when the social cost exceeds the private cost. For example, homeowners who improve their buildings and grounds periodically contribute a positive externality to the neighborhood while a factory which pollutes the air produces a negative externality to the community nearby.

Externalities cause problems in achieving economic efficiency when indirect benefits and costs are excluded in making resource allocation decisions. If MIB = 0 in Eq. (17.1) and MIC = 0 in Eq. (17.2), it can be seen that MSB = MPB and MSC = MPC as for the case of a perfectly competitive private market system shown earlier in Fig. 17.3. However, if MIB > 0, it means that MSB > MPB, and there is a tendency for underproduction under such circumstances. But if MIC > 0, it means that MSC > MPC, and there is an incentive for overproduction. In either case, the net social benefit is not maximized.

Since positive or negative externalities often have widespread geographic effects far beyond the sources from which they are generated, these effects are referred to as *spillovers*. The allocation function of the fiscal policy established by the government can provide a criterion for resolving the inefficiency problems of externalities. For example, positive externalities may be encouraged and negative externalities discouraged by regulations and/or by economic incentives and disincentives. The externalities may also be internalized if the indirect benefit (or cost) is accrued (or charged) to consumers. In the case of neighborhood improvement, for example, the local government may pass an ordinance requiring that all buildings and grounds in the neighborhood be maintained following certain standards; or in the case of air pollution from a plant, the factory may be required to spend a considerable amount of money for air pollution abatement, and it may charge this expense to customers who purchase its products. The difficulty in implementing such policies is to determine the appropriate level of government regulation so that the cure will not be worse than the problem. We shall illustrate qualitatively how the problem can be tackled with two examples.

Example 17.1

The private market for low-cost new housing units in a city is represented by the demand schedule (same as the MPB curve) and the supply schedule (same as the MPC curve) in Fig. 17.4. The market price and the quantity produced at the market equilibrium point 1 are P_1 and Q_1, respectively. However, the city council desires to encourage the construction of a larger new housing stock to the level of Q_2 to maximize the social benefits to low-income families. What professional advice can you offer to the city council?

In the private market system, the net private benefit is maximized at the

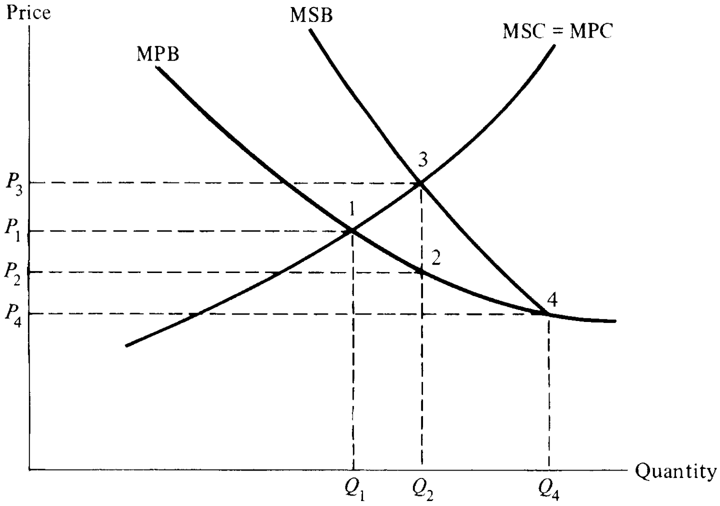


Figure 17.4 Positive Externality.

market equilibrium point 1. Given that MSC = MPC, the net social benefit will be maximized at point 3 for which the quantity produced will be Q_2. However, to produce the quantity Q_2, the price of each housing unit will go up to P_3 because higher costs must be paid to bid for additional resources that are committed to other uses. On the other hand, to find enough buyers for the quantity Q_2, the market price must be set at P_2. Thus, the city council must be prepared to raise funds to pay a subsidy of $(P_3 - P_2)$ per unit either to the producer or to the buyer if it considers the construction of Q_2 new housing units is a socially desirable goal.

Note that $(P_3 - P_2)$ represents the positive marginal externality to the city when Q_2 units of new housing are produced. If a quantity greater than Q_2 is produced, not only the price of each unit must be reduced below P_2 to find sufficient buyers, but the positive marginal externality also becomes smaller. If the quantity produced exceeds Q_4, no positive externality exists since the marginal social benefit cannot be less than the marginal private benefit.

Example 17.2

Suppose that the curves in Fig. 17.4 for Example 17.1 can be approximated by straight lines. The values corresponding to the intersections of these straight lines are given as follows: P_1 = \$70,000, P_3 = \$80,000, P_4 = \$50,000, Q_1 = 500 units, Q_2 = 600 units, and Q_4 = 1,000 units. Determine the subsidy per unit that the city must provide in order to encourage the construction of 600 new housing units.

The straight-line approximation of Fig. 17.4 is shown in Fig. 17.5 from which we find

$$P_1 = 70{,}000$$

$$P_2 = 50{,}000 + (70{,}000 - 50{,}000)\frac{(1{,}000 - 600)}{(1{,}000 - 500)} = 66{,}000$$

$$P_3 - P_2 = 80{,}000 - 66{,}000 = 14{,}000$$

Thus, the city must provide a subsidy of \$14,000 per housing unit.

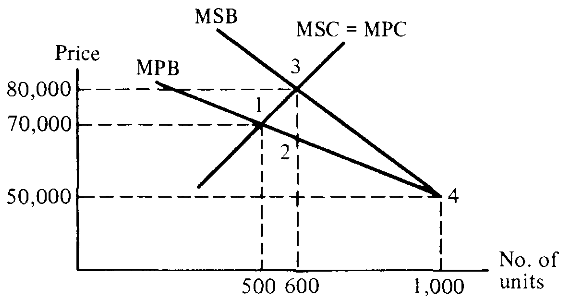


Figure 17.5 Example of positive externality.

Example 17.3

A cement factory that supplies its products to many builders in the region discharges its industrial waste into a river, causing pollution downstream. The private market for cement is represented by the demand schedule (same as the MPB curve) and the supply schedule (same as the MPC curve) in Fig. 17.6. The market price and the quantity produced at the market equilibrium point 1 are P_1 and Q_1, respectively. To reduce the level of water pollution, the state government is considering levying a tax for each sack of cement produced beyond the quantity Q_2, which is considered the quantity necessary for satisfying the needs of the builders in the state. What is a reasonable amount for this tax?

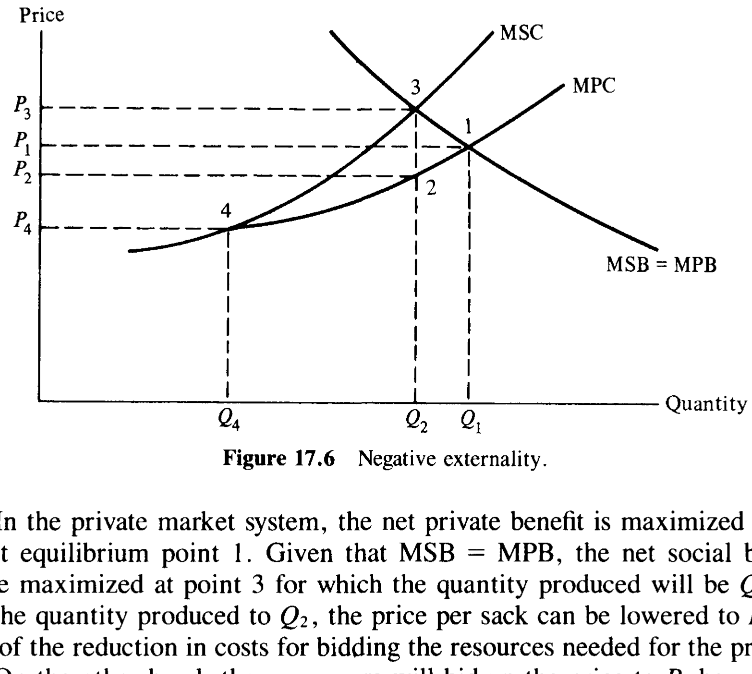


Figure 17.6 Negative externality.

In the private market system, the net private benefit is maximized at the market equilibrium point 1. Given that MSB = MPB, the net social benefit will be maximized at point 3 for which the quantity produced will be Q_2. To limit the quantity produced to Q_2, the price per sack can be lowered to P_2 because of the reduction in costs for bidding the resources needed for the production. On the other hand, the consumers will bid up the price to P_3 because of scarcity. Consequently, if the state government levies a tax of $(P_3 - P_2)$ for each sack of cement produced beyond the quantity Q_2, the company may comply with the limit of production voluntarily rather than pay the tax imposed by the government.

Note that $(P_3 - P_2)$ represents the negative marginal externality to users of the river water downstream. If a quantity less than Q_2 is produced, not only the negative marginal externality becomes smaller but the price per sack also becomes lower. If the quantity produced is below Q_4, no negative externality exists since the marginal social cost cannot be less than the marginal private cost.

Example 17.4

Suppose that the curves in Fig. 17.6 for Example 17.3 can be approximated by straight lines. The values corresponding to the intersections of these straight lines are given as follows: P_1 = \$10, P_3 = \$12, P_4 = \$6, Q_1 = 10,000 sacks, Q_2 = 9,000 sacks, and Q_4 = 6,000 sacks. Determine the tax per sack of cement that the state should levy to discourage the production beyond the level of 9,000 sacks.

The straight-line approximation of Fig. 17.6 is shown in Fig. 17.7 from which we find

$$P_1 = 10$$

$$P_2 = 10 - (10 - 6)\frac{(10{,}000 - 9{,}000)}{(10{,}000 - 6{,}000)} = 9$$

$$P_3 - P_2 = 12 - 9 = 3$$

Hence, the state should levy a tax of \$3 per sack of cement.

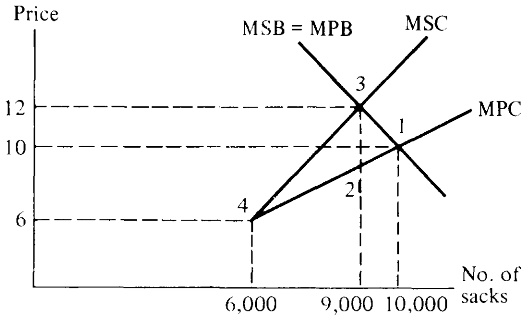


Figure 17.7 Example of negative externality.

17.5 CONSUMERS' SURPLUS

Let the market price and the quantity produced at the market equilibrium point 1 be represented by P_1 and Q_1, respectively, as shown in Fig. 17.8(a). The area under the demand curve up to the point Q_1 represents the amount that the consumers would be willing to pay for Q_1 units, and the total revenue R_1 to producers is the product $P_1 Q_1$ as represented by the rectangular area. The shaded area S_1, which represents the difference between the area under the demand curve up to Q_1 and the rectangular area $P_1 Q_1$, is referred to as the *consumers' surplus*.

Suppose that the producers produce Q_2 units and sell them at a price P_2 as shown in Fig. 17.8(b), which is analogous to the case of positive externality in Fig. 17.4. Then the total revenue to the producers is $R_2 = P_2 Q_2$, and the consumer's surplus is S_2. By increasing the production from Q_1 to Q_2 units, the increase in private benefits to the producers is

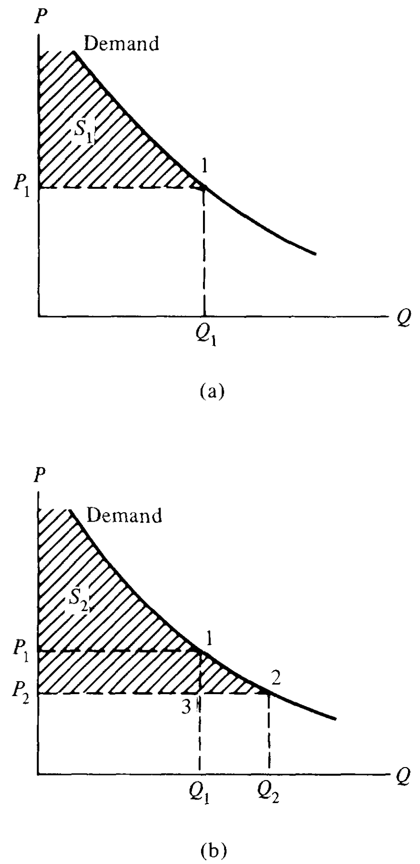


Figure 17.8 Change in consumers' surplus.

$$R_2 - R_1 = P_2Q_2 - P_1Q_1 \tag{17.3}$$

and the increase in the consumers' surplus is

$$S_2 - S_1 = (P_1 - P_2)Q_1 + \tfrac{1}{2}(P_1 - P_2)(Q_2 - Q_1) \tag{17.4}$$

The last term in Eq. (17.4) is an approximation of the area 1–2–3 in Fig. 17.8(b) by a triangle. Since this term is very small when $(Q_2 - Q_1)$ is small, it is negligible compared to $(P_1 - P_2)Q_1$.

If the production is increased by one unit above Q_1, $(R_2 - R_1)$ represents the marginal private benefit to the producers, i,e.,

$$\text{MPB} = P_2Q_2 - P_1Q_1 \tag{17.5}$$

The sum of $(R_2 - R_1)$ and $(S_2 - S_1)$ represents the marginal social benefit to society (both producers and consumers), i.e., by neglecting the higher order term,

$$\text{MSB} = (P_2Q_2 - P_1Q_1) + (P_1 - P_2)Q_1 \tag{17.6}$$

What is the implication of the consumers' surplus? First, if the producer is a monopolist, it will stop production at a quantity Q_2 when $R_2 - R_1$ becomes zero, thus creating the situation of underproduction. Second, in making decisions on a public project, the consumers' surplus is an indication of the benefits to a group of potential users of the proposed public project. In each situation, a knowledge of the consumers' surplus helps to analyze the consequences of the action.

The consumers' surplus represents the benefits to a group of consumers who otherwise would be willing to pay a higher price. If the producer is a monopolist, it will stop production at a quantity when there will be no more increase in private benefit to the producer.

Example 17.5

Suppose that for the demand curve in Figs. 17.8(a) and (b), it is known that an increase of production from $Q_1 = 100$ units to $Q_2 = 101$ units will cause $P_1 = \$10.00$ to drop to $P_2 = \$9.95$. Determine the marginal private benefit and the marginal social benefit.

Using Eqs.(17.5) and (17.6), we obtain

$$\text{MPB} = (9.95)(101) - (10.00)(100) = \$4.95$$

$$\text{MSB} = 4.95 + (10.00 - 9.95)(100) = \$9.95$$

Example 17.6

The annual number of users in a section of toll road is 4.42 million vehicles. A proposed improvement on some sharp curves is expected to increase the annual number of users to 5.84 million vehicles. Because of the anticipated larger number of users, the toll charge for this section will be reduced from the current price of \$0.74 per vehicle to \$0.68 per vehicle. Find the change in consumers' surplus.

For this problem, $Q_1 = 4.42 \times 10^6$, $Q_2 = 5.84 \times 10^6$, $P_1 = \$0.74$, and $P_2 = \$0.68$. Hence, the consumers' surplus is

$$S_2 - S_1 = \tfrac{1}{2}(5.84 - 4.42)(10^6)(0.74 + 0.68) = \$1{,}008{,}200$$

17.6 SOCIAL EQUITY AND INCOME DISTRIBUTION

There are social and political as well as economic considerations in public investment decisions, even if the private market functions efficiently. A competitive market can respond only to the effective demand of consumers under the existing income distribution. However, a society may judge whether the prevailing income distribution is fair, and such a judgment is a matter for social and political debate rather than economic analysis.

Through taxation and fiscal policies, the government performs the distributive function of both costs and benefits in administering government programs among individuals. The general principles of fairness for taxation and social welfare programs are *horizontal equity* and *vertical equity*. In simple terms, horizontal equity means equal treatment of equals, and vertical equity means unequal treatment of unequals. Most frequently, personal incomes are used as the scale in classifying the "equal" and "unequal" groups.

Historically, these principles were first applied to taxation, thus requiring those in the same income group to pay the same tax and those belonging to higher income groups to pay higher taxes. When these principles were later applied to recipients of social welfare programs, they were extended to mean that those in the same income group receive the same benefits and those belonging to higher income groups receive fewer benefits.

Theoretically, it is desirable to provide a precise accounting of the groups who ultimately bear the burden of a tax or who are the eventual beneficiaries of a social program. Then, the general principles can be applied to measure the net benefits of both taxes and social programs to different groups of individuals. However, this concept is difficult to carry out in practice.

There are certain goods which are regarded as meritorious and their production is encouraged since they tend to produce positive externality to society. Such goods are referred to as *merit wants*. However, the distribution of merit wants often raises the question of fairness because they can be provided either through *supply-based* or *demand-based subsidies*. An example of merit want in the form of supply-based subsidy is the public housing for low-income families. Housing has long been singled out for government action because of the general belief that every person is entitled to a decent home. However, public housing units are rival and excludable in consumption, and not all low-income families can receive such benefits if they want to. Consequently, the provision of such merit wants violates the principles of social equity. On the other hand, merit wants in the form of demand-based subsidy, such as food stamps or any transfer payment in kind of goods and services (other than cash), can be distributed more fairly. Whether the merit wants are provided in the form of supply-based subsidy or demand-based subsidy, the government exercises the value judgment that these merit wants are good for the recipients who cannot legally exchange such goods for something else they prefer. Hence, the provision of merit wants is not justified on the ground of economic efficiency, nor even necessarily on the ground of social equity.

17.7 DIFFERENT VIEWS ON SOCIAL DISCOUNT RATES

There are many factors affecting the choice of social discount rates, and these factors are often related to the fiscal policy objectives. However, the strongest arguments are centered on the issues of decision under uncertainty and the proper way in dealing with the time-risk adjustments in the public sector in order to avoid misallocation of resources. The substantially different views on the determination of the social discount rates have profound implications for the evaluation of public projects. Consequently, the basic arguments supporting different views are summarized for consideration:

1. The first position is to treat risk in the public sector in the same way as that in the private sector in order to avoid overinvestment in the public sector at the expense of private investments. Since investments are discounted with respect to both time and risk in perfect capital markets, the market-determined rates of return should be used to evaluate public investment opportunities.[2] This position suffers from the presupposition of the existence of perfect markets, and the conclusion also needs to be modified for the consideration of corporate income tax.
2. The second position is developed along the same approach as the first one but with added refinements.[3] It is demonstrated that when the risks associated with a public investment are publicly borne, the total cost of risk bearing is insignificant. Therefore, the government should ignore uncertainty in evaluating public investments, and the choice of the rate of discount should in this case be independent of considerations of risk. However, many of the risks associated with public investments are borne by private individuals, and in such cases it is appropriate to discount for risk as would these individuals. This position seems more defensible than the argument that the government should ignore uncertainty and behave as if it were indifferent to risk because the government invests in a greater number of diverse projects and is able to pool risks to a much greater extent than private investors.[4]
3. The third position asserts that time and risk preferences relevant for government action should be established as a matter of policy.[5] This position is based on the arguments that the government has a broader responsibility than that of its individual members and that capital markets are so imperfect that they do not reflect the time and risk preferences of individuals. Therefore, the appropriate discount rate should be inferred from the national objectives. The weakness of this position is that the adoption of an artificially low discount rate for public investments as a national policy constitutes a redistribution of income from present to future generations as well as misallocation of resources between the public and private sectors.

It appears that the second position advanced by Arrow and Lind[6] is most attractive from the theoretical viewpoint. We shall therefore explore this approach fur-

[2] See Refs. 17.9 and 17.10.

[3] See Ref. 17.1.

[4] See Ref. 17.16.

[5] See Ref. 17.14.

[6] See Ref. 17.1.

ther in its application. If the uncertain returns from a given public investment are independent of other components of national income, and if the size of the share of investment borne by each taxpayer is a negligible component of income, then the cost of risk bearing associated with holding it will be small. Under these conditions, the total cost of risk bearing is also small, since the risk associated with any investment is distributed among a large number of people, not because the government is able to pool investments. Therefore, the choice of social discount rate should be independent of risk. If private investments are replaced by public investments yielding a lower return as a result of the choice, the government should serve the role of a supplier of insurance by subsidizing private investments where appropriate. Although the philosophical foundation of this approach is by no means universally accepted, more economists seem to agree with the recommendation that *public investments should be evaluated according to their net present values computed on the basis of expected net returns, using a social discount rate equivalent to the market-determined risk-free rate for private investments.*

Another important aspect of the position suggested by Arrow and Lind has received less attention than it deserves, perhaps because of potential difficulties in its application. However, it provides some useful insights on cases where the risks of public investments are borne by private individuals. Under such circumstances, some benefits and costs will accrue to the government and the uncertainties involved will be publicly borne; other benefits and costs will accrue to individuals and the attendant uncertainties will be borne privately. Hence, two different streams of benefits and costs should be treated differently with respect to uncertainty, using a risk-free rate for discounting the stream of benefits and costs accruing to government and higher rates reflecting the risk for the stream accruing directly to individuals. Such a procedure causes some difficulties in making the distinction between private and public benefits and costs, but the problem does not appear to be insurmountable. For the typical case where costs are borne publicly and benefits accrue privately, this procedure calls for the use of a risk-free discount rate for the costs but higher discount rates for the benefits, depending on the nature of the uncertainty and time-risk preferences of the individuals who receive these benefits.

The practical implications of the cases where the risks of public investments are borne by private individuals can easily be understood with some simple illustrations. For example, the public education system in the United States has been viewed as an investment in human capital where the social costs include the provision of physical facilities and staffing and the social benefits include the improvement of the character and ability of its citizens. It is generally agreed that the benefits and costs which accrue to the society should be discounted at a low social discount rate. However, the benefits and costs that accrue to individual students will be viewed differently by them and their parents. For those who value their future highly will use a low discount rate for future expected returns in computing their net present values and thus regard high school education as the key to upward mobility; whereas those who value the present as being far more important than the future will choose a high discount rate for future expected returns in computing their net present values and thus conclude that it is to their own advantage to drop out of high school. It has been argued by Banfield[7] that the division of social groups according to their attitudes toward present and future, i.e., their choice of social discount rates, is far more significant than other forms of grouping in order to understand the persistence of poverty in society.

It is ironic that for the typical case where costs are borne publicly and benefits accrue privately, the procedure of adopting a lower discount rate for public costs and a higher discount rate for private benefits will lead to a lower total net present value than a procedure using a single rate (either the higher or the lower) to discount both benefits and costs. However, the concept of treating cases in which risks are associated with public investment differently from those cases where the risks are publicly borne is still an ideal which is seldom put into practice.

Example 17.7

A small water project requires an initial investment of \$340,000 and an annual operating cost of \$50,000 in the next 5 years with no salvage value at the end of 5 years. The annual benefit will be \$150,000 at the end of the first year and will decrease by \$10,000 linearly in each succeeding year. Determine the net present value on each of the following conditions for the discount rate: (1) both benefits and costs are discounted at 3%, (2) both are discounted at 6%, and (3) benefits that will accrue to individuals are discounted at 6% and costs that will be borne by the public are discounted at 3%.

The net present value is obtained by discounting the benefits at i_b and the costs at i_c as follows:

$$\text{NPV} = -340{,}000 - (50{,}000)(P \mid U, i_c, 5) + (150{,}000)(P \mid U, i_b, 5) - (10{,}000)(P \mid G, i_b, 5)$$

1. For $i_b = 3\%$ and $i_c = 3\%$

$$\text{NPV} = -340{,}000 + (100{,}000)(4.5797) - (10{,}000)(8.8887) = 9{,}083$$

2. For $i_b = 6\%$ and $i_c = 6\%$

$$\text{NPV} = -340{,}000 + (100{,}000)(4.2124) - (10{,}000)(7.9345) = 1{,}895$$

3. For $i_b = 6\%$ and $i_c = 3\%$

$$\text{NPV} = -340{,}000 - (50{,}000)(4.5797) + (150{,}000)(4.2124) - (10{,}000)(7.9345) = -16{,}470$$

[7] See Ref. 17.2, pp. 46–54.

17.8 BENEFIT-COST RATIO FOR UNCONVENTIONAL CASH FLOWS

Since the benefit-cost ratio is a popular profit measure for public projects, it is important to emphasize again that such a measure is affected by the way that benefits and costs are assessed. In particular, when an unconventional cash flow profile is encountered, we must take great care to ensure that the discounted benefits and costs are treated properly.

In practice, the net cash flow profile of a project is sometimes the only investment information available. In that case, the benefits and costs are identified with the net receipts and disbursements respectively in the profile. When the net cash flow profile consists of two or more sign changes, the transition point $t = e$ in the profile A_t must be determined according to the principle developed in Section 16.7. From Eqs. (16.15) and (16.16), we get for a project x

$$\text{CPV}_x = -P_e = -\sum_{t=0}^{e} A_t(P \mid F, i, t) \tag{17.7}$$

$$\text{BPV}_x = F_e(P \mid F, i, n) = (P \mid F, i, n) \sum_{t=e+1}^{n} A_t(F \mid P, i, n - t) \tag{17.8}$$

Then, these values of BPV_x and CPV_x should be used in the computation of the ratio $\text{BPV}_x/\text{CPV}_x$.

It can be seen that the values of CPV_x and BPV_x as computed by Eqs. (17.7) and (17.8) are unique; hence, $\text{BPV}_x/\text{CPV}_x$ computed on this basis is also unique.

Example 17.8

Find the benefit-cost ratio of an investment project as represented by the cash flow profile in millions of dollars as follows: $A_0 = +18$, $A_1 = +10$, $A_2 = -40$, $A_3 = -60$, $A_4 = +30$, and $A_5 = +50$. The MARR of the organization is 8%.

Since $t = 3$ is the only time period at which the sign of A_t changes from negative to positive in the following period, $t = 3$ is the transition point. Then,

$$\begin{aligned}\text{CPV} = -P_e &= -[18 + (10)(P \mid F, 8\%, 1) \\ &\qquad - (40)(P \mid F, 8\%, 2) - (60)(P \mid F, 8\%, 3)] \\ &= 54.665\end{aligned}$$

$$\begin{aligned}\text{BPV} &= (P \mid F, 8\%, 5)[(30)(P \mid F, 8\%, 1) + 50] \\ &= (0.6806)(84.2) = 57.306\end{aligned}$$

Hence, the benefit-cost ratio is

$$\frac{\text{BPV}}{\text{CPV}} = 1.048$$

17.9 MULTIOBJECTIVES IN THE CHOICE OF SOCIAL DISCOUNT RATES

There are strong arguments that the private market will not make adequate provision for public investment because of problems of externalities and other market imperfections. The classical arguments in favor of a larger share of the national wealth for the public sectors are familiar ones. In the words of Galbraith,[8]

> [W]e must find a way to remedy the poverty which afflicts us in public services and which is in such increasingly bizarre contrast with our affluence in private goods. This is necessary to temper and, more hopefully, to eliminate the social disorders which are the counterpart of the present imbalance. It is necessary in the long run for promoting the growth of private output itself. Such balance is a matter of elementary common sense in a country in which need is becoming so exiguous that it must be cherished where it exists and nurtured where it does not.

However, the proponents of this view often regard the choice of lower social discount rates as a substitute for public subsidies in serving the distribution and stabilization functions of the fiscal policy objectives. Hence, the choice of social discount rates has created an unnecessary issue in favor of public projects with long-term commitments.

In the discussion of social discount rate for public projects, Marglin[9] has inadvertently also raised the question of investment for present versus future generations. Simply put: "why do governments require citizens to sacrifice current consumption in order to undertake investments that will not yield their benefits until those called upon to make the sacrifice are all dead?" Thus, we are confronted with the ethical issues as well as the economic issues in considering long-term versus short-term public projects. Both of these issues have been examined by a number of economists who have clarified the nature of the problems.[10]

A balanced view toward the allocation of investment for the public sector between long-term and short-term projects has been presented by Baumol.[11] Citing the argument of Tullock[12] that the next generation is likely going to be wealthier than this generation and that we may want to redistribute income in favor of the present poor rather than the future poor, he correctly points out that an argument against a lower social discount rate is not necessarily an argument against activities in the public sector. It may well be in society's interest to spend more on short-term projects in solving today's problems and less on the long-term projects that will bring more wealth to future generations, given the total amount for the public sector. Such

[8] See Ref. 17.6, p. 271.

[9] See Ref. 17.14, p. 95.

[10] See Refs. 17.4, 17.11, and 17.18.

[11] See Ref. 17.4.

[12] See Ref. 17.18.

spending may serve the stabilization function of the fiscal policy by expanding employment and social order. Baumol summarizes his view succinctly[13]:

> However, this does not mean that the future should in every respect be left to the mercy of the free market. There are important externalities and investments of the public goods variety which cry for special attention. Irreversibilities constitute a prime example. If we poison our soil so that never again will it be the same, if we destroy the Grand Canyon and turn it into a hydroelectric plant, we give up assets which like Goldsmith's bold peasantry, ". . . their country's pride, when once destroy'd can never be supplied." All the wealth and resources of future generations will not suffice to restore them. Investment in the preservation of such items then seems perfectly proper, but for this purpose the appropriate instrument would appear to be a set of selective subsidies rather than a low general discount rate that encourages indiscriminately all sorts of investment programs whether or not they are relevant.
>
> Moreover, one can envision circumstances in which a more general program of encouragement to investment commends itself to us. In a country which is stagnating and where only a major restriction of current consumption can put life into its development program, one may well wish to make the sacrifice for tomorrow, for in such a case, without it the future generation will be as impoverished as the present.

So far, we have considered the fiscal policy for the public sector strictly from the national point of view. Actually, the fiscal system is decentralized among federal, state, and local governments. Fiscal decentralization permits adaptation of budget patterns to the preferences of the residents of particular jurisdictions. Ideally, the decentralized fiscal system encourages national services provided collectively on a nationwide basis, regional services provided for regionally, and local services accounted for locally. Except for national defense and other items which benefit the entire population in the nation, most public projects generate benefits which are spatially limited. However, because of the power of the federal government to levy tax, many public projects are provided for at the local level even though the funding may be supplied fully or partially by state and federal appropriations. This practice often produces the net effect of promoting stabilization of employment in some localities and regions at the expense of economic efficiency for resource allocation.

17.10 THE CHOICE OF SOCIAL DISCOUNT RATES IN PRACTICE

In recent years, there has been a greater recognition of the importance of social discount rate in the allocation function of government. Consequently, the minimum attractive rates of return on the public investments have been adjusted upward in order to avoid misallocation of resources.

The Office of Management and Budget (OMB) requires since 1972 that, with certain exceptions, federal agencies must use a real discount rate of 10% to evaluate

[13] See Ref. 17.4, p. 801.

federal investment decisions.[14] This real rate of 10%, excluding inflation, is comparable to the opportunity of cost of capital in the private sector.

To avoid misallocation of resources among different federal agencies, a consistent policy should be applied uniformly to all government programs. However, for various reasons, some of which are political, certain programs are exempt from the OMB requirements. An example of the exception to the OMB requirements is the discount rate used for water projects which was set by the U.S. Congress over the years. A brief review of the history of the discount rates established for water projects will give some insight into the controversy that arises from the choice of social discount rates.[15]

Prior to 1968, the discount rate for water projects was set at the coupon rate of interest on U.S. government securities that had *original* terms of maturity of 15 years or more. This rule resulted in the low discount rate of $2\frac{5}{8}\%$ in the 1950s and early 1960s. By the late 1960s the yields on government securities were considerably higher than the average of the original coupon rates on outstanding government securities. In 1968, the rule for computing discount rates was changed to the yield during the preceding fiscal year on interest-bearing marketable securities of the United States, which at the time the computation is made have terms of 15 years or more remaining to maturity. This change had the effect of rapidly increasing the discount rate. To dampen this effect, a ceiling was introduced in 1974 such that "in no event shall the rate be raised or lowered more than one-quarter of one percent for any year." A grandfather clause in the Water Resources Development Act of 1974 also specifies that the discount rate used at the time of a project's authorization is binding, even though economic conditions at the time a project finally receives funding may have changed. Since many years may elapse between project authorization and appropriations for a project, the official discount rate at the time of appropriations is often higher than that at the time of projection authorization. The grandfather clause has the net effect of keeping the discount rate lower.

17.11 SUMMARY AND STUDY GUIDE

This chapter has presented a cursory treatment of the broad social and political issues which often enter into the evaluation of public projects and has provided some understanding of the most important factors affecting public investment decisions.

We began with a discussion of the provision of public projects and pointed out the distinction between the provision of public projects and public production. We also mentioned briefly the types of market failure that have prompted government regulation and the types of goods and services which are the reasons for government action.

[14] OMB Circular A-94, "Discount Rates to Be used in Evaluating Time-Distributed Costs and Benefits," Executive Office of the President, March 27, 1972.

[15] See Ref. 17.8.

We then examined the fiscal policy objectives for the public sector, including the allocation function, the distribution function, and the stabilization function. We emphasized the significance of economic efficiency and the effects of externalities, and rationale for dealing with such problems is illustrated qualitatively with examples.

We also discussed the general principles of horizontal equity and vertical equity for taxation and social welfare programs, particularly with respect to income distribution. Furthermore, we considered the supply-based and demand-based subsidies in providing merit wants.

The problems of selecting the appropriate social discount rates and their significance in public investment decisions were examined. The multiobjectives in the choice of social discount rates were also discussed. Finally, the problems associated with the choice of social discount rates in practice as well as the problems in connection with the estimation of costs and benefits for public projects were also reviewed briefly.

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PROBLEMS

P17.1 A private caterer that provides lunch service for a university cafeteria offers a standard menu at \$2 per person and has attracted 1,000 students to contract for the meal. To supply the same meal to 2,000 students, the caterer would have to raise the price to \$2.50 per person because it has to forego other outside commitments. However, the university likes to encourage more students to participate in the lunch for social and other benefits and is willing to provide subsidy to 2,000 students. If it wants to subsidize the meal for all of its 4,000 students, the price per person would have to be \$0.50 to find all takers since some students do not want to eat in the cafeteria unless it is a bargain. What amount should the university be willing to subsidize the price per student in order to produce the maximum social benefits? Assume that the MSB, MSC, MPB, and MPC curves can be approximated by straight lines.

P17.2 A sewage treatment plant that processes large quantities of sewage materials from local communities emits an unpleasant odor in its surrounding area. The maximum capacity of the plant for efficient operation is 250 mgd (million gallons per day), and it charges the customers a fee of \$0.30 per gallon of discharge. If the plant operates at 100 mgd, there will be no emission of unpleasant odor, and it can charge \$0.12 per gallon of discharge. A reasonable compromise is to process 200 mgd and charge the customers a fee of \$0.42 per gallon of discharge. What fee per gallon of discharge should the local government assess the sewage plant for maintaining the desirable level of 200 mgd in order to minimize the social costs? Assume that the MSB, MSC, MPB, and MPC curves can be approximated by straight lines.

P17.3 Referring to Fig. 17.8 in this chapter, determine the increase in private benefits to the producers and the increase in consumers' surplus for a product if P_1 = \$20, P_2 = \$15, Q_1 = 200 units, and Q_2 = 300 units.

P17.4 Referring to Fig. 17.8 in this chapter, determine the marginal private benefit to the producers and the marginal social benefit to society if P_1 = \$80, P_2 = \$75, Q_1 = 15 units, and Q_2 = 16 units. If the producer is a monopolist, is there any incentive for him or her to produce more than 15 units?

P17.5 The annual number of barges that pass through a lock in an inland waterway is 14,400 and the user fee is \$20 per barge. A proposed improvement is expected to increase the annual capacity to handle 18,000 barges and the user fee will be reduced to \$18 per barge. Find the change in consumers' surplus.

P17.6 The cost of a college education to an average student attending a state-supported university is estimated to be as shown in the following table. Upon graduation, an average student is expected to earn $5,000 more per year in the next 45 years in comparison with what his or her earnings would be if he or she chooses not to attend college. Determine the net present value of the cash flows depicting the costs and benefits of this college education if the MARR of the student is (1) 6% and (2) 12%.

	Private cost				
Year	Tuition	Expenses	Foregone earning	Total	Private benefit (increased earning)
0	$2,000	$4,000	0	$6,000	0
1-3 (each)	$2,000	$4,000	$9,000	$15,000	0
4	0	0	$9,000	$9,000	0
5-49 (each)	0	0	0	0	$5,000

P17.7 In allocating state funds to state-supported colleges, the state legislature considers the social costs and benefits of this human capital investment and comes up with the data shown in the following table. The public subsidy per student is $8,000 per year above the total private cost, which includes tuition, expenses, and foregone earnings of the student. The benefit to the state is better educated citizens who, among other things, can generate more job opportunities for others as well as increase their own private gains. If the MARR used by the state legislature is 6%, determine the net present value of social benefit over social cost per student supported by the state.

	Social cost per student			Social benefit per student		
Year	Private cost	State subsidy	Total	Private	Benefit to state	Total
0	$6,000	$8,000	$14,000	0	0	0
1-3 (each)	$15,000	$8,000	$23,000	0	0	0
4	$9,000	0	$9,000	0	0	0
5-49 (each)	0	0	0	$5,000	$2,000	$7,000

P17.8 A public project that requires an initial investment of $180,000 and a uniform annual operating cost of $40,000 in the next 40 years is expected to generate a uniform annual benefit of $63,000 over the planning horizon of 40 years. There will be no salvage value. The opportunity cost for similar projects in the private sector is approximately 12%, allowing for the difference in taxes. The social discount rate for similar projects has been set at 6% in the past. Determine the benefit-cost ratio of this project on the basis of each of the following conditions: (1) both costs and benefits are discounted at 12%, (2) both are discounted at 6%, and (3) costs that will be borne by the public are discounted at 6% and benefits that will accrue to individuals are discounted at 12%.

P.17.9 Find the benefit-cost ratio of an investment project with the following unconventional cash flow profile: $A_0 = -\$100{,}000$, $A_1 = +\$20{,}000$, $A_2 = -\$24{,}200$, $A_3 = +\$10{,}000$, $A_4 = +\$75{,}000$, and $A_5 = +\$90{,}000$. The MARR of the government agency is 10%.

P17.10 Find the benefit-cost ratio of an investment project with the following unconventional cash flow profile: $A_0 = -\$80{,}000$, $A_1 = +\$60{,}000$, $A_2 = +\$25{,}000$, $A_3 = -\$90{,}000$, $A_4 = +\$100{,}000$, and $A_5 = +\$27{,}241$. The MARR of the public agency is 10%.

APPENDIX

A Discrete Compound Interest Tables

TABLE A.1 DISCRETE COMPOUND INTEREST TABLE *i* = 0.5%

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.0050	0.9950	1.0000	1.0000	0.9950	1.0050	0.0000
2	1.0100	0.9901	2.0050	0.4988	1.9851	0.5038	0.9893
3	1.0151	0.9851	3.0150	0.3317	2.9702	0.3367	2.9589
4	1.0202	0.9802	4.0301	0.2481	3.9505	0.2531	5.8992
5	1.0253	0.9754	5.0502	0.1980	4.9259	0.2030	9.8003
6	1.0304	0.9705	6.0755	0.1646	5.8964	0.1696	14.6522
7	1.0355	0.9657	7.1059	0.1407	6.8621	0.1457	20.4456
8	1.0407	0.9609	8.1414	0.1228	7.8229	0.1278	27.1712
9	1.0459	0.9561	9.1821	0.1089	8.7790	0.1139	34.8193
10	1.0511	0.9513	10.2280	0.0978	9.7304	0.1028	43.3813
11	1.0564	0.9466	11.2791	0.0887	10.6770	0.0937	52.8469
12	1.0617	0.9419	12.3355	0.0811	11.6189	0.0861	63.2069
13	1.0670	0.9372	13.3972	0.0746	12.5561	0.0796	74.4536
14	1.0723	0.9326	14.4642	0.0691	13.4887	0.0741	86.5757
15	1.0777	0.9279	15.5365	0.0644	14.4166	0.0694	99.5659
16	1.0831	0.9233	16.6142	0.0602	15.3399	0.0652	113.4152
17	1.0885	0.9187	17.6973	0.0565	16.2586	0.0615	128.1139
18	1.0939	0.9141	18.7857	0.0532	17.1727	0.0582	143.6539
19	1.0994	0.9096	19.8797	0.0503	18.0823	0.0553	160.0257
20	1.1049	0.9051	20.9791	0.0477	18.9874	0.0527	177.2214
21	1.1104	0.9006	22.0840	0.0453	19.8879	0.0503	195.2325
22	1.1160	0.8961	23.1944	0.0431	20.7840	0.0481	214.0491
23	1.1216	0.8916	24.3103	0.0411	21.6756	0.0461	233.6642
24	1.1272	0.8872	25.4319	0.0393	22.5628	0.0443	254.0688
25	1.1328	0.8828	26.5590	0.0377	23.4456	0.0427	275.2549
26	1.1385	0.8784	27.6918	0.0361	24.3240	0.0411	297.2139
27	1.1442	0.8740	28.8303	0.0347	25.1980	0.0397	319.9378
28	1.1499	0.8697	29.9744	0.0334	26.0676	0.0384	343.4179
29	1.1556	0.8653	31.1243	0.0321	26.9330	0.0371	367.6473
30	1.1614	0.8610	32.2799	0.0310	27.7940	0.0360	392.6160
35	1.1907	0.8398	38.1453	0.0262	32.0353	0.0312	528.2935
40	1.2208	0.8191	44.1587	0.0226	36.1721	0.0276	681.3133
45	1.2516	0.7990	50.3240	0.0199	40.2071	0.0249	850.7396
50	1.2832	0.7793	56.6450	0.0177	44.1427	0.0227	1035.6700
55	1.3156	0.7601	63.1256	0.0158	47.9813	0.0208	1235.2395
60	1.3488	0.7414	69.7698	0.0143	51.7254	0.0193	1448.6141
70	1.4178	0.7053	83.5658	0.0120	58.9393	0.0170	1913.6063
80	1.4903	0.6710	98.0674	0.0102	65.8022	0.0152	2424.6044
90	1.5666	0.6383	113.3105	0.0088	72.3311	0.0138	2976.0311
100	1.6467	0.6073	129.3333	0.0077	78.5425	0.0127	3562.7440

TABLE A.2 DISCRETE COMPOUND INTEREST TABLE $i = 1.0\%$

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.0100	0.9901	1.0000	1.0000	0.9901	1.0100	0.0000
2	1.0201	0.9803	2.0100	0.4975	1.9704	0.5075	0.9803
3	1.0303	0.9706	3.0301	0.3300	2.9410	0.3400	2.9214
4	1.0406	0.9610	4.0604	0.2463	3.9020	0.2563	5.8043
5	1.0510	0.9515	5.1010	0.1960	4.8534	0.2060	9.6101
6	1.0615	0.9420	6.1520	0.1625	5.7955	0.1725	14.3203
7	1.0721	0.9327	7.2135	0.1386	6.7282	0.1486	19.9165
8	1.0829	0.9235	8.2857	0.1207	7.6517	0.1307	26.3809
9	1.0937	0.9143	9.3685	0.1067	8.5660	0.1167	33.6956
10	1.1046	0.9053	10.4622	0.0956	9.4713	0.1056	41.8431
11	1.1157	0.8963	11.5668	0.0865	10.3676	0.0965	50.8063
12	1.1268	0.8874	12.6825	0.0788	11.2551	0.0888	60.5682
13	1.1381	0.8787	13.8093	0.0724	12.1337	0.0824	71.1121
14	1.1495	0.8700	14.9474	0.0669	13.0037	0.0769	82.4215
15	1.1610	0.8613	16.0969	0.0621	13.8650	0.0721	94.4803
16	1.1726	0.8528	17.2579	0.0579	14.7179	0.0679	107.2726
17	1.1843	0.8444	18.4304	0.0543	15.5622	0.0643	120.7827
18	1.1961	0.8360	19.6147	0.0510	16.3983	0.0610	134.9949
19	1.2081	0.8277	20.8109	0.0481	17.2260	0.0581	149.8941
20	1.2202	0.8195	22.0190	0.0454	18.0455	0.0554	165.4655
21	1.2324	0.8114	23.2392	0.0430	18.8570	0.0530	181.6940
22	1.2447	0.8034	24.4716	0.0409	19.6604	0.0509	198.5653
23	1.2572	0.7954	25.7163	0.0389	20.4558	0.0489	216.0648
24	1.2697	0.7876	26.9735	0.0371	21.2434	0.0471	234.1789
25	1.2824	0.7798	28.2432	0.0354	22.0231	0.0454	252.8934
26	1.2953	0.7720	29.5256	0.0339	22.7952	0.0439	272.1944
27	1.3082	0.7644	30.8209	0.0324	23.5596	0.0424	292.0690
28	1.3213	0.7568	32.1291	0.0311	24.3164	0.0411	312.5034
29	1.3345	0.7493	33.4504	0.0299	25.0658	0.0399	333.4850
30	1.3478	0.7419	34.7849	0.0287	25.8077	0.0387	355.0007
35	1.4166	0.7059	41.6603	0.0240	29.4086	0.0340	470.1567
40	1.4889	0.6717	48.8863	0.0205	32.8347	0.0305	596.8543
45	1.5648	0.6391	56.4810	0.0177	36.0945	0.0277	733.7017
50	1.6446	0.6080	64.4631	0.0155	39.1961	0.0255	879.4155
55	1.7285	0.5785	72.8524	0.0137	42.1472	0.0237	1032.8123
60	1.8167	0.5504	81.6696	0.0122	44.9550	0.0222	1192.8036
70	2.0068	0.4983	100.6763	0.0099	50.1685	0.0199	1528.6446
80	2.2167	0.4511	121.6714	0.0082	54.8882	0.0182	1879.8739
90	2.4486	0.4084	144.8632	0.0069	59.1609	0.0169	2240.5641
100	2.7048	0.3697	170.4812	0.0059	63.0289	0.0159	2605.7721

TABLE A.3 DISCRETE COMPOUND INTEREST TABLE $i = 2.0\%$

	COMPOUND AMOUNT FACTOR	DISCOUNT AMOUNT FACTOR	COMPOUND UNIFORM SERIES FACTOR	SINKING FUND FACTOR	DISCOUNT UNIFORM SERIES FACTOR	CAPITAL RECOVERY FACTOR	DISCOUNT GRADIENT FACTOR
n	(F|P,i,n)	(P|F,i,n)	(F|U,i,n)	(U|F,i,n)	(P|U,i,n)	(U|P,i,n)	(P|G,i,n)
1	1.0200	0.9804	1.0000	1.0000	0.9804	1.0200	0.0000
2	1.0404	0.9612	2.0200	0.4950	1.9416	0.5150	0.9611
3	1.0612	0.9423	3.0604	0.3268	2.8839	0.3468	2.8457
4	1.0824	0.9238	4.1216	0.2426	3.8077	0.2626	5.6173
5	1.1041	0.9057	5.2040	0.1922	4.7135	0.2122	9.2402
6	1.1262	0.8880	6.3081	0.1585	5.6014	0.1785	13.6800
7	1.1487	0.8706	7.4343	0.1345	6.4720	0.1545	18.9033
8	1.1717	0.8535	8.5830	0.1165	7.3255	0.1365	24.8777
9	1.1951	0.8368	9.7546	0.1025	8.1622	0.1225	31.5718
10	1.2190	0.8203	10.9497	0.0913	8.9826	0.1113	38.9549
11	1.2434	0.8043	12.1687	0.0822	9.7868	0.1022	46.9975
12	1.2682	0.7885	13.4121	0.0746	10.5753	0.0946	55.6709
13	1.2936	0.7730	14.6803	0.0681	11.3484	0.0881	64.9472
14	1.3195	0.7579	15.9739	0.0626	12.1062	0.0826	74.7996
15	1.3459	0.7430	17.2934	0.0578	12.8493	0.0778	85.2017
16	1.3728	0.7284	18.6393	0.0537	13.5777	0.0737	96.1284
17	1.4002	0.7142	20.0121	0.0500	14.2919	0.0700	107.5550
18	1.4282	0.7002	21.4123	0.0467	14.9920	0.0667	119.4577
19	1.4568	0.6864	22.8405	0.0438	15.6785	0.0638	131.8134
20	1.4859	0.6730	24.2974	0.0412	16.3514	0.0612	144.5998
21	1.5157	0.6598	25.7833	0.0388	17.0112	0.0588	157.7954
22	1.5460	0.6468	27.2990	0.0366	17.6580	0.0566	171.3789
23	1.5769	0.6342	28.8449	0.0347	18.2922	0.0547	185.3303
24	1.6084	0.6217	30.4218	0.0329	18.9139	0.0529	199.6299
25	1.6406	0.6095	32.0303	0.0312	19.5234	0.0512	214.2587
26	1.6734	0.5976	33.6709	0.0297	20.1210	0.0497	229.1981
27	1.7069	0.5859	35.3443	0.0283	20.7069	0.0483	244.4305
28	1.7410	0.5744	37.0512	0.0270	21.2813	0.0470	259.9386
29	1.7758	0.5631	38.7922	0.0258	21.8444	0.0458	275.7057
30	1.8114	0.5521	40.5681	0.0246	22.3964	0.0446	291.7158
35	1.9999	0.5000	49.9944	0.0200	24.9986	0.0400	374.8819
40	2.2080	0.4529	60.4019	0.0166	27.3555	0.0366	461.9923
45	2.4379	0.4102	71.8927	0.0139	29.4902	0.0339	551.5643
50	2.6916	0.3715	84.5793	0.0118	31.4236	0.0318	642.3596
55	2.9717	0.3365	98.5864	0.0101	33.1748	0.0301	733.3516
60	3.2810	0.3048	114.0514	0.0088	34.7609	0.0288	823.6965
70	3.9996	0.2500	149.9778	0.0067	37.4986	0.0267	999.8332
80	4.8754	0.2051	193.7718	0.0052	39.7445	0.0252	1166.7856
90	5.9431	0.1683	247.1564	0.0040	41.5869	0.0240	1322.1690
100	7.2446	0.1380	312.2319	0.0032	43.0983	0.0232	1464.7516

TABLE A.4 DISCRETE COMPOUND INTEREST TABLE *i* = 3.0%

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.0300	0.9709	1.0000	1.0000	0.9709	1.0300	0.0000
2	1.0609	0.9426	2.0300	0.4926	1.9135	0.5226	0.9426
3	1.0927	0.9151	3.0909	0.3235	2.8286	0.3535	2.7729
4	1.1255	0.8885	4.1836	0.2390	3.7171	0.2690	5.4383
5	1.1593	0.8626	5.3091	0.1884	4.5797	0.2184	8.8887
6	1.1941	0.8375	6.4684	0.1546	5.4172	0.1846	13.0761
7	1.2299	0.8131	7.6625	0.1305	6.2303	0.1605	17.9547
8	1.2668	0.7894	8.8923	0.1125	7.0197	0.1425	23.4805
9	1.3048	0.7664	10.1591	0.0984	7.7861	0.1284	29.6119
10	1.3439	0.7441	11.4639	0.0872	8.5302	0.1172	36.3087
11	1.3842	0.7224	12.8078	0.0781	9.2526	0.1081	43.5329
12	1.4258	0.7014	14.1920	0.0705	9.9540	0.1005	51.2481
13	1.4685	0.6810	15.6178	0.0640	10.6350	0.0940	59.4195
14	1.5126	0.6611	17.0863	0.0585	11.2961	0.0885	68.0140
15	1.5580	0.6419	18.5989	0.0538	11.9379	0.0838	77.0001
16	1.6047	0.6232	20.1569	0.0496	12.5611	0.0796	86.3475
17	1.6528	0.6050	21.7616	0.0460	13.1661	0.0760	96.0278
18	1.7024	0.5874	23.4144	0.0427	13.7535	0.0727	106.0135
19	1.7535	0.5703	25.1169	0.0398	14.3238	0.0698	116.2786
20	1.8061	0.5537	26.8704	0.0372	14.8775	0.0672	126.7985
21	1.8603	0.5375	28.6765	0.0349	15.4150	0.0649	137.5495
22	1.9161	0.5219	30.5368	0.0327	15.9369	0.0627	148.5092
23	1.9736	0.5067	32.4529	0.0308	16.4436	0.0608	159.6564
24	2.0328	0.4919	34.4265	0.0290	16.9355	0.0590	170.9709
25	2.0938	0.4776	36.4592	0.0274	17.4131	0.0574	182.4334
26	2.1566	0.4637	38.5530	0.0259	17.8768	0.0559	194.0258
27	2.2213	0.4502	40.7096	0.0246	18.3270	0.0546	205.7307
28	2.2879	0.4371	42.9309	0.0233	18.7641	0.0533	217.5317
29	2.3566	0.4243	45.2188	0.0221	19.1885	0.0521	229.4134
30	2.4273	0.4120	47.5754	0.0210	19.6004	0.0510	241.3610
35	2.8139	0.3554	60.4621	0.0165	21.4872	0.0465	301.6264
40	3.2620	0.3066	75.4012	0.0133	23.1148	0.0433	361.7497
45	3.7816	0.2644	92.7198	0.0108	24.5187	0.0408	420.6322
50	4.3839	0.2281	112.7968	0.0089	25.7298	0.0389	477.4800
55	5.0821	0.1968	136.0715	0.0073	26.7744	0.0373	531.7408
60	5.8916	0.1697	163.0533	0.0061	27.6756	0.0361	583.0523
70	7.9178	0.1263	230.5939	0.0043	29.1234	0.0343	676.0866
80	10.6409	0.0940	321.3627	0.0031	30.2008	0.0331	756.0863
90	14.3005	0.0699	443.3485	0.0023	31.0024	0.0323	823.6300
100	19.2186	0.0520	607.2871	0.0016	31.5989	0.0316	879.8538

TABLE A.5 DISCRETE COMPOUND INTEREST TABLE $i = 4.0\%$

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.0400	0.9615	1.0000	1.0000	0.9615	1.0400	0.0000
2	1.0816	0.9246	2.0400	0.4902	1.8861	0.5302	0.9246
3	1.1249	0.8890	3.1216	0.3203	2.7751	0.3603	2.7026
4	1.1699	0.8548	4.2465	0.2355	3.6299	0.2755	5.2670
5	1.2167	0.8219	5.4163	0.1846	4.4518	0.2246	8.5547
6	1.2653	0.7903	6.6330	0.1508	5.2421	0.1908	12.5063
7	1.3159	0.7599	7.8983	0.1266	6.0021	0.1666	17.0658
8	1.3686	0.7307	9.2142	0.1085	6.7327	0.1485	22.1806
9	1.4233	0.7026	10.5828	0.0945	7.4353	0.1345	27.8013
10	1.4802	0.6756	12.0061	0.0833	8.1109	0.1233	33.8814
11	1.5395	0.6496	13.4864	0.0741	8.7605	0.1141	40.3772
12	1.6010	0.6246	15.0258	0.0666	9.3851	0.1066	47.2477
13	1.6651	0.6006	16.6268	0.0601	9.9856	0.1001	54.4546
14	1.7317	0.5775	18.2919	0.0547	10.5631	0.0947	61.9618
15	1.8009	0.5553	20.0236	0.0499	11.1184	0.0899	69.7355
16	1.8730	0.5339	21.8245	0.0458	11.6523	0.0858	77.7441
17	1.9479	0.5134	23.6975	0.0422	12.1657	0.0822	85.9581
18	2.0258	0.4936	25.6454	0.0390	12.6593	0.0790	94.3498
19	2.1068	0.4746	27.6712	0.0361	13.1339	0.0761	102.8934
20	2.1911	0.4564	29.7781	0.0336	13.5903	0.0736	111.5647
21	2.2788	0.4388	31.9692	0.0313	14.0292	0.0713	120.3414
22	2.3699	0.4220	34.2480	0.0292	14.4511	0.0692	129.2024
23	2.4647	0.4057	36.6179	0.0273	14.8568	0.0673	138.1284
24	2.5633	0.3901	39.0826	0.0256	15.2470	0.0656	147.1012
25	2.6658	0.3751	41.6459	0.0240	15.6221	0.0640	156.1040
26	2.7725	0.3607	44.3117	0.0226	15.9828	0.0626	165.1213
27	2.8834	0.3468	47.0842	0.0212	16.3296	0.0612	174.1385
28	2.9987	0.3335	49.9676	0.0200	16.6631	0.0600	183.1424
29	3.1187	0.3207	52.9663	0.0189	16.9837	0.0589	192.1206
30	3.2434	0.3083	56.0849	0.0178	17.2920	0.0578	201.0619
35	3.9461	0.2534	73.6522	0.0136	18.6646	0.0536	244.8768
40	4.8010	0.2083	95.0255	0.0105	19.7928	0.0505	286.5303
45	5.8412	0.1712	121.0294	0.0083	20.7200	0.0483	325.4028
50	7.1067	0.1407	152.6671	0.0066	21.4822	0.0466	361.1639
55	8.6464	0.1157	191.1592	0.0052	22.1086	0.0452	393.6890
60	10.5196	0.0951	237.9907	0.0042	22.6235	0.0442	422.9967
70	15.5716	0.0642	364.2905	0.0027	23.3945	0.0427	472.4789
80	23.0498	0.0434	551.2451	0.0018	23.9154	0.0418	511.1162
90	34.1193	0.0293	827.9835	0.0012	24.2673	0.0412	540.7369
100	50.5050	0.0198	1237.6239	0.0008	24.5050	0.0408	563.1249

TABLE A.6 DISCRETE COMPOUND INTEREST TABLE $i = 5.0\%$

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.0500	0.9524	1.0000	1.0000	0.9524	1.0500	0.0000
2	1.1025	0.9070	2.0500	0.4878	1.8594	0.5378	0.9070
3	1.1576	0.8638	3.1525	0.3172	2.7232	0.3672	2.6347
4	1.2155	0.8227	4.3101	0.2320	3.5460	0.2820	5.1028
5	1.2763	0.7835	5.5256	0.1810	4.3295	0.2310	8.2369
6	1.3401	0.7462	6.8019	0.1470	5.0757	0.1970	11.9680
7	1.4071	0.7107	8.1420	0.1228	5.7864	0.1728	16.2321
8	1.4775	0.6768	9.5491	0.1047	6.4632	0.1547	20.9699
9	1.5513	0.6446	11.0266	0.0907	7.1078	0.1407	26.1268
10	1.6289	0.6139	12.5779	0.0795	7.7217	0.1295	31.6520
11	1.7103	0.5847	14.2068	0.0704	8.3064	0.1204	37.4988
12	1.7959	0.5568	15.9171	0.0628	8.8633	0.1128	43.6240
13	1.8856	0.5303	17.7130	0.0565	9.3936	0.1065	49.9879
14	1.9799	0.5051	19.5986	0.0510	9.8986	0.1010	56.5538
15	2.0789	0.4810	21.5786	0.0463	10.3797	0.0963	63.2880
16	2.1829	0.4581	23.6575	0.0423	10.8378	0.0923	70.1597
17	2.2920	0.4363	25.8404	0.0387	11.2741	0.0887	77.1404
18	2.4066	0.4155	28.1324	0.0355	11.6896	0.0855	84.2043
19	2.5269	0.3957	30.5390	0.0327	12.0853	0.0827	91.3275
20	2.6533	0.3769	33.0659	0.0302	12.4622	0.0802	98.4884
21	2.7860	0.3589	35.7192	0.0280	12.8212	0.0780	105.6672
22	2.9253	0.3418	38.5052	0.0260	13.1630	0.0760	112.8461
23	3.0715	0.3256	41.4305	0.0241	13.4886	0.0741	120.0086
24	3.2251	0.3101	44.5020	0.0225	13.7986	0.0725	127.1402
25	3.3864	0.2953	47.7271	0.0210	14.0939	0.0710	134.2275
26	3.5557	0.2812	51.1134	0.0196	14.3752	0.0696	141.2585
27	3.7335	0.2678	54.6691	0.0183	14.6430	0.0683	148.2225
28	3.9201	0.2551	58.4026	0.0171	14.8981	0.0671	155.1101
29	4.1161	0.2429	62.3227	0.0160	15.1411	0.0660	161.9126
30	4.3219	0.2314	66.4388	0.0151	15.3725	0.0651	168.6225
35	5.5160	0.1813	90.3203	0.0111	16.3742	0.0611	200.5806
40	7.0400	0.1420	120.7997	0.0083	17.1591	0.0583	229.5451
45	8.9850	0.1113	159.7001	0.0063	17.7741	0.0563	255.3145
50	11.4674	0.0872	209.3479	0.0048	18.2559	0.0548	277.9147
55	14.6356	0.0683	272.7125	0.0037	18.6335	0.0537	297.5104
60	18.6792	0.0535	353.5836	0.0028	18.9293	0.0528	314.3431
70	30.4264	0.0329	588.5283	0.0017	19.3427	0.0517	340.8409
80	49.5614	0.0202	971.2283	0.0010	19.5965	0.0510	359.6460
90	80.7303	0.0124	1594.6064	0.0006	19.7523	0.0506	372.7488
100	131.5012	0.0076	2610.0236	0.0004	19.8479	0.0504	381.7492

TABLE A.7 DISCRETE COMPOUND INTEREST TABLE $i = 6.0\%$

	COMPOUND AMOUNT FACTOR	DISCOUNT AMOUNT FACTOR	COMPOUND UNIFORM SERIES FACTOR	SINKING FUND FACTOR	DISCOUNT UNIFORM SERIES FACTOR	CAPITAL RECOVERY FACTOR	DISCOUNT GRADIENT FACTOR
n	(F|P,i,n)	(P|F,i,n)	(F|U,i,n)	(U|F,i,n)	(P|U,i,n)	(U|P,i,n)	(P|G,i,n)
1	1.0600	0.9434	1.0000	1.0000	0.9434	1.0600	0.0000
2	1.1236	0.8900	2.0600	0.4854	1.8334	0.5454	0.8900
3	1.1910	0.8396	3.1836	0.3141	2.6730	0.3741	2.5692
4	1.2625	0.7921	4.3746	0.2286	3.4651	0.2886	4.9455
5	1.3382	0.7473	5.6371	0.1774	4.2124	0.2374	7.9345
6	1.4185	0.7050	6.9753	0.1434	4.9173	0.2034	11.4593
7	1.5036	0.6651	8.3938	0.1191	5.5824	0.1791	15.4497
8	1.5938	0.6274	9.8975	0.1010	6.2098	0.1610	19.8416
9	1.6895	0.5919	11.4913	0.0870	6.8017	0.1470	24.5768
10	1.7908	0.5584	13.1808	0.0759	7.3601	0.1359	29.6023
11	1.8983	0.5268	14.9716	0.0668	7.8869	0.1268	34.8702
12	2.0122	0.4970	16.8699	0.0593	8.3838	0.1193	40.3368
13	2.1329	0.4688	18.8821	0.0530	8.8527	0.1130	45.9629
14	2.2609	0.4423	21.0151	0.0476	9.2950	0.1076	51.7128
15	2.3966	0.4173	23.2760	0.0430	9.7122	0.1030	57.5545
16	2.5404	0.3936	25.6725	0.0390	10.1059	0.0990	63.4592
17	2.6928	0.3714	28.2129	0.0354	10.4773	0.0954	69.4011
18	2.8543	0.3503	30.9056	0.0324	10.8276	0.0924	75.3569
19	3.0256	0.3305	33.7600	0.0296	11.1581	0.0896	81.3061
20	3.2071	0.3118	36.7856	0.0272	11.4699	0.0872	87.2304
21	3.3996	0.2942	39.9927	0.0250	11.7641	0.0850	93.1135
22	3.6035	0.2775	43.3923	0.0230	12.0416	0.0830	98.9411
23	3.8197	0.2618	46.9958	0.0213	12.3034	0.0813	104.7007
24	4.0489	0.2470	50.8156	0.0197	12.5504	0.0797	110.3812
25	4.2919	0.2330	54.8645	0.0182	12.7834	0.0782	115.9731
26	4.5494	0.2198	59.1564	0.0169	13.0032	0.0769	121.4684
27	4.8223	0.2074	63.7058	0.0157	13.2105	0.0757	126.8600
28	5.1117	0.1956	68.5281	0.0146	13.4062	0.0746	132.1420
29	5.4184	0.1846	73.6398	0.0136	13.5907	0.0736	137.3096
30	5.7435	0.1741	79.0582	0.0126	13.7648	0.0726	142.3588
35	7.6861	0.1301	111.4348	0.0090	14.4982	0.0690	165.7427
40	10.2857	0.0972	154.7619	0.0065	15.0463	0.0665	185.9568
45	13.7646	0.0727	212.7435	0.0047	15.4558	0.0647	203.1096
50	18.4201	0.0543	290.3358	0.0034	15.7619	0.0634	217.4574
55	24.6503	0.0406	394.1719	0.0025	15.9905	0.0625	229.3222
60	32.9877	0.0303	533.1280	0.0019	16.1614	0.0619	239.0428
70	59.0759	0.0169	967.9318	0.0010	16.3845	0.0610	253.3271
80	105.7959	0.0095	1746.5991	0.0006	16.5091	0.0606	262.5493
90	189.4644	0.0053	3141.0735	0.0003	16.5787	0.0603	268.3946
100	339.3019	0.0029	5638.3647	0.0002	16.6175	0.0602	272.0471

TABLE A.8 DISCRETE COMPOUND INTEREST TABLE $i = 7.0\%$

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.0700	0.9346	1.0000	1.0000	0.9346	1.0700	0.0000
2	1.1449	0.8734	2.0700	0.4831	1.8080	0.5531	0.8734
3	1.2250	0.8163	3.2149	0.3111	2.6243	0.3811	2.5060
4	1.3108	0.7629	4.4399	0.2252	3.3872	0.2952	4.7947
5	1.4026	0.7130	5.7507	0.1739	4.1002	0.2439	7.6467
6	1.5007	0.6663	7.1533	0.1398	4.7665	0.2098	10.9784
7	1.6058	0.6227	8.6540	0.1156	5.3893	0.1856	14.7149
8	1.7182	0.5820	10.2598	0.0975	5.9713	0.1675	18.7889
9	1.8385	0.5439	11.9780	0.0835	6.5152	0.1535	23.1404
10	1.9672	0.5083	13.8164	0.0724	7.0236	0.1424	27.7156
11	2.1049	0.4751	15.7836	0.0634	7.4987	0.1334	32.4665
12	2.2522	0.4440	17.8885	0.0559	7.9427	0.1259	37.3506
13	2.4098	0.4150	20.1406	0.0497	8.3577	0.1197	42.3302
14	2.5785	0.3878	22.5505	0.0443	8.7455	0.1143	47.3718
15	2.7590	0.3624	25.1290	0.0398	9.1079	0.1098	52.4461
16	2.9522	0.3387	27.8881	0.0359	9.4466	0.1059	57.5271
17	3.1588	0.3166	30.8402	0.0324	9.7632	0.1024	62.5923
18	3.3799	0.2959	33.9990	0.0294	10.0591	0.0994	67.6220
19	3.6165	0.2765	37.3790	0.0268	10.3356	0.0968	72.5991
20	3.8697	0.2584	40.9955	0.0244	10.5940	0.0944	77.5091
21	4.1406	0.2415	44.8652	0.0223	10.8355	0.0923	82.3393
22	4.4304	0.2257	49.0057	0.0204	11.0612	0.0904	87.0793
23	4.7405	0.2109	53.4361	0.0187	11.2722	0.0887	91.7201
24	5.0724	0.1971	58.1767	0.0172	11.4693	0.0872	96.2545
25	5.4274	0.1842	63.2490	0.0158	11.6536	0.0858	100.6765
26	5.8074	0.1722	68.6765	0.0146	11.8258	0.0846	104.9814
27	6.2139	0.1609	74.4838	0.0134	11.9867	0.0834	109.1656
28	6.6488	0.1504	80.6977	0.0124	12.1371	0.0824	113.2264
29	7.1143	0.1406	87.3465	0.0114	12.2777	0.0814	117.1622
30	7.6123	0.1314	94.4608	0.0106	12.4090	0.0806	120.9718
35	10.6766	0.0937	138.2369	0.0072	12.9477	0.0772	138.1353
40	14.9745	0.0668	199.6351	0.0050	13.3317	0.0750	152.2928
45	21.0025	0.0476	285.7493	0.0035	13.6055	0.0735	163.7559
50	29.4570	0.0339	406.5290	0.0025	13.8007	0.0725	172.9051
55	41.3150	0.0242	575.9286	0.0017	13.9399	0.0717	180.1243
60	57.9464	0.0173	813.5204	0.0012	14.0392	0.0712	185.7677
65	81.2729	0.0123	1146.7552	0.0009	14.1099	0.0709	190.1452
70	113.9894	0.0088	1614.1343	0.0006	14.1604	0.0706	193.5185
75	159.8760	0.0063	2269.6576	0.0004	14.1964	0.0704	196.1035
80	224.2344	0.0045	3189.0629	0.0003	14.2220	0.0703	198.0748

TABLE A.9 DISCRETE COMPOUND INTEREST TABLE $i = 8.0\%$

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.0800	0.9259	1.0000	1.0000	0.9259	1.0800	0.0000
2	1.1664	0.8573	2.0800	0.4808	1.7833	0.5608	0.8573
3	1.2597	0.7938	3.2464	0.3080	2.5771	0.3880	2.4450
4	1.3605	0.7350	4.5061	0.2219	3.3121	0.3019	4.6501
5	1.4693	0.6806	5.8666	0.1705	3.9927	0.2505	7.3724
6	1.5869	0.6302	7.3359	0.1363	4.6229	0.2163	10.5233
7	1.7138	0.5835	8.9228	0.1121	5.2064	0.1921	14.0242
8	1.8509	0.5403	10.6366	0.0940	5.7466	0.1740	17.8061
9	1.9990	0.5002	12.4876	0.0801	6.2469	0.1601	21.8081
10	2.1589	0.4632	14.4866	0.0690	6.7101	0.1490	25.9768
11	2.3316	0.4289	16.6455	0.0601	7.1390	0.1401	30.2657
12	2.5182	0.3971	18.9771	0.0527	7.5361	0.1327	34.6339
13	2.7196	0.3677	21.4953	0.0465	7.9038	0.1265	39.0463
14	2.9372	0.3405	24.2149	0.0413	8.2442	0.1213	43.4723
15	3.1722	0.3152	27.1521	0.0368	8.5595	0.1168	47.8857
16	3.4259	0.2919	30.3243	0.0330	8.8514	0.1130	52.2640
17	3.7000	0.2703	33.7502	0.0296	9.1216	0.1096	56.5883
18	3.9960	0.2502	37.4502	0.0267	9.3719	0.1067	60.8425
19	4.3157	0.2317	41.4463	0.0241	9.6036	0.1041	65.0134
20	4.6610	0.2145	45.7620	0.0219	9.8181	0.1019	69.0898
21	5.0338	0.1987	50.4229	0.0198	10.0168	0.0998	73.0629
22	5.4365	0.1839	55.4567	0.0180	10.2007	0.0980	76.9256
23	5.8715	0.1703	60.8933	0.0164	10.3711	0.0964	80.6726
24	6.3412	0.1577	66.7647	0.0150	10.5288	0.0950	84.2997
25	6.8485	0.1460	73.1059	0.0137	10.6748	0.0937	87.8041
26	7.3964	0.1352	79.9544	0.0125	10.8100	0.0925	91.1841
27	7.9881	0.1252	87.3507	0.0114	10.9352	0.0914	94.4390
28	8.6271	0.1159	95.3388	0.0105	11.0511	0.0905	97.5687
29	9.3173	0.1073	103.9659	0.0096	11.1584	0.0896	100.5738
30	10.0627	0.0994	113.2832	0.0088	11.2578	0.0888	103.4558
35	14.7853	0.0676	172.3168	0.0058	11.6546	0.0858	116.0920
40	21.7245	0.0460	259.0564	0.0039	11.9246	0.0839	126.0422
45	31.9204	0.0313	386.5055	0.0026	12.1084	0.0826	133.7331
50	46.9016	0.0213	573.7699	0.0017	12.2335	0.0817	139.5928
55	68.9138	0.0145	848.9228	0.0012	12.3186	0.0812	144.0064
60	101.2570	0.0099	1253.2127	0.0008	12.3766	0.0808	147.3000
65	148.7798	0.0067	1847.2471	0.0005	12.4160	0.0805	149.7387
70	218.6063	0.0046	2720.0785	0.0004	12.4428	0.0804	151.5326
75	321.2043	0.0031	4002.5542	0.0002	12.4611	0.0802	152.8448
80	471.9545	0.0021	5886.9316	0.0002	12.4735	0.0802	153.8001

TABLE A.10 DISCRETE COMPOUND INTEREST TABLE *i* = 9.0%

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.0900	0.9174	1.0000	1.0000	0.9174	1.0900	0.0000
2	1.1881	0.8417	2.0900	0.4785	1.7591	0.5685	0.8417
3	1.2950	0.7722	3.2781	0.3051	2.5313	0.3951	2.3860
4	1.4116	0.7084	4.5731	0.2187	3.2397	0.3087	4.5113
5	1.5386	0.6499	5.9847	0.1671	3.8897	0.2571	7.1110
6	1.6771	0.5963	7.5233	0.1329	4.4859	0.2229	10.0924
7	1.8280	0.5470	9.2004	0.1087	5.0330	0.1987	13.3746
8	1.9926	0.5019	11.0285	0.0907	5.5348	0.1807	16.8877
9	2.1719	0.4604	13.0210	0.0768	5.9952	0.1668	20.5711
10	2.3674	0.4224	15.1929	0.0658	6.4177	0.1558	24.3728
11	2.5804	0.3875	17.5603	0.0569	6.8052	0.1469	28.2481
12	2.8127	0.3555	20.1407	0.0497	7.1607	0.1397	32.1590
13	3.0658	0.3262	22.9534	0.0436	7.4869	0.1336	36.0731
14	3.3417	0.2992	26.0192	0.0384	7.7862	0.1284	39.9633
15	3.6425	0.2745	29.3609	0.0341	8.0607	0.1241	43.8069
16	3.9703	0.2519	33.0034	0.0303	8.3126	0.1203	47.5849
17	4.3276	0.2311	36.9737	0.0270	8.5436	0.1170	51.2821
18	4.7171	0.2120	41.3013	0.0242	8.7556	0.1142	54.8860
19	5.1417	0.1945	46.0185	0.0217	8.9501	0.1117	58.3868
20	5.6044	0.1784	51.1601	0.0195	9.1285	0.1095	61.7770
21	6.1088	0.1637	56.7645	0.0176	9.2922	0.1076	65.0509
22	6.6586	0.1502	62.8733	0.0159	9.4424	0.1059	68.2048
23	7.2579	0.1378	69.5319	0.0144	9.5802	0.1044	71.2359
24	7.9111	0.1264	76.7898	0.0130	9.7066	0.1030	74.1433
25	8.6231	0.1160	84.7009	0.0118	9.8226	0.1018	76.9265
26	9.3992	0.1064	93.3240	0.0107	9.9290	0.1007	79.5863
27	10.2451	0.0976	102.7231	0.0097	10.0266	0.0997	82.1241
28	11.1671	0.0895	112.9682	0.0089	10.1161	0.0989	84.5419
29	12.1722	0.0822	124.1353	0.0081	10.1983	0.0981	86.8422
30	13.2677	0.0754	136.3075	0.0073	10.2737	0.0973	89.0280
35	20.4140	0.0490	215.7107	0.0046	10.5668	0.0946	98.3590
40	31.4094	0.0318	337.8824	0.0030	10.7574	0.0930	105.3762
45	48.3273	0.0207	525.8586	0.0019	10.8812	0.0919	110.5561
50	74.3575	0.0134	815.0834	0.0012	10.9617	0.0912	114.3251
55	114.4082	0.0087	1260.0915	0.0008	11.0140	0.0908	117.0362

TABLE A.11 DISCRETE COMPOUND INTEREST TABLE $i = 10.0\%$

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.1000	0.9091	1.0000	1.0000	0.9091	1.1000	0.0000
2	1.2100	0.8264	2.1000	0.4762	1.7355	0.5762	0.8264
3	1.3310	0.7513	3.3100	0.3021	2.4869	0.4021	2.3291
4	1.4641	0.6830	4.6410	0.2155	3.1699	0.3155	4.3781
5	1.6105	0.6209	6.1051	0.1638	3.7908	0.2638	6.8618
6	1.7716	0.5645	7.7156	0.1296	4.3553	0.2296	9.6842
7	1.9487	0.5132	9.4872	0.1054	4.8684	0.2054	12.7631
8	2.1436	0.4665	11.4359	0.0874	5.3349	0.1874	16.0287
9	2.3579	0.4241	13.5795	0.0736	5.7590	0.1736	19.4214
10	2.5937	0.3855	15.9374	0.0627	6.1446	0.1627	22.8913
11	2.8531	0.3505	18.5312	0.0540	6.4951	0.1540	26.3963
12	3.1384	0.3186	21.3843	0.0468	6.8137	0.1468	29.9012
13	3.4523	0.2897	24.5227	0.0408	7.1034	0.1408	33.3772
14	3.7975	0.2633	27.9750	0.0357	7.3667	0.1357	36.8005
15	4.1772	0.2394	31.7725	0.0315	7.6061	0.1315	40.1520
16	4.5950	0.2176	35.9497	0.0278	7.8237	0.1278	43.4164
17	5.0545	0.1978	40.5447	0.0247	8.0216	0.1247	46.5819
18	5.5599	0.1799	45.5992	0.0219	8.2014	0.1219	49.6395
19	6.1159	0.1635	51.1591	0.0195	8.3649	0.1195	52.5827
20	6.7275	0.1486	57.2750	0.0175	8.5136	0.1175	55.4069
21	7.4002	0.1351	64.0025	0.0156	8.6487	0.1156	58.1095
22	8.1403	0.1228	71.4027	0.0140	8.7715	0.1140	60.6893
23	8.9543	0.1117	79.5430	0.0126	8.8832	0.1126	63.1462
24	9.8497	0.1015	88.4973	0.0113	8.9847	0.1113	65.4813
25	10.8347	0.0923	98.3470	0.0102	9.0770	0.1102	67.6964
26	11.9182	0.0839	109.1817	0.0092	9.1609	0.1092	69.7940
27	13.1100	0.0763	121.0999	0.0083	9.2372	0.1083	71.7772
28	14.4210	0.0693	134.2099	0.0075	9.3066	0.1075	73.6495
29	15.8631	0.0630	148.6309	0.0067	9.3696	0.1067	75.4146
30	17.4494	0.0573	164.4940	0.0061	9.4269	0.1061	77.0766
35	28.1024	0.0356	271.0243	0.0037	9.6442	0.1037	83.9871
40	45.2592	0.0221	442.5924	0.0023	9.7791	0.1023	88.9525
45	72.8904	0.0137	718.9045	0.0014	9.8628	0.1014	92.4544
50	117.3908	0.0085	1163.9079	0.0009	9.9148	0.1009	94.8889
55	189.0590	0.0053	1880.5903	0.0005	9.9471	0.1005	96.5619

TABLE A.12 DISCRETE COMPOUND INTEREST TABLE $i = 11.0\%$

	COMPOUND AMOUNT FACTOR	DISCOUNT AMOUNT FACTOR	COMPOUND UNIFORM SERIES FACTOR	SINKING FUND FACTOR	DISCOUNT UNIFORM SERIES FACTOR	CAPITAL RECOVERY FACTOR	DISCOUNT GRADIENT FACTOR
n	(F|P,i,n)	(P|F,i,n)	(F|U,i,n)	(U|F,i,n)	(P|U,i,n)	(U|P,i,n)	(P|G,i,n)
1	1.1100	0.9009	1.0000	1.0000	0.9009	1.1100	0.0000
2	1.2321	0.8116	2.1100	0.4739	1.7125	0.5839	0.8116
3	1.3676	0.7312	3.3421	0.2992	2.4437	0.4092	2.2740
4	1.5181	0.6587	4.7097	0.2123	3.1024	0.3223	4.2502
5	1.6851	0.5935	6.2278	0.1606	3.6959	0.2706	6.6240
6	1.8704	0.5346	7.9129	0.1264	4.2305	0.2364	9.2972
7	2.0762	0.4817	9.7833	0.1022	4.7122	0.2122	12.1872
8	2.3045	0.4339	11.8594	0.0843	5.1461	0.1943	15.2246
9	2.5580	0.3909	14.1640	0.0706	5.5370	0.1806	18.3520
10	2.8394	0.3522	16.7220	0.0598	5.8892	0.1698	21.5217
11	3.1518	0.3173	19.5614	0.0511	6.2065	0.1611	24.6945
12	3.4985	0.2858	22.7132	0.0440	6.4924	0.1540	27.8388
13	3.8833	0.2575	26.2116	0.0382	6.7499	0.1482	30.9290
14	4.3104	0.2320	30.0949	0.0332	6.9819	0.1432	33.9449
15	4.7846	0.2090	34.4054	0.0291	7.1909	0.1391	36.8709
16	5.3109	0.1883	39.1899	0.0255	7.3792	0.1355	39.6953
17	5.8951	0.1696	44.5008	0.0225	7.5488	0.1325	42.4094
18	6.5436	0.1528	50.3959	0.0198	7.7016	0.1298	45.0074
19	7.2633	0.1377	56.9395	0.0176	7.8393	0.1276	47.4856
20	8.0623	0.1240	64.2028	0.0156	7.9633	0.1256	49.8423
21	8.9492	0.1117	72.2651	0.0138	8.0751	0.1238	52.0771
22	9.9336	0.1007	81.2143	0.0123	8.1757	0.1223	54.1912
23	11.0263	0.0907	91.1479	0.0110	8.2664	0.1210	56.1864
24	12.2392	0.0817	102.1741	0.0098	8.3481	0.1198	58.0656
25	13.5855	0.0736	114.4133	0.0087	8.4217	0.1187	59.8322
26	15.0799	0.0663	127.9988	0.0078	8.4881	0.1178	61.4900
27	16.7386	0.0597	143.0786	0.0070	8.5478	0.1170	63.0433
28	18.5799	0.0538	159.8173	0.0063	8.6016	0.1163	64.4965
29	20.6237	0.0485	178.3972	0.0056	8.6501	0.1156	65.8542
30	22.8923	0.0437	199.0208	0.0050	8.6938	0.1150	67.1210
35	38.5748	0.0259	341.5895	0.0029	8.8552	0.1129	72.2538
40	65.0009	0.0154	581.8259	0.0017	8.9511	0.1117	75.7789
45	109.5302	0.0091	986.6383	0.0010	9.0079	0.1110	78.1551
50	184.5648	0.0054	1668.7707	0.0006	9.0417	0.1106	79.7340
55	311.0024	0.0032	2818.2034	0.0004	9.0617	0.1104	80.7712

TABLE A.13 DISCRETE COMPOUND INTEREST TABLE i = 12.0%

	COMPOUND AMOUNT FACTOR	DISCOUNT AMOUNT FACTOR	COMPOUND UNIFORM SERIES FACTOR	SINKING FUND FACTOR	DISCOUNT UNIFORM SERIES FACTOR	CAPITAL RECOVERY FACTOR	DISCOUNT GRADIENT FACTOR
n	(F|P,i,n)	(P|F,i,n)	(F|U,i,n)	(U|F,i,n)	(P|U,i,n)	(U|P,i,n)	(P|G,i,n)
1	1.1200	0.8929	1.0000	1.0000	0.8929	1.1200	0.0000
2	1.2544	0.7972	2.1200	0.4717	1.6901	0.5917	0.7972
3	1.4049	0.7118	3.3744	0.2963	2.4018	0.4163	2.2208
4	1.5735	0.6355	4.7793	0.2092	3.0373	0.3292	4.1273
5	1.7623	0.5674	6.3528	0.1574	3.6048	0.2774	6.3970
6	1.9738	0.5066	8.1152	0.1232	4.1114	0.2432	8.9302
7	2.2107	0.4523	10.0890	0.0991	4.5638	0.2191	11.6443
8	2.4760	0.4039	12.2997	0.0813	4.9676	0.2013	14.4714
9	2.7731	0.3606	14.7757	0.0677	5.3282	0.1877	17.3563
10	3.1058	0.3220	17.5487	0.0570	5.6502	0.1770	20.2541
11	3.4785	0.2875	20.6546	0.0484	5.9377	0.1684	23.1288
12	3.8960	0.2567	24.1331	0.0414	6.1944	0.1614	25.9523
13	4.3635	0.2292	28.0291	0.0357	6.4235	0.1557	28.7024
14	4.8871	0.2046	32.3926	0.0309	6.6282	0.1509	31.3624
15	5.4736	0.1827	37.2797	0.0268	6.8109	0.1468	33.9202
16	6.1304	0.1631	42.7533	0.0234	6.9740	0.1434	36.3670
17	6.8660	0.1456	48.8837	0.0205	7.1196	0.1405	38.6973
18	7.6900	0.1300	55.7497	0.0179	7.2497	0.1379	40.9080
19	8.6128	0.1161	63.4397	0.0158	7.3658	0.1358	42.9979
20	9.6463	0.1037	72.0524	0.0139	7.4694	0.1339	44.9676
21	10.8038	0.0926	81.6987	0.0122	7.5620	0.1322	46.8188
22	12.1003	0.0826	92.5026	0.0108	7.6446	0.1308	48.5543
23	13.5523	0.0738	104.6029	0.0096	7.7184	0.1296	50.1776
24	15.1786	0.0659	118.1552	0.0085	7.7843	0.1285	51.6929
25	17.0001	0.0588	133.3339	0.0075	7.8431	0.1275	53.1046
26	19.0401	0.0525	150.3339	0.0067	7.8957	0.1267	54.4177
27	21.3249	0.0469	169.3740	0.0059	7.9426	0.1259	55.6369
28	23.8839	0.0419	190.6989	0.0052	7.9844	0.1252	56.7674
29	26.7499	0.0374	214.5827	0.0047	8.0218	0.1247	57.8141
30	29.9599	0.0334	241.3327	0.0041	8.0552	0.1241	58.7821
35	52.7996	0.0189	431.6634	0.0023	8.1755	0.1223	62.6052
40	93.0510	0.0107	767.0913	0.0013	8.2438	0.1213	65.1159
45	163.9876	0.0061	1358.2298	0.0007	8.2825	0.1207	66.7342
50	289.0021	0.0035	2400.0178	0.0004	8.3045	0.1204	67.7624
55	509.3205	0.0020	4236.0043	0.0002	8.3170	0.1202	68.4082

TABLE A.14 DISCRETE COMPOUND INTEREST TABLE i = 15.0%

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.1500	0.8696	1.0000	1.0000	0.8696	1.1500	0.0000
2	1.3225	0.7561	2.1500	0.4651	1.6257	0.6151	0.7561
3	1.5209	0.6575	3.4725	0.2880	2.2832	0.4380	2.0712
4	1.7490	0.5718	4.9934	0.2003	2.8550	0.3503	3.7864
5	2.0114	0.4972	6.7424	0.1483	3.3522	0.2983	5.7751
6	2.3131	0.4323	8.7537	0.1142	3.7845	0.2642	7.9368
7	2.6600	0.3759	11.0668	0.0904	4.1604	0.2404	10.1924
8	3.0590	0.3269	13.7268	0.0729	4.4873	0.2229	12.4807
9	3.5179	0.2843	16.7858	0.0596	4.7716	0.2096	14.7548
10	4.0456	0.2472	20.3037	0.0493	5.0188	0.1993	16.9795
11	4.6524	0.2149	24.3493	0.0411	5.2337	0.1911	19.1289
12	5.3503	0.1869	29.0017	0.0345	5.4206	0.1845	21.1849
13	6.1528	0.1625	34.3519	0.0291	5.5831	0.1791	23.1352
14	7.0757	0.1413	40.5047	0.0247	5.7245	0.1747	24.9725
15	8.1371	0.1229	47.5804	0.0210	5.8474	0.1710	26.6930
16	9.3576	0.1069	55.7175	0.0179	5.9542	0.1679	28.2960
17	10.7613	0.0929	65.0751	0.0154	6.0472	0.1654	29.7828
18	12.3755	0.0808	75.8364	0.0132	6.1280	0.1632	31.1565
19	14.2318	0.0703	88.2118	0.0113	6.1982	0.1613	32.4213
20	16.3665	0.0611	102.4436	0.0098	6.2593	0.1598	33.5822
21	18.8215	0.0531	118.8101	0.0084	6.3125	0.1584	34.6448
22	21.6447	0.0462	137.6316	0.0073	6.3587	0.1573	35.6150
23	24.8915	0.0402	159.2764	0.0063	6.3988	0.1563	36.4988
24	28.6252	0.0349	184.1678	0.0054	6.4338	0.1554	37.3023
25	32.9190	0.0304	212.7930	0.0047	6.4641	0.1547	38.0314
26	37.8568	0.0264	245.7120	0.0041	6.4906	0.1541	38.6918
27	43.5353	0.0230	283.5688	0.0035	6.5135	0.1535	39.2890
28	50.0656	0.0200	327.1041	0.0031	6.5335	0.1531	39.8283
29	57.5755	0.0174	377.1697	0.0027	6.5509	0.1527	40.3146
30	66.2118	0.0151	434.7451	0.0023	6.5660	0.1523	40.7526

TABLE A.15 DISCRETE COMPOUND INTEREST TABLE i = 18.0%

	COMPOUND AMOUNT FACTOR	DISCOUNT AMOUNT FACTOR	COMPOUND UNIFORM SERIES FACTOR	SINKING FUND FACTOR	DISCOUNT UNIFORM SERIES FACTOR	CAPITAL RECOVERY FACTOR	DISCOUNT GRADIENT FACTOR
n	(F|P,i,n)	(P|F,i,n)	(F|U,i,n)	(U|F,i,n)	(P|U,i,n)	(U|P,i,n)	(P|G,i,n)
1	1.1800	0.8475	1.0000	1.0000	0.8475	1.1800	0.0000
2	1.3924	0.7182	2.1800	0.4587	1.5656	0.6387	0.7182
3	1.6430	0.6086	3.5724	0.2799	2.1743	0.4599	1.9354
4	1.9388	0.5158	5.2154	0.1917	2.6901	0.3717	3.4828
5	2.2878	0.4371	7.1542	0.1398	3.1272	0.3198	5.2312
6	2.6996	0.3704	9.4420	0.1059	3.4976	0.2859	7.0834
7	3.1855	0.3139	12.1415	0.0824	3.8115	0.2624	8.9670
8	3.7589	0.2660	15.3270	0.0652	4.0776	0.2452	10.8292
9	4.4355	0.2255	19.0859	0.0524	4.3030	0.2324	12.6329
10	5.2338	0.1911	23.5213	0.0425	4.4941	0.2225	14.3525
11	6.1759	0.1619	28.7551	0.0348	4.6560	0.2148	15.9716
12	7.2876	0.1372	34.9311	0.0286	4.7932	0.2086	17.4811
13	8.5994	0.1163	42.2187	0.0237	4.9095	0.2037	18.8765
14	10.1472	0.0985	50.8180	0.0197	5.0081	0.1997	20.1576
15	11.9737	0.0835	60.9653	0.0164	5.0916	0.1964	21.3269
16	14.1290	0.0708	72.9390	0.0137	5.1624	0.1937	22.3885
17	16.6722	0.0600	87.0680	0.0115	5.2223	0.1915	23.3482
18	19.6733	0.0508	103.7403	0.0096	5.2732	0.1896	24.2123
19	23.2144	0.0431	123.4135	0.0081	5.3162	0.1881	24.9877
20	27.3930	0.0365	146.6280	0.0068	5.3527	0.1868	25.6813
21	32.3238	0.0309	174.0210	0.0057	5.3837	0.1857	26.3000
22	38.1421	0.0262	206.3448	0.0048	5.4099	0.1848	26.8506
23	45.0076	0.0222	244.4869	0.0041	5.4321	0.1841	27.3394
24	53.1090	0.0188	289.4945	0.0035	5.4509	0.1835	27.7725
25	62.6686	0.0160	342.6035	0.0029	5.4669	0.1829	28.1555
26	73.9490	0.0135	405.2721	0.0025	5.4804	0.1825	28.4935
27	87.2598	0.0115	479.2211	0.0021	5.4919	0.1821	28.7915
28	102.9666	0.0097	566.4809	0.0018	5.5016	0.1818	29.0537
29	121.5006	0.0082	669.4475	0.0015	5.5098	0.1815	29.2842
30	143.3706	0.0070	790.9481	0.0013	5.5168	0.1813	29.4864

TABLE A.16 DISCRETE COMPOUND INTEREST TABLE i = 20.0%

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.2000	0.8333	1.0000	1.0000	0.8333	1.2000	0.0000
2	1.4400	0.6944	2.2000	0.4545	1.5278	0.6545	0.6944
3	1.7280	0.5787	3.6400	0.2747	2.1065	0.4747	1.8519
4	2.0736	0.4823	5.3680	0.1863	2.5887	0.3863	3.2986
5	2.4883	0.4019	7.4416	0.1344	2.9906	0.3344	4.9061
6	2.9860	0.3349	9.9299	0.1007	3.3255	0.3007	6.5806
7	3.5832	0.2791	12.9159	0.0774	3.6046	0.2774	8.2551
8	4.2998	0.2326	16.4991	0.0606	3.8372	0.2606	9.8831
9	5.1598	0.1938	20.7989	0.0481	4.0310	0.2481	11.4335
10	6.1917	0.1615	25.9587	0.0385	4.1925	0.2385	12.8871
11	7.4301	0.1346	32.1504	0.0311	4.3271	0.2311	14.2330
12	8.9161	0.1122	39.5805	0.0253	4.4392	0.2253	15.4667
13	10.6993	0.0935	48.4966	0.0206	4.5327	0.2206	16.5883
14	12.8392	0.0779	59.1959	0.0169	4.6106	0.2169	17.6008
15	15.4070	0.0649	72.0351	0.0139	4.6755	0.2139	18.5095
16	18.4884	0.0541	87.4421	0.0114	4.7296	0.2114	19.3208
17	22.1861	0.0451	105.9305	0.0094	4.7746	0.2094	20.0419
18	26.6233	0.0376	128.1167	0.0078	4.8122	0.2078	20.6805
19	31.9480	0.0313	154.7400	0.0065	4.8435	0.2065	21.2439
20	38.3376	0.0261	186.6880	0.0054	4.8696	0.2054	21.7395
21	46.0051	0.0217	225.0256	0.0044	4.8913	0.2044	22.1742
22	55.2061	0.0181	271.0307	0.0037	4.9094	0.2037	22.5546
23	66.2474	0.0151	326.2368	0.0031	4.9245	0.2031	22.8867
24	79.4968	0.0126	392.4842	0.0025	4.9371	0.2025	23.1760
25	95.3962	0.0105	471.9810	0.0021	4.9476	0.2021	23.4276
26	114.4754	0.0087	567.3772	0.0018	4.9563	0.2018	23.6460
27	137.3705	0.0073	681.8527	0.0015	4.9636	0.2015	23.8353
28	164.8446	0.0061	819.2232	0.0012	4.9697	0.2012	23.9991
29	197.8136	0.0051	984.0678	0.0010	4.9747	0.2010	24.1406
30	237.3763	0.0042	1181.8814	0.0008	4.9789	0.2008	24.2628

TABLE A.17 DISCRETE COMPOUND INTEREST TABLE *i* = 25.0%

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.2500	0.8000	1.0000	1.0000	0.8000	1.2500	0.0000
2	1.5625	0.6400	2.2500	0.4444	1.4400	0.6944	0.6400
3	1.9531	0.5120	3.8125	0.2623	1.9520	0.5123	1.6640
4	2.4414	0.4096	5.7656	0.1734	2.3616	0.4234	2.8928
5	3.0518	0.3277	8.2070	0.1218	2.6893	0.3718	4.2035
6	3.8147	0.2621	11.2588	0.0888	2.9514	0.3388	5.5142
7	4.7684	0.2097	15.0735	0.0663	3.1611	0.3163	6.7725
8	5.9605	0.1678	19.8419	0.0504	3.3289	0.3004	7.9469
9	7.4506	0.1342	25.8023	0.0388	3.4631	0.2888	9.0207
10	9.3132	0.1074	33.2529	0.0301	3.5705	0.2801	9.9870
11	11.6415	0.0859	42.5661	0.0235	3.6564	0.2735	10.8460
12	14.5519	0.0687	54.2077	0.0184	3.7251	0.2684	11.6020
13	18.1899	0.0550	68.7596	0.0145	3.7801	0.2645	12.2617
14	22.7374	0.0440	86.9495	0.0115	3.8241	0.2615	12.8334
15	28.4217	0.0352	109.6868	0.0091	3.8593	0.2591	13.3260
16	35.5271	0.0281	138.1085	0.0072	3.8874	0.2572	13.7482
17	44.4089	0.0225	173.6357	0.0058	3.9099	0.2558	14.1085
18	55.5112	0.0180	218.0446	0.0046	3.9279	0.2546	14.4147
19	69.3889	0.0144	273.5558	0.0037	3.9424	0.2537	14.6741
20	86.7362	0.0115	342.9447	0.0029	3.9539	0.2529	14.8932
21	108.4202	0.0092	429.6809	0.0023	3.9631	0.2523	15.0777
22	135.5253	0.0074	538.1011	0.0019	3.9705	0.2519	15.2326
23	169.4066	0.0059	673.6263	0.0015	3.9764	0.2515	15.3625
24	211.7582	0.0047	843.0329	0.0012	3.9811	0.2512	15.4711
25	264.6978	0.0038	1054.7912	0.0009	3.9849	0.2509	15.5618
26	330.8722	0.0030	1319.4890	0.0008	3.9879	0.2508	15.6373
27	413.5903	0.0024	1650.3612	0.0006	3.9903	0.2506	15.7002
28	516.9879	0.0019	2063.9515	0.0005	3.9923	0.2505	15.7524
29	646.2348	0.0015	2580.9394	0.0004	3.9938	0.2504	15.7957
30	807.7935	0.0012	3227.1742	0.0003	3.9950	0.2503	15.8316

TABLE A.18 DISCRETE COMPOUND INTEREST TABLE $i = 30.0\%$

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.3000	0.7692	1.0000	1.0000	0.7692	1.3000	0.0000
2	1.6900	0.5917	2.3000	0.4348	1.3609	0.7348	0.5917
3	2.1970	0.4552	3.9900	0.2506	1.8161	0.5506	1.5020
4	2.8561	0.3501	6.1870	0.1616	2.1662	0.4616	2.5524
5	3.7129	0.2693	9.0431	0.1106	2.4356	0.4106	3.6297
6	4.8268	0.2072	12.7560	0.0784	2.6427	0.3784	4.6656
7	6.2749	0.1594	17.5828	0.0569	2.8021	0.3569	5.6218
8	8.1573	0.1226	23.8577	0.0419	2.9247	0.3419	6.4800
9	10.6045	0.0943	32.0150	0.0312	3.0190	0.3312	7.2343
10	13.7858	0.0725	42.6195	0.0235	3.0915	0.3235	7.8872
11	17.9216	0.0558	56.4053	0.0177	3.1473	0.3177	8.4452
12	23.2981	0.0429	74.3269	0.0135	3.1903	0.3135	8.9173
13	30.2875	0.0330	97.6250	0.0102	3.2233	0.3102	9.3135
14	39.3738	0.0254	127.9125	0.0078	3.2487	0.3078	9.6437
15	51.1859	0.0195	167.2863	0.0060	3.2682	0.3060	9.9172
16	66.5417	0.0150	218.4722	0.0046	3.2832	0.3046	10.1426
17	86.5041	0.0116	285.0138	0.0035	3.2948	0.3035	10.3276
18	112.4554	0.0089	371.5180	0.0027	3.3037	0.3027	10.4788
19	146.1920	0.0068	483.9734	0.0021	3.3105	0.3021	10.6019
20	190.0496	0.0053	630.1654	0.0016	3.3158	0.3016	10.7019
21	247.0645	0.0040	820.2150	0.0012	3.3198	0.3012	10.7828
22	321.1838	0.0031	1067.2794	0.0009	3.3230	0.3009	10.8482
23	417.5390	0.0024	1388.4633	0.0007	3.3254	0.3007	10.9009
24	542.8007	0.0018	1806.0023	0.0006	3.3272	0.3006	10.9433
25	705.6409	0.0014	2348.8029	0.0004	3.3286	0.3004	10.9773
26	917.3331	0.0011	3054.4438	0.0003	3.3297	0.3003	11.0045
27	1192.5331	0.0008	3971.7769	0.0003	3.3305	0.3003	11.0263
28	1550.2930	0.0006	5164.3099	0.0002	3.3312	0.3002	11.0437
29	2015.3809	0.0005	6714.6029	0.0001	3.3317	0.3001	11.0576
30	2619.9951	0.0004	8729.9836	0.0001	3.3321	0.3001	11.0687

TABLE A.19 DISCRETE COMPOUND INTEREST TABLE i = 40.0%

	COMPOUND AMOUNT FACTOR	DISCOUNT AMOUNT FACTOR	COMPOUND UNIFORM SERIES FACTOR	SINKING FUND FACTOR	DISCOUNT UNIFORM SERIES FACTOR	CAPITAL RECOVERY FACTOR	DISCOUNT GRADIENT FACTOR
n	(F|P,i,n)	(P|F,i,n)	(F|U,i,n)	(U|F,i,n)	(P|U,i,n)	(U|P,i,n)	(P|G,i,n)
1	1.4000	0.7143	1.0000	1.0000	0.7143	1.4000	0.0000
2	1.9600	0.5102	2.4000	0.4167	1.2245	0.8167	0.5102
3	2.7440	0.3644	4.3600	0.2294	1.5889	0.6294	1.2391
4	3.8416	0.2603	7.1040	0.1408	1.8492	0.5408	2.0200
5	5.3782	0.1859	10.9456	0.0914	2.0352	0.4914	2.7637
6	7.5295	0.1328	16.3238	0.0613	2.1680	0.4613	3.4278
7	10.5414	0.0949	23.8534	0.0419	2.2628	0.4419	3.9970
8	14.7579	0.0678	34.3947	0.0291	2.3306	0.4291	4.4713
9	20.6610	0.0484	49.1526	0.0203	2.3790	0.4203	4.8585
10	28.9255	0.0346	69.8137	0.0143	2.4136	0.4143	5.1696
11	40.4957	0.0247	98.7391	0.0101	2.4383	0.4101	5.4166
12	56.6939	0.0176	139.2348	0.0072	2.4559	0.4072	5.6106
13	79.3715	0.0126	195.9287	0.0051	2.4685	0.4051	5.7618
14	111.1201	0.0090	275.3002	0.0036	2.4775	0.4036	5.8788
15	155.5681	0.0064	386.4202	0.0026	2.4839	0.4026	5.9688
16	217.7953	0.0046	541.9883	0.0018	2.4885	0.4018	6.0376
17	304.9135	0.0033	759.7837	0.0013	2.4918	0.4013	6.0901
18	426.8789	0.0023	1064.6972	0.0009	2.4941	0.4009	6.1299
19	597.6304	0.0017	1491.5760	0.0007	2.4958	0.4007	6.1601
20	836.6826	0.0012	2089.2065	0.0005	2.4970	0.4005	6.1828

TABLE A.20 DISCRETE COMPOUND INTEREST TABLE $i = 50.0\%$

n	COMPOUND AMOUNT FACTOR (F|P,i,n)	DISCOUNT AMOUNT FACTOR (P|F,i,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,i,n)	SINKING FUND FACTOR (U|F,i,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,i,n)	CAPITAL RECOVERY FACTOR (U|P,i,n)	DISCOUNT GRADIENT FACTOR (P|G,i,n)
1	1.5000	0.6667	1.0000	1.0000	0.6667	1.5000	0.0000
2	2.2500	0.4444	2.5000	0.4000	1.1111	0.9000	0.4444
3	3.3750	0.2963	4.7500	0.2105	1.4074	0.7105	1.0370
4	5.0625	0.1975	8.1250	0.1231	1.6049	0.6231	1.6296
5	7.5938	0.1317	13.1875	0.0758	1.7366	0.5758	2.1564
6	11.3906	0.0878	20.7813	0.0481	1.8244	0.5481	2.5953
7	17.0859	0.0585	32.1719	0.0311	1.8829	0.5311	2.9465
8	25.6289	0.0390	49.2578	0.0203	1.9220	0.5203	3.2196
9	38.4434	0.0260	74.8867	0.0134	1.9480	0.5134	3.4277
10	57.6650	0.0173	113.3301	0.0088	1.9653	0.5088	3.5838
11	86.4976	0.0116	170.9951	0.0058	1.9769	0.5058	3.6994
12	129.7463	0.0077	257.4927	0.0039	1.9846	0.5039	3.7842
13	194.6195	0.0051	387.2390	0.0026	1.9897	0.5026	3.8459
14	291.9293	0.0034	581.8585	0.0017	1.9931	0.5017	3.8904
15	437.8939	0.0023	873.7878	0.0011	1.9954	0.5011	3.9224
16	656.8408	0.0015	1311.6817	0.0008	1.9970	0.5008	3.9452
17	985.2613	0.0010	1968.5225	0.0005	1.9980	0.5005	3.9614
18	1477.8919	0.0007	2953.7838	0.0003	1.9986	0.5003	3.9729
19	2216.8378	0.0005	4431.6756	0.0002	1.9991	0.5002	3.9811
20	3325.2567	0.0003	6648.5134	0.0002	1.9994	0.5002	3.9868

APPENDIX

B Basic Factors for Continuous Compounding

TABLE B BASIC FACTORS FOR CONTINUOUS COMPOUNDING

rn	e^{rn}	e^{-rn}	rn	e^{rn}	e^{-rn}
.01	1.0101	.9900	.51	1.6653	.6005
.02	1.0202	.9802	.52	1.6820	.5945
.03	1.0305	.9704	.53	1.6989	.5886
.04	1.0408	.9608	.54	1.7160	.5827
.05	1.0513	.9512	.55	1.7333	.5769
.06	1.0618	.9418	.56	1.7507	.5712
.07	1.0725	.9324	.57	1.7683	.5655
.08	1.0833	.9231	.58	1.7860	.5599
.09	1.0942	.9139	.59	1.8040	.5543
.10	1.1052	.9048	.60	1.8221	.5488
.11	1.1163	.8958	.61	1.8404	.5434
.12	1.1275	.8869	.62	1.8589	.5379
.13	1.1388	.8781	.63	1.8776	.5326
.14	1.1503	.8694	.64	1.8965	.5273
.15	1.1618	.8607	.65	1.9155	.5220
.16	1.1735	.8521	.66	1.9348	.5169
.17	1.1853	.8437	.67	1.9542	.5117
.18	1.1972	.8353	.68	1.9739	.5066
.19	1.2092	.8270	.69	1.9937	.5016
.20	1.2214	.8187	.70	2.0138	.4966
.21	1.2337	.8106	.71	2.0340	.4916
.22	1.2461	.8025	.72	2.0544	.4868
.23	1.2586	.7945	.73	2.0751	.4819
.24	1.2712	.7866	.74	2.0959	.4771
.25	1.2840	.7788	.75	2.1170	.4724
.26	1.2969	.7711	.76	2.1383	.4677
.27	1.3100	.7634	.77	2.1598	.4630
.28	1.3231	.7558	.78	2.1815	.4584
.29	1.3364	.7483	.79	2.2034	.4538
.30	1.3499	.7408	.80	2.2255	.4493
.31	1.3634	.7334	.81	2.2479	.4449
.32	1.3771	.7261	.82	2.2705	.4404
.33	1.3910	.7189	.83	2.2933	.4360
.34	1.4049	.7118	.84	2.3164	.4317
.35	1.4191	.7047	.85	2.3396	.4274
.36	1.4333	.6977	.86	2.3632	.4232
.37	1.4477	.6907	.87	2.3869	.4190
.38	1.4623	.6839	.88	2.4109	.4148
.39	1.4770	.6771	.89	2.4351	.4107
.40	1.4918	.6703	.90	2.4596	.4066
.41	1.5068	.6637	.91	2.4843	.4025
.42	1.5220	.6570	.92	2.5093	.3985
.43	1.5373	.6505	.93	2.5345	.3946
.44	1.5527	.6440	.94	2.5600	.3906
.45	1.5683	.6376	.95	2.5857	.3867
.46	1.5841	.6313	.96	2.6117	.3829
.47	1.6000	.6250	.97	2.6379	.3791
.48	1.6161	.6188	.98	2.6645	.3753
.49	1.6323	.6126	.99	2.6912	.3716
.50	1.6487	.6065	1.00	2.7183	.3679

APPENDIX

C Continuous Compound Interest Tables

TABLE C.1 CONTINUOUS COMPOUND INTEREST TABLE $r = 2.0\%$

n	COMPOUND AMOUNT FACTOR (F|P,r,n)	DISCOUNT AMOUNT FACTOR (P|F,r,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,r,n)	SINKING FUND FACTOR (U|F,r,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,r,n)	CAPITAL RECOVERY FACTOR (U|P,r,n)	DISCOUNT GRADIENT FACTOR (P|G,r,n)
1	1.0202	0.9802	1.0000	1.0000	0.9802	1.0202	0.0000
2	1.0408	0.9608	2.0202	0.4950	1.9410	0.5152	0.9609
3	1.0618	0.9418	3.0610	0.3267	2.8828	0.3469	2.8444
4	1.0833	0.9231	4.1229	0.2426	3.8059	0.2628	5.6138
5	1.1052	0.9048	5.2061	0.1921	4.7107	0.2123	9.2332
6	1.1275	0.8869	6.3113	0.1584	5.5976	0.1786	13.6678
7	1.1503	0.8694	7.4388	0.1344	6.4670	0.1546	18.8840
8	1.1735	0.8521	8.5891	0.1164	7.3191	0.1366	24.8491
9	1.1972	0.8353	9.7626	0.1024	8.1544	0.1226	31.5313
10	1.2214	0.8187	10.9598	0.0912	8.9731	0.1114	38.8999
11	1.2461	0.8025	12.1812	0.0821	9.7757	0.1023	46.9251
12	1.2712	0.7866	13.4273	0.0745	10.5623	0.0947	55.5780
13	1.2969	0.7711	14.6985	0.0680	11.3333	0.0882	64.8307
14	1.3231	0.7558	15.9955	0.0625	12.0891	0.0827	74.6559
15	1.3499	0.7408	17.3186	0.0577	12.8299	0.0779	85.0274
16	1.3771	0.7261	18.6685	0.0536	13.5561	0.0738	95.9197
17	1.4049	0.7118	20.0456	0.0499	14.2679	0.0701	107.3080
18	1.4333	0.6977	21.4505	0.0466	14.9655	0.0668	119.1686
19	1.4623	0.6839	22.8839	0.0437	15.6494	0.0639	131.4781
20	1.4918	0.6703	24.3462	0.0411	16.3197	0.0613	144.2142
21	1.5220	0.6570	25.8380	0.0387	16.9768	0.0589	157.3552
22	1.5527	0.6440	27.3599	0.0365	17.6208	0.0568	170.8800
23	1.5841	0.6313	28.9127	0.0346	18.2521	0.0548	184.7683
24	1.6161	0.6188	30.4967	0.0328	18.8709	0.0530	199.0003
25	1.6487	0.6065	32.1128	0.0311	19.4774	0.0513	213.5570
26	1.6820	0.5945	33.7615	0.0296	20.0719	0.0498	228.4201
27	1.7160	0.5827	35.4436	0.0282	20.6547	0.0484	243.5716
28	1.7507	0.5712	37.1596	0.0269	21.2259	0.0471	258.9943
29	1.7860	0.5599	38.9102	0.0257	21.7858	0.0459	274.6714
30	1.8221	0.5488	40.6963	0.0246	22.3346	0.0448	290.5871
35	2.0138	0.4966	50.1825	0.0199	24.9199	0.0401	373.2131
40	2.2255	0.4493	60.6664	0.0165	27.2592	0.0367	459.6726
45	2.4596	0.4066	72.2528	0.0138	29.3758	0.0340	548.4876
50	2.7183	0.3679	85.0579	0.0118	31.2910	0.0320	638.4270
55	3.0042	0.3329	99.2096	0.0101	33.0240	0.0303	728.4724
60	3.3201	0.3012	114.8497	0.0087	34.5921	0.0289	817.7891
70	4.0552	0.2466	151.2376	0.0066	37.2947	0.0268	991.6649
80	4.9530	0.2019	195.6818	0.0051	39.5075	0.0253	1156.1499
90	6.0496	0.1653	249.9661	0.0040	41.3191	0.0242	1308.9352
100	7.3891	0.1353	316.2691	0.0032	42.8024	0.0234	1448.8578

TABLE C.2 CONTINUOUS COMPOUND INTEREST TABLE $r = 4.0\%$

	COMPOUND AMOUNT FACTOR	DISCOUNT AMOUNT FACTOR	COMPOUND UNIFORM SERIES FACTOR	SINKING FUND FACTOR	DISCOUNT UNIFORM SERIES FACTOR	CAPITAL RECOVERY FACTOR	DISCOUNT GRADIENT FACTOR
n	(F|P,r,n)	(P|F,r,n)	(F|U,r,n)	(U|F,r,n)	(P|U,r,n)	(U|P,r,n)	(P|G,r,n)
1	1.0408	0.9608	1.0000	1.0000	0.9608	1.0408	0.0000
2	1.0833	0.9231	2.0408	0.4900	1.8839	0.5308	0.9231
3	1.1275	0.8869	3.1241	0.3201	2.7708	0.3609	2.6970
4	1.1735	0.8521	4.2516	0.2352	3.6230	0.2760	5.2534
5	1.2214	0.8187	5.4251	0.1843	4.4417	0.2251	8.5283
6	1.2712	0.7866	6.6465	0.1505	5.2283	0.1913	12.4615
7	1.3231	0.7558	7.9178	0.1263	5.9841	0.1671	16.9962
8	1.3771	0.7261	9.2409	0.1082	6.7103	0.1490	22.0792
9	1.4333	0.6977	10.6180	0.0942	7.4079	0.1350	27.6607
10	1.4918	0.6703	12.0513	0.0830	8.0783	0.1238	33.6936
11	1.5527	0.6440	13.5432	0.0738	8.7223	0.1146	40.1339
12	1.6161	0.6188	15.0959	0.0662	9.3411	0.1071	46.9405
13	1.6820	0.5945	16.7120	0.0598	9.9356	0.1006	54.0748
14	1.7507	0.5712	18.3940	0.0544	10.5068	0.0952	61.5005
15	1.8221	0.5488	20.1447	0.0496	11.0556	0.0905	69.1839
16	1.8965	0.5273	21.9668	0.0455	11.5829	0.0863	77.0933
17	1.9739	0.5066	23.8633	0.0419	12.0895	0.0827	85.1992
18	2.0544	0.4868	25.8371	0.0387	12.5763	0.0795	93.4739
19	2.1383	0.4677	27.8916	0.0359	13.0440	0.0767	101.8919
20	2.2255	0.4493	30.0298	0.0333	13.4933	0.0741	110.4292
21	2.3164	0.4317	32.2554	0.0310	13.9250	0.0718	119.0634
22	2.4109	0.4148	34.5718	0.0289	14.3398	0.0697	127.7739
23	2.5093	0.3985	36.9827	0.0270	14.7383	0.0679	136.5413
24	2.6117	0.3829	39.4919	0.0253	15.1212	0.0661	145.3478
25	2.7183	0.3679	42.1036	0.0238	15.4891	0.0646	154.1769
26	2.8292	0.3535	44.8219	0.0223	15.8425	0.0631	163.0133
27	2.9447	0.3396	47.6511	0.0210	16.1821	0.0618	171.8428
28	3.0649	0.3263	50.5958	0.0198	16.5084	0.0606	180.6524
29	3.1899	0.3135	53.6607	0.0186	16.8219	0.0594	189.4300
30	3.3201	0.3012	56.8506	0.0176	17.1231	0.0584	198.1646
35	4.0552	0.2466	74.8626	0.0134	18.4609	0.0542	240.8677
40	4.9530	0.2019	96.8625	0.0103	19.5562	0.0511	281.3067
45	6.0496	0.1653	123.7332	0.0081	20.4530	0.0489	318.8991
50	7.3891	0.1353	156.5532	0.0064	21.1872	0.0472	353.3482
55	9.0250	0.1108	196.6396	0.0051	21.7883	0.0459	384.5583
60	11.0232	0.0907	245.6013	0.0041	22.2804	0.0449	412.5718
70	16.4446	0.0608	378.4454	0.0026	23.0133	0.0435	459.5990
80	24.5325	0.0408	576.6256	0.0017	23.5045	0.0425	496.0347
90	36.5982	0.0273	872.2756	0.0011	23.8338	0.0420	523.7511
100	54.5981	0.0183	1313.3336	0.0008	24.0545	0.0416	544.5373

TABLE C.3 CONTINUOUS COMPOUND INTEREST TABLE $r = 6.0\%$

n	COMPOUND AMOUNT FACTOR (F|P,r,n)	DISCOUNT AMOUNT FACTOR (P|F,r,n)	COMPOUND UNIFORM SERIES FACTOR (F|U,r,n)	SINKING FUND FACTOR (U|F,r,n)	DISCOUNT UNIFORM SERIES FACTOR (P|U,r,n)	CAPITAL RECOVERY FACTOR (U|P,r,n)	DISCOUNT GRADIENT FACTOR (P|G,r,n)
1	1.0618	0.9418	1.0000	1.0000	0.9418	1.0618	0.0000
2	1.1275	0.8869	2.0618	0.4850	1.8287	0.5468	0.8869
3	1.1972	0.8353	3.1893	0.3135	2.6640	0.3754	2.5575
4	1.2712	0.7866	4.3866	0.2280	3.4506	0.2898	4.9174
5	1.3499	0.7408	5.6578	0.1767	4.1914	0.2386	7.8806
6	1.4333	0.6977	7.0077	0.1427	4.8891	0.2045	11.3690
7	1.5220	0.6570	8.4410	0.1185	5.5461	0.1803	15.3113
8	1.6161	0.6188	9.9630	0.1004	6.1649	0.1622	19.6428
9	1.7160	0.5827	11.5790	0.0864	6.7477	0.1482	24.3048
10	1.8221	0.5488	13.2950	0.0752	7.2965	0.1371	29.2441
11	1.9348	0.5169	15.1172	0.0662	7.8133	0.1280	34.4126
12	2.0544	0.4868	17.0519	0.0586	8.3001	0.1205	39.7669
13	2.1815	0.4584	19.1064	0.0523	8.7585	0.1142	45.2677
14	2.3164	0.4317	21.2878	0.0470	9.1902	0.1088	50.8800
15	2.4596	0.4066	23.6042	0.0424	9.5968	0.1042	56.5719
16	2.6117	0.3829	26.0638	0.0384	9.9797	0.1002	62.3153
17	2.7732	0.3606	28.6755	0.0349	10.3402	0.0967	68.0849
18	2.9447	0.3396	31.4487	0.0318	10.6798	0.0936	73.8580
19	3.1268	0.3198	34.3934	0.0291	10.9997	0.0909	79.6147
20	3.3201	0.3012	37.5202	0.0267	11.3009	0.0885	85.3374
21	3.5254	0.2837	40.8403	0.0245	11.5845	0.0863	91.0105
22	3.7434	0.2671	44.3657	0.0225	11.8516	0.0844	96.6203
23	3.9749	0.2516	48.1091	0.0208	12.1032	0.0826	102.1551
24	4.2207	0.2369	52.0840	0.0192	12.3402	0.0810	107.6044
25	4.4817	0.2231	56.3047	0.0178	12.5633	0.0796	112.9595
26	4.7588	0.2101	60.7864	0.0165	12.7734	0.0783	118.2129
27	5.0531	0.1979	65.5452	0.0153	12.9713	0.0771	123.3583
28	5.3656	0.1864	70.5983	0.0142	13.1577	0.0760	128.3904
29	5.6973	0.1755	75.9639	0.0132	13.3332	0.0750	133.3050
30	6.0496	0.1653	81.6612	0.0122	13.4985	0.0741	138.0986
35	8.1662	0.1225	115.8889	0.0086	14.1913	0.0705	160.1863
40	11.0232	0.0907	162.0915	0.0062	14.7046	0.0680	179.1156
45	14.8797	0.0672	224.4584	0.0045	15.0848	0.0663	195.0399
50	20.0855	0.0498	308.6450	0.0032	15.3665	0.0651	208.2454
55	27.1126	0.0369	422.2849	0.0024	15.5752	0.0642	219.0716
60	36.5982	0.0273	575.6828	0.0017	15.7298	0.0636	227.8649
70	66.6863	0.0150	1062.2575	0.0009	15.9292	0.0628	240.6259
80	121.5104	0.0082	1948.8545	0.0005	16.0386	0.0623	248.7235
90	221.4064	0.0045	3564.3392	0.0003	16.0986	0.0621	253.7680
100	403.4288	0.0025	6507.9447	0.0002	16.1316	0.0620	256.8660

TABLE C.4 CONTINUOUS COMPOUND INTEREST TABLE $r = 8.0\%$

	COMPOUND AMOUNT FACTOR	DISCOUNT AMOUNT FACTOR	COMPOUND UNIFORM SERIES FACTOR	SINKING FUND FACTOR	DISCOUNT UNIFORM SERIES FACTOR	CAPITAL RECOVERY FACTOR	DISCOUNT GRADIENT FACTOR
n	(F|P,r,n)	(P|F,r,n)	(F|U,r,n)	(U|F,r,n)	(P|U,r,n)	(U|P,r,n)	(P|G,r,n)
1	1.0833	0.9231	1.0000	1.0000	0.9231	1.0833	0.0000
2	1.1735	0.8521	2.0833	0.4800	1.7753	0.5633	0.8521
3	1.2712	0.7866	3.2568	0.3071	2.5619	0.3903	2.4254
4	1.3771	0.7261	4.5280	0.2208	3.2880	0.3041	4.6039
5	1.4918	0.6703	5.9052	0.1693	3.9584	0.2526	7.2851
6	1.6161	0.6188	7.3970	0.1352	4.5771	0.2185	10.3791
7	1.7507	0.5712	9.0131	0.1109	5.1484	0.1942	13.8063
8	1.8965	0.5273	10.7637	0.0929	5.6756	0.1762	17.4974
9	2.0544	0.4868	12.6602	0.0790	6.1624	0.1623	21.3914
10	2.2255	0.4493	14.7147	0.0680	6.6117	0.1512	25.4353
11	2.4109	0.4148	16.9402	0.0590	7.0265	0.1423	29.5832
12	2.6117	0.3829	19.3511	0.0517	7.4094	0.1350	33.7950
13	2.8292	0.3535	21.9628	0.0455	7.7629	0.1288	38.0364
14	3.0649	0.3263	24.7920	0.0403	8.0891	0.1236	42.2781
15	3.3201	0.3012	27.8569	0.0359	8.3903	0.1192	46.4948
16	3.5966	0.2780	31.1770	0.0321	8.6684	0.1154	50.6654
17	3.8962	0.2567	34.7736	0.0288	8.9250	0.1120	54.7719
18	4.2207	0.2369	38.6698	0.0259	9.1620	0.1091	58.7997
19	4.5722	0.2187	42.8905	0.0233	9.3807	0.1066	62.7365
20	4.9530	0.2019	47.4627	0.0211	9.5826	0.1044	66.5726
21	5.3656	0.1864	52.4158	0.0191	9.7689	0.1024	70.3000
22	5.8124	0.1720	57.7813	0.0173	9.9410	0.1006	73.9130
23	6.2965	0.1588	63.5938	0.0157	10.0998	0.0990	77.4070
24	6.8210	0.1466	69.8903	0.0143	10.2464	0.0976	80.7789
25	7.3891	0.1353	76.7113	0.0130	10.3817	0.0963	84.0270
26	8.0045	0.1249	84.1003	0.0119	10.5067	0.0952	87.1502
27	8.6711	0.1153	92.1048	0.0109	10.6220	0.0941	90.1487
28	9.3933	0.1065	100.7759	0.0099	10.7285	0.0932	93.0231
29	10.1757	0.0983	110.1693	0.0091	10.8267	0.0924	95.7747
30	11.0232	0.0907	120.3449	0.0083	10.9174	0.0916	98.4055
35	16.4446	0.0608	185.4387	0.0054	11.2765	0.0887	109.8393
40	24.5325	0.0408	282.5473	0.0035	11.5172	0.0868	118.7070
45	36.5982	0.0273	427.4161	0.0023	11.6786	0.0856	125.4581
50	54.5981	0.0183	643.5351	0.0016	11.7868	0.0848	130.5242
55	81.4509	0.0123	965.9468	0.0010	11.8593	0.0843	134.2826

TABLE C.5 CONTINUOUS COMPOUND INTEREST TABLE $r = 10.0\%$

	COMPOUND AMOUNT FACTOR	DISCOUNT AMOUNT FACTOR	COMPOUND UNIFORM SERIES FACTOR	SINKING FUND FACTOR	DISCOUNT UNIFORM SERIES FACTOR	CAPITAL RECOVERY FACTOR	DISCOUNT GRADIENT FACTOR
n	(F|P,r,n)	(P|F,r,n)	(F|U,r,n)	(U|F,r,n)	(P|U,r,n)	(U|P,r,n)	(P|G,r,n)
1	1.1052	0.9048	1.0000	1.0000	0.9048	1.1052	0.0000
2	1.2214	0.8187	2.1052	0.4750	1.7236	0.5802	0.8187
3	1.3499	0.7408	3.3266	0.3006	2.4644	0.4058	2.3004
4	1.4918	0.6703	4.6764	0.2138	3.1347	0.3190	4.3113
5	1.6487	0.6065	6.1683	0.1621	3.7412	0.2673	6.7375
6	1.8221	0.5488	7.8170	0.1279	4.2900	0.2331	9.4815
7	2.0138	0.4966	9.6391	0.1037	4.7866	0.2089	12.4610
8	2.2255	0.4493	11.6529	0.0858	5.2360	0.1910	15.6063
9	2.4596	0.4066	13.8784	0.0721	5.6425	0.1772	18.8589
10	2.7183	0.3679	16.3380	0.0612	6.0104	0.1664	22.1698
11	3.0042	0.3329	19.0563	0.0525	6.3433	0.1576	25.4985
12	3.3201	0.3012	22.0604	0.0453	6.6445	0.1505	28.8117
13	3.6693	0.2725	25.3806	0.0394	6.9170	0.1446	32.0820
14	4.0552	0.2466	29.0499	0.0344	7.1636	0.1396	35.2878
15	4.4817	0.2231	33.1051	0.0302	7.3867	0.1354	38.4116
16	4.9530	0.2019	37.5867	0.0266	7.5886	0.1318	41.4401
17	5.4739	0.1827	42.5398	0.0235	7.7713	0.1287	44.3630
18	6.0496	0.1653	48.0137	0.0208	7.9366	0.1260	47.1731
19	6.6859	0.1496	54.0634	0.0185	8.0862	0.1237	49.8653
20	7.3891	0.1353	60.7493	0.0165	8.2215	0.1216	52.4367
21	8.1662	0.1225	68.1383	0.0147	8.3440	0.1198	54.8858
22	9.0250	0.1108	76.3045	0.0131	8.4548	0.1183	57.2127
23	9.9742	0.1003	85.3295	0.0117	8.5550	0.1169	59.4184
24	11.0232	0.0907	95.3037	0.0105	8.6458	0.1157	61.5049
25	12.1825	0.0821	106.3269	0.0094	8.7278	0.1146	63.4749
26	13.4637	0.0743	118.5094	0.0084	8.8021	0.1136	65.3318
27	14.8797	0.0672	131.9731	0.0076	8.8693	0.1127	67.0791
28	16.4446	0.0608	146.8528	0.0068	8.9301	0.1120	68.7210
29	18.1741	0.0550	163.2975	0.0061	8.9852	0.1113	70.2616
30	20.0855	0.0498	181.4716	0.0055	9.0349	0.1107	71.7055
35	33.1155	0.0302	305.3644	0.0033	9.2212	0.1084	77.6289
40	54.5981	0.0183	509.6291	0.0020	9.3342	0.1071	81.7865
45	90.0171	0.0111	846.4045	0.0012	9.4027	0.1064	84.6508
50	148.4132	0.0067	1401.6534	0.0007	9.4443	0.1059	86.5959
55	244.6919	0.0041	2317.1041	0.0004	9.4695	0.1056	87.9017

APPENDIX

D Use of Electronic Spreadsheets as Computational Aids

GENERAL INFORMATION

An electronic spreadsheet is a versatile tool for various computational purposes and can be conveniently used for calculation in engineering economics. Graphs and figures can also be generated from spreadsheets. Virtually all personal computers can accommodate one of the commercial software packages for operating on a spreadsheet. Typical software packages include:

Lotus 123 (copyrighted by Lotus Development Corporation)
Microsoft Excel (copyrighted by Microsoft Corporation)

The basic information for using these software packages can be found in the respective reference manuals published by the distributors or in other descriptive books.

Spreadsheets have the capability for storage and calculations on repeated data so that special purpose problem-solving *template* files can be developed. For example, the formulas to calculate net present value for a range of interest rates can be entered into a spreadsheet template, and this template can be retrieved to insert new data without necessarily altering the calculating formulas. More advanced applications can add small programs (called *macros*) or customized user menus.

Financial functions are available in these software packages for use in calculation formulas. These built-in functions are particularly useful for solving problems in engineering economics. A number of such functions are listed in Table D for illustration. In using these built-in functions, users should be aware of their requirements and limitations. For example, the internal rate of return (IRR) function will return only one value each trial even though multiple roots of IRR may exist. As another example, the net present value (NPV) function input range may assume values in years 1 to n so cash flows in the present (year 0) must be added separately.

Several examples in this text are reworked in Appendix D using electronic spreadsheets. The example numbers are identical to those in Chapters 3, 4, and 5 of the text.

TABLE D COMMON FINANCIAL FUNCTIONS

Function	Result of computation
FV	The future value of a uniform series of payments over a number of time periods at the specified interest rate.
PV	The present value of a uniform series of payments over a number of time periods at the specified interest rate.
PMT	The amount of uniform payment to pay off principal and interest over a number of time periods.
NPV	The present value of a series of cash flows discounted to the present at the specified interest rate.
IRR	The internal rate of return for a series of cash flows, based on the approximate percentage guess of the IRR.
SLN	Straight-line depreciation for an asset.
SYD	Sum-of-the-Years' digits depreciation for an asset.
DDB	Depreciation of an asset using the double declining balance method.

Example 3.14

The following notation is used for the problem:

P = sum of money to be deposited at t=0
i = interest rate
U = uniform amount to be withdrawn at t=7,t=8,t=9, and t=10

The sum of money to be deposited is expressed as follows:

```
P=U[((1+i)^-7)+((1+i)^-8)+((1+i)^-9)+((1+i)^-10)]
```

In order to solve for P, we must know the amount to be received, U, and the interest rate, i. A good way to set up a spreadsheet would be to have i and U in the first two columns with the final P in the final column. U and i will be entered by the user.

Amount Withdrawn U	Interest Rate i	Deposited Sum P
$4,000.00	6.00%	$9,771.05

The formula in cell [C19] is as follows:

```
+$A19*(((1+$B19)^-7)+((1+$B19)^-8)+
   ((1+$B19)^-9)+((1+$B19)^-10))
```

By treating the problem as the difference between two uniform series, the answer P can be expressed as follows:

```
P=U[(((1+i)^4)-1)/(i(1+i)^10)]
```

Amount Withdrawn U	Interest Rate i	Deposited Sum P
$4,000.00	6.00%	$9,771.05

The formula in cell [C31] is as follows:

```
+$A31*(((1+$B31)^4)-1)/($B31*(1+$B31)^10)
```

Example 4.7

In this problem the NPV of the cash flows is computed for each of the interest rates to determine the IRR.

Year	Cash Flow	Interest Rates				
		0%	10%	20%	25%	30%
t	A(t)	0	0.1	0.2	0.25	0.3
0	-77	-77.000	-77.000	-77.000	-77.000	-77.000
1	38	38.000	34.545	31.667	30.400	29.231
2	32	32.000	26.446	22.222	20.480	18.935
3	26	26.000	19.534	15.046	13.312	11.834
4	20	20.000	13.660	9.645	8.192	7.003
5	14	14.000	8.693	5.626	4.588	3.771
NPV		53.000	25.879	7.207	-0.028	-6.227

From observation it is seen that the IRR lies somewhere between i = 20% and i = 25%. One can guess that it is probably quite close to 25%. Interpolation shows that the IRR is indeed 24.98%.

Note:

The formula at the intersection of Year t = 0 and Interest Rate = 0 is (At*(1+i)^(-t)) or +$B13*(1+C$11)^(-$A13) This formula was just copied throughout the entire table without the need for any specific changes in any of the cells. In using the function for computing NPV, retain the cash flow range and alter only the interest rate for each column.

The internal rate of return can be computed directly by using the available financial function for finding IRR. Using Lotus 123, the function is

@IRR (*guess, range*)

The guess should be realistic but not necessarily very accurate. For example, a guess of 0.1 or 10% will lead to a solution. Thus,

@IRR (0.1, C2..C6) yields 0.2498

Example 4.9

In this problem the NPV of the cash flows is computed for each of the interest rates to determine the IRR.

	Cash		Interest Rates				
Year	Flow	0%	10%	20%	30%	40%	50%
t	A(t)	0	0.1	0.2	0.3	0.4	0.5
0	18	18.000	18.000	18.000	18.000	18.000	18.000
1	10	10.000	9.091	8.333	7.692	7.143	6.667
2	-40	-40.000	-33.058	-27.778	-23.669	-20.408	-17.778
3	-60	-60.000	-45.079	-34.722	-27.310	-21.866	-17.778
4	30	30.000	20.490	14.468	10.504	7.809	5.926
5	50	50.000	31.046	20.094	13.466	9.297	6.584
NPV		8.000	0.491	-1.605	-1.316	-0.025	1.621

From observation it is seen that a value of the IRR lies between i = 10% and i = 20% and another value lies between i = 40% and i = 50%.

The internal rate of return is computed directly by using the available financial function for finding IRR. Since two values of IRR are expected, we must make two guesses. Each guess must be closer to the value of targeted IRR than others when two or more values of IRR exist. For this problem, four guesses and their results are shown below using the financial function of Lotus 123.

```
@IRR (0.1, B2..B6)  yields  0.1130
@IRR (0.2, B2..B6)  yields  0.1130
@IRR (0.3, B2..B6)  yields  0.4016
@Irr (0.5, B2..B6)  yields  0.4016
```

The two values of IRR are 11.30% and 40.15%. Note that the guesses of 0.1 and 0.2 converge toward 0.1130 while the guesses of 0.4 and 0.5 converge toward 0.4016. If a user only makes guesses in the range below 20%, the second value will be missed.

Example 4.12

Find the year by year unrecovered balance for an interest rate of 25.00% per year.

Time t	Cash Flow A(t)	Interest R(t-1)*i I(t)	Change in Balance A(t)+I(t) Q(t)	Balance R(t-1)+Q(t) R(t)
0.00	($77.00)	$0.00	$0.00	($77.00)
1.00	$38.00	($19.25)	$18.75	($58.25)
2.00	$32.00	($14.56)	$17.44	($40.81)
3.00	$26.00	($10.20)	$15.80	($25.02)
4.00	$20.00	($6.25)	$13.75	($11.27)
5.00	$14.00	($2.82)	$11.18	($0.09)

The formulas for computing the above balances on a spreadsheet are given below.

Time t	Cash Flow A(t)	Interest I(t)	Change in Balance Q(t)	Balance R(t)
0	($77.00)	------------	------------	+B14
1	$38.00	+$E26*$D$7	+$B27+$C27	+$E26+$D27
2	$32.00	+$E27*$D$7	+$B28+$C28	+$E27+$D28
3	$26.00	+$E28*$D$7	+$B29+$C29	+$E28+$D29
4	$20.00	+$E29*$D$7	+$B30+$C30	+$E29+$D30
5	$14.00	+$E30*$D$7	+$B31+$C31	+$E30+$D31

Example 5.19

Find the variation of the NPV with respect to variations in other parameters using C(0) = $8 million.

NPV = -C(0) + (B(t)-C(t))(P|U,i,n)

Figure 5.5 in the text depicting the result of computation in this example has been generated from this spreadsheet.

B(t)	C(t)	i	n	NPV	% change NPV	% change B(t)
1.50	0.50	0.10	40.00	1.78	0.00	100.00
1.80	0.50	0.10	40.00	4.71	164.90	120.00
1.65	0.50	0.10	40.00	3.25	82.45	110.00
1.35	0.50	0.10	40.00	0.31	-82.45	90.00
1.20	0.50	0.10	40.00	-1.15	-164.90	80.00

B(t)	C(t)	i	n	NPV	% change NPV	% change C(t)
1.50	0.50	0.10	40.00	1.78	0.00	100.00
1.50	0.60	0.10	40.00	0.80	-54.97	120.00
1.50	0.55	0.10	40.00	1.29	-27.48	110.00
1.50	0.45	0.10	40.00	2.27	27.48	90.00
1.50	0.40	0.10	40.00	2.76	54.97	80.00

B(t)	C(t)	i	n	NPV	% change NPV	% change i
1.50	0.50	0.10	40.00	1.78	0.00	100.00
1.50	0.50	0.12	40.00	0.24	-86.30	120.00
1.50	0.50	0.11	40.00	0.95	-46.54	110.00
1.50	0.50	0.09	40.00	2.76	54.99	90.00
1.50	0.50	0.08	40.00	3.92	120.60	80.00

B(t)	C(t)	i	n	NPV	% change NPV	% change n
1.50	0.50	0.10	40.00	1.78	0.00	100.00
1.50	0.50	0.10	50.00	1.91	7.63	125.00
1.50	0.50	0.10	45.00	1.86	4.71	112.50
1.50	0.50	0.10	35.00	1.64	-7.58	87.50
1.50	0.50	0.10	30.00	1.43	-19.79	75.00

APPENDIX

E

Table of Standard Normal Probability Distribution

TABLE E NORMAL DISTRIBUTION FUNCTION

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-(1/2)t^2}\, dt$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998									
4.0	0.99997									
5.0	0.9999997									
6.0	0.999999999									

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Answers to Selected Problems

Chapter 1

P1.1 and P1.2 No specific answers.

Chapter 2

P2.1 NFV = −320 (<0).
IRR = 11.11% (<12%).

P2.3 IRR = 66.67% (>15%).

P2.5 3,000 units for a maximum profit $6 million.

P2.7 (a) plan 2; (b) plan 1.

P2.9 Proposal 1. (a) NFV_1 = $3,700;
(b) IRR_2 = 15%, $IIRR_{1-2}$ = 10%.

P2.11 Rank: (1) NFV_2 = $1,440;
(2) NFV_3 = $1,400;
(3) NFV_1 = $1,200.

P2.13 $200.

Chapter 3

P3.1 $9,594.

P3.3 $14,737 (<$15,000).

P3.5 $6,145 (>$5,000).

P3.7 $3,057.

P3.9 $2,210.

P3.11 $4,630.

P3.13 $1,829,631.

P3.15 $197,736.

P3.17 $18,635.

P3.19 $1,860.

P3.21 2.57%.

P3.23 8.15%.

P3.25 10.57%.

P3.27 $1,885 and $1,822, respectively.

P3.29 $1,997.

P3.31 34.01%.

P3.33 $1,759,185.

Chapter 4

P4.1 −$428.

P4.3 −3.

P4.5 −632.

P4.7 NPV = 0 for i = 0%, 30%, and 40%.

P4.9 2.53%.

P4.11 11.93%.

P4.13 7.46% for Ms. Sand.

P4.15 (a) IRR = 5.98%; (b) ORR = 6.23%.

P4.17 5.29%.

P4.19 i = 7.44%. Annual interests: $409, $380, $327, $247, and $139.

P4.21 9.79%.

P4.23 (a) Annual interests: $8,800, $7,680, $6,453, $5,083, $3,562, and $1,874; (b) I_3 = $3,562, Q_3 = −$15,348; I_4 = $1,874, Q_4 = −$17,036; (c) I_3 = $3,153, Q_3 = −$15,757; I_4 = $1,420, Q_4 = −$12,911.

P4.25 ORR = 17.7%.

Chapter 5

P5.1 NPV = 2,334; NFV = 4,608; EUAV = 568.

P5.3 (a) $70.4; (b) $48.9.

P5.5 $467 (>0).

P5.7 NPV = $239.

P5.9 $82.9, $30.5, $36.5, and $44.7 (all in millions).

P5.11 (a) −$207, −$67.35, $78.28, $233.24, $394.90, $564.65, $742.88, and −$55.98 (all in thousands); (b) −$241, −$127.15, $3.78, $154.35, $327.50, $526.63, $755.63, and $32.97 (all in thousands).

P5.13 i = 9% is unique since $R_t < 0$ for $t = 0$ through $t = 5$, and $R_6 = 0$.

P5.14 (a) $1,456, (b) $0, and (c) −$1,365.

P5.16 (a) NPV = $0.3864, (b) NPV = −$0.6522, (c) NPV = $1.4250, all in millions.

P5.17 NPV = −$24.

Chapter 6

P6.1 Proposal 3 with NPV = $67.

P6.3 (1) 8%, (2) 7.7%, (3) 8.94%.

P6.5 First pump with NPV = $13,799. ORR = 22.75% and 21.5% for $x = 1$ and $x = 2$, respectively.

P6.7 i_{3-2} = 15%, i_{3-1} = 22.5%, and i_{2-1} = 30%.

P6.9 IRR = 20%, 12%, and 10% for x = 1, 2, and 3, respectively. i_{3-1} = 4.29%, i_{3-2} = 7.93%, $i_{2-1} < 0$.

P6.11 (a) At i = 0%, NPV_1 = $2,100, NPV_2 = $2,800, NPV_3 = $3,210; at i = 10%, NPV_1 = $1,000, NPV_2 = $1,200, NPV_3 = $1,100; at i = 20%, NPV_1 = $83, NPV_2 = −$133, NPV_3 = −$658. (b) $IIRR_{2-1}$ = 14.58%, $IIRR_{3-2}$ = 7.88%, and $IIRR_{3-1}$ = 11.11%.

P6.13 Select type II with NPV = $1,653.

P6.15 Select the second alternative with CPV = $5,914.

P6.17 Single-stage construction with NPV = $192,460.

P6.19 Select proposal 2 with CPV = $6,483.

Chapter 7

P7.1 Maximum N = 1 at C = 1, both in $ million.

P7.3 Maximum N = 1.5 at C = 3, both in $ million.

P7.5 Proposal 3 or 2, with $\Delta B_{3-0}/\Delta C_{3-0}$ = 1.05 and $\Delta B_{2-3}/\Delta C_{2-3}$ = 1.0.

P7.7 (a) Proposal 1 with $\Delta B_{1-0}/\Delta C_{1-0}$ = 1.31, (b) proposal 1 with $\Delta B_{1-0}/\Delta C_{1-0}$ = 1.19, and (c) proposal 1 with $\Delta B_{1-0}/\Delta C_{1-0}$ = 1.09.

P7.9 (a) Select x = 1 with $\Delta B_{1-2}/\Delta C_{1-2}$ = 1.65; (b) select x = 1 with $\Delta B_{1-0}/\Delta C_{1-0}$ = 1.73.

P7.11 Select $x = 2$ with $\Delta B_{2-1}/\Delta C_{2-1} = 1.04$.

Chapter 8

P8.1 Proposal 3, using ascending order of $x = 2$, $x = 1$, and $x = 3$ in incremental analysis.

P8.3 Proposal 1, using ascending order of $x = 2$ and $x = 1$ in incremental analysis.

P8.5 Proposal 1, using ascending order of $x = 2$, $x = 3$, and $x = 1$ in incremental analysis, with $[dN_x/di]_{0\%}$ to break the tie for ordering all alternatives.

P8.7 Proposal 3, using ascending order of $x = 1$, $x = 2$, and $x = 3$ in incremental analysis, with proposal 3, using ascending order $[dN_x/di]_{0\%}$ to break the tie for ordering $x = 1$ and $x = 2$.

P8.9 For borrowing cash flows, arrange proposals in descending order of $x = 3$, $x = 2$, and $x = 1$ in incremental analysis. For MARR = 6%, all proposals are rejected.

Chapter 9

P9.1 (a) Proposal 3 with NPV = \$25,000; (b) Proposal 1 with ORR = 5.3%.

P9.3 11.51%.

P9.5 (a) $x = 4$, $x = 3$, $x = 8$, $x = 6$, and $x = 2$, with CPV = \$225 million and NPV = \$94 million; (b) $x = 4$, $x = 3$, $x = 8$, and $x = 2$ with CPV = \$200 million and NPV = \$84 million.

P9.7 (a) $x = 1$, $x = 5$, $x = 4$, and $x = 6$, with CPV = \$4.8 million and NPV = 6.75 million; (b) two combinations are tied for CPV = \$6.0 million and NPV = \$7.8 million.

P9.9 $B/C = 2.08$, 1.40, 1.48, and 1.48 for alternatives 1, 2, 3, and 4, respectively.

P9.11 Select Plan 2 with CPV = \$88,745.

P9.13 Use the purchase plan with NPV = \$572.

Chapter 10

P10.1 (a) $I(1980) = 13.5\%$, $I(1981) = 10.3\%$, $I(1982) = 6.2\%$; (b) same as (a).

P10.3 \$40,852.

P10.5 NPV = \$47,542.

P10.7 (a) 3.7%; (b) \$12,439.

P10.9 (a) NPV = −\$2,083; (b) NPV = \$48.

P10.11 (a) and (b): Both yield NPV = \$20,975.

P10.13 Present value of saving = \$82.16 in base-year dollars.

Chapter 11

P11.1 (a) $D_1 = \$3{,}000$, $B_1 = \$14{,}000$; (b) $D_1 = \$5{,}000$, $B_1 = \$12{,}000$.

P11.3 $D_1 = \$5{,}120$, $B_1 = \$15{,}360$; $D_2 = \$3{,}840$, $B_2 = \$11{,}520$.

P11.5 $D_1 = \$10{,}000$, $B_1 = \$15{,}000$; $D_4 = \$400$, $B_4 = \$5{,}000$.

P11.7 (a) $D_2 = \$350$, $B_2 = \$1{,}700$; (b) $D_3 = \$400$, $B_3 = \$900$; (c) $D_4 = \$237$, $B_4 = \$474$.

P11.8 $D_6 = \$297{,}000$, $B_6 = \$889{,}000$; $D_7' = \$295{,}000$, $B_7' = \$595{,}000$.

P11.10 $A_1 - D_1 = \$16{,}000$; $A_2 - D_2 = \$32{,}000$; $A_8 - D_8' = \$75{,}828$; $A_9 - D_9' = \$80{,}000$.

P11.12 $A_5 - D_5 = \$3{,}000$;
$A_6 - D_6 = \$6{,}000$.

P11.14 $D_1 = \$162{,}000$, $B_1 = \$567{,}000$;
$D_8' = \$40{,}461$, $B_8' = \$80{,}462$;
$D_9' = \$40{,}461$, $B_9' = \$40{,}001$.

P11.16 $D_4 = \$3{,}200$; $D_5 = \$2{,}400$.

Chapter 12

P12.1 \$3,047.

P12.3 NPV = −\$9,354.

P12.5 \$6,592.

P12.7 $P = \$51{,}263$.

P12.9 Alternative 1 with NPV = \$2.362 million.

P12.11 NPV = −\$2,880.

P12.13 NPV = \$79,979.

P12.15 \$2,400.

P12.17 (a) For $N = 32$, NPV = \$665,365; (b) for $N = 15$, NPV = \$774,825.

P12.19 NPV = \$381,889.

Chapter 13

P13.1 $MC = 27 - 18x + 3x^2$; and $AC = (108/x) + 27 - 9x + x^2$.

P13.3 $MC = -45 + 12x$; and $AC = (216/x) - 45 + 6x$.

P13.5 $MC = -15 + 12x$; and $AC = (150/x) - 15 + 6x$.

P13.7 800.

P13.9 $dC/dx = 500$ and $dB/dx = 500$ at $x = 20$.

P13.11 $dC/dx = 600$ and $dB/dx = 600$ at $x = 6$.

Chapter 14

P14.1 (a) 1.82 and 1.14; (b) 4 and 1.17; (c) 3.5%, 6%, \$10 per share and 0.33.

P14.3 A contingent legal liability will make the firm less attractive.

P14.5 Sources of funds, \$680,000; uses of funds, \$270,000; increase in working capital, \$410,000.

P14.7 (a) 1.84 and 0.89; (b) 3 and 1.67; (c) 2%, 3.3%, \$0.50 per share and 0%.

P14.9 Net changes in components \$33,000.

P14.11 Sources of funds, \$3,600; uses of funds, \$16,000; decrease in working capital, \$12,400.

Chapter 15

P15.1 $\mu = \$6{,}700$; $\sigma = \$1{,}350$.

P15.3 Chance factor for the "to bid" branch; no uncertainty in the "not to bid" branch.

P15.5 (a) $p = 0.1$; (b) $9{,}000p$ for $0 \le p \le 0.1$, and $1{,}000(1 - p)$ for $0.1 \le p \le 1$.

P15.7 (a) $p = 0.04$; (b) $240{,}000p$ for $0 \le p \le 0.04$, and $10{,}000(1 - p)$ for $0.04 \le p \le 1$; (c) $50{,}000p$ for $0 \le p \le 0.04$, $10{,}000 - 200{,}000p$ for $0.04 \le p \le 0.05$, and 0 for $0.05 \le p \le 1$.

P15.9 16.36%.

P15.11 $\beta_a = 0.99$; $r_a = 16.19\%$.

Chapter 16

P16.1 \$17,000 for both (a) and (b).

P16.3 $r_0 = 13.07\%$.

P16.5 $r_0 = 22.17\%$.

P16.7 A: 12.15%, B: 14.64%, C: 17.13%, and D: 19.62%.

P16.9 APV = \$37.20 million.

P16.11 $r_0 = 12.74\%$.

P16.13 18.15%.

P16.15 16.15%.

P16.17 1. $R'_0 = 1{,}818$, $R'_5 = 184{,}600$; 2. $R'_0 = 0$, $R'_5 = 181{,}672$; and 3. $R'_0 = 0$, $R'_5 = 149{,}462$.

P16.19 (a) 14.06%; (b) $R'_0 = -1{,}004$, $R'_2 = 0$, and $R'_5 = -19{,}274$.

Chapter 17

P17.1 \$1.00 per person.

P17.3 $R_2 - R_1 = \$500$; $S_2 - S_1 = \$1{,}250$.

P17.5 \$32,400.

P17.7 \$3,090.

P17.9 B/C = 1.126.

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